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ON POSITIVE MULTILINEAR MAPS

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1. Introduction and Preliminaries

Let $\mathcal{E}$ be a vector space over $\mathbf{C}$. Throughout this paper let $M_{m,n}(\mathcal{E})$ denote the vector space of $m \times n$ matrices with entries from $\mathcal{E}$, let $M_{m,n}$ denote the $m \times n$ complex matrices with C^*-norm. We set $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$ and $M_n = M_{n,n}$.

If $\mathcal{B}$ is a C^*-algebra and $\mathcal{E}$ is a subspace, then we call $\mathcal{E}$ an operator space. If $\mathcal{E}$ is a subset of a C^*-algebra $\mathcal{B}$, then we set

$$\mathcal{E}^* = \{a : \ a^* \in \mathcal{E}\},$$

and we call $\mathcal{E}$ self-adjoint when $\mathcal{E} = \mathcal{E}^*$. If $\mathcal{B}$ has a unit I and $\mathcal{E}$ is a self-adjoint subspace of $\mathcal{B}$ containing I, then we call $\mathcal{E}$ an operator system.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are operator spaces and $\phi : \mathcal{E} \to \mathcal{F}$ is a linear map. We define the map $\phi_n : M_n(\mathcal{E}) \to M_n(\mathcal{F})$ by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$ for $[x_{ij}] \in M_n(\mathcal{E})$. We write $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbf{N}\}$, where $\|\phi\| = \sup\{\|\phi(x)\| : x \in \mathcal{E}, \|x\| = 1\}$. We call ϕ completely bounded if $\|\phi\|_{cb} < \infty$, and completely contractive if $\|\phi\|_{cb} \leq 1$. We call ϕ a complete isometry if for each $n \in \mathbf{N}$, $\phi_n : M_n(\mathcal{E}) \to M_n(\mathcal{F})$ is an isometry.

Let $\mathcal{B}$ and $\mathcal{C}$ be two C^*-algebras, let $\mathcal{S}$ be an operator system of $\mathcal{B}$, and let $\phi : \mathcal{S} \to \mathcal{C}$ be a linear map. We call ϕ n-positive if ϕ_n is positive and we call ϕ completely positive if ϕ is n-positive for all positive integers n.

Many people have studied the positive linear maps and the completely positive linear maps ([1], [2], [3] e.t.c.).

Throughout the paper $\mathcal{B}$, $\mathcal{B}_k$, and $\mathcal{C}$ will denote unital C^*-algebras, $\mathcal{S}$ will denote operator system, and $\bar{\mathcal{S}}$ will denote the norm closure of $\mathcal{S}$. And $C(\mathcal{X}_k)$ will denote the set of all continuous functions on a compact space $\mathcal{X}_k$, $\mathcal{H}$ will denote a Hilbert space.

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For $P=(p_1,\dots,p_n), Q=(q_1,\dots,q_n)\in \mathcal{S}_1\times\cdots\times\mathcal{S}_n$, set $P\le Q$ if $p_k\le q_k$ for $1\le k\le n$, and $\|P\|=(\|p_1\|,\dots,\|p_n\|)$.

If $P=(p_1,\dots,p_n)\in \mathcal{S}_1\times\cdots\times\mathcal{S}_n$, let $P^*=(p_1^*,\dots,p_n^*)$ and P is said to be self-adjoint if $P=P^*$, and positive if $p_k\ge 0$ for $1\le k\le n$.

In this paper we introduce the positive multilinear maps and the completely positive multilinear maps, and we study some basic properties.

2. Positive Multilinear Maps

DEFINITION 1. A multilinear map $\phi:\mathcal{S}_1\times\cdots\times\mathcal{S}_n\to\mathcal{B}$ is said to be self-adjoint if $\phi(x_1^*,\dots,x_n^*)=\phi(x_1,\dots,x_n)^*$ for $x_k\in\mathcal{S}_k$ ($1\le k\le n$). A multilinear map $\phi:\mathcal{S}_1\times\cdots\times\mathcal{S}_n\to\mathcal{A}$ is said to be positive if $\phi(x_1,\dots,x_n)$ is positive whenever x_k is positive in $\mathcal{S}_k$ for $1\le k\le n$, and bounded if $\|\phi\|=sup\{\|\phi(x_1,\dots,x_n)\|:x_k\in\mathcal{S}_k,\|x_k\|\le 1\}$ is finite.

LEMMA 1. *Let $\phi:\mathcal{S}_1\times\cdots\times\mathcal{S}_n\to\mathcal{B}$ be positive multilinear. For a self-adjoint element H in $\mathcal{S}_1\times\cdots\times\mathcal{S}_k$, define $\phi_H:\mathcal{S}_{k+1}\times\cdots\times\mathcal{S}_n\to\mathcal{B}$ by $\phi_H(x_{k+1},\dots,x_n)=\phi(H,x_{k+1},\dots,x_n)$ for $1\le k\le n-1$. Then ϕ_H is self-adjoint.*

Proof. By induction, it is trivial.

PROPOSITION 2. *If $\phi:\mathcal{S}_1\times\cdots\times\mathcal{S}_n\to\mathcal{B}$ is positive multilinear, then ϕ is self-adjoint.*

Proof. Let $(x,X)\in\mathcal{S}_1\times\cdots\times\mathcal{S}_n$ and $h=\frac{x+x^*}{2}$, $k=\frac{x-x^*}{2i}$. Then $\phi(x^*,X^*)=\phi_h(X^*)-i\phi_k(X^*)=[\phi_h(X)]^*-i[\phi_k(X)]^*=[\phi(h,X)]^*+[\phi(ik,X)]^*=[\phi(x,X)]^*$. Hence ϕ is self-adjoint.

PROPOSITION 3. *If $\phi:\mathcal{S}_1\times\cdots\times\mathcal{S}_n\to\mathcal{B}$ is positive multilnear, then ϕ is bounded and $\|\phi\|\le 2^n\|\phi(1,\dots,1)\|$* (cf. [13, Proposition 2.1]).

Proof. Let $P=(p_1,\dots,p_n), Q=(q_1,\dots,q_n)$ with $P\le Q$ and $A_k=(p_1,\dots,p_k,q_{k+1},\dots,q_n)$ for $0\le k\le n$. Then $\phi(A_{k-1}-A_k)=\phi(A_{k-1})-\phi(A_k)$ is positive and $\phi(P)\le\phi(Q)$. Note that if p and q are positive, then $\|p-q\|\le\max\{\|p\|,\|q\|\}$. Let $H=(h_1,\dots,h_n)$ be self-adjoint in $\mathcal{S}_1\times\cdots\times\mathcal{S}_n$. Then $\phi(H)\le\phi(\|H\|)$ and $-\phi(H)=\phi(-h_1,h_2,\dots,h_n)\le\phi(\|H\|)$. Hence $-\phi(\|H\|)\le\phi(H)\le\phi(\|H\|)$ and $\|\phi(H)\|\le\|\phi(\|H\|)\|=\|h_1\|\cdots\|h_n\|\cdot\|\phi(1,\dots,1)\|$.

Finally, let $A = (a_1, \ldots, a_n)$ be an arbitrary element of $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Put $h_m = \frac{a_m + a_m^*}{2}$ and $k_m = \frac{a_m - a_m^*}{2i}$. Then $\|h_m\| \leq \|a_m\|$ and $\|k_m\| \leq \|a_m\|$ and $\|\phi(A)\| = \|\sum_{l_m \in \{h_m, ik_m\}, 1 \leq m \leq n} \phi(l_1, \ldots, l_n)\| \leq 2^n \|\phi(\|A\|)\|$.

Next an example shows that 2^n is the best constant in the above Proposition.

EXAMPLE 4. Let T denote the unit circle in the complex plane, $C(T)$ the continuous functions on T, z the coordinate function, and $\mathcal{S} \in C(T)$ the subspace spanned by $1, z, \bar{z}$. We define $\phi : [\mathcal{S}]^n \to M_2$ by

$$\phi(a_1 + b_1 z + c_1 \bar{z}, \ldots, a_n + b_n z + c_n \bar{z}) = \begin{bmatrix} a_1 \cdots a_n & 2^n b_1 \cdots b_n \\ 2^n c_1 \cdots c_n & a_1 \cdots a_n \end{bmatrix}$$

By elementary calculation, $a + bz + c\bar{z}$ of S_k is positive if and only if $c = \bar{b}$ and $a \geq 2|b|$. They are well-known that a self -adjoint element of M_2 is positive if and only if its diagonal entries and its determinant are non-negative real numbers, and the Schur product of two positive matrices is positive. Combining these facts, it is clear that ϕ is positive . However, $\phi(z, \ldots, z) = \begin{bmatrix} 0 & 2^n \\ 2^n & 0 \end{bmatrix}$ and $2^n \|\phi(1, \ldots, 1)\| = 2^n = \|\phi(z, \ldots, z)\| \leq \|\phi\|$, so that $\|\phi\| = 2^n \|\phi(1, \ldots, 1)\|$ (cf. [13, Example 2.2]).

PROPOSITION 5. *Let $\phi : C(\mathcal{X}_1) \times \cdots \times C(\mathcal{X}_n) \to \mathcal{B}$ be positive multilinear. Then $\|\phi\| = \|\phi(1, \ldots, 1)\|$* (cf. [13, Theorem 2.4]).

Proof. We may assume that $\phi(1, \ldots, 1) \leq 1$. Let $f_k \in C(\mathcal{X}_k), \|f_k\| \leq 1$, and let $\varepsilon > 0$ be given. Choose a finite open covering $\{U_{ki}\}_{i=1}^{l_k}$ of $\mathcal{X}_k$ and a finite subset $\{y_{ki}\}_{i=1}^{l_k}$ of $\mathcal{X}_k$ such that $|f_k(x) - f_k(y_{ki})| < \varepsilon$ for $x \in U_{ki}$, and let $\{P_{ki}\}$ be a partition of unity subordinate to the covering. Set $\lambda_{ki} = f_k(y_{ki})$ and $p_k = \sum_i \lambda_{ki} P_{ki}$. Then $\|p_k\| \leq 1$ and $\|\phi(p_1, \ldots, p_n)\| \leq 1$ by [13, Lemma 2.3]. Note that if $P_{ki}(x) \neq 0$ for some i, then $|f_k(x) - \lambda_{ki}| < \varepsilon$. Hence, for any $x, |f_k(x) - p_k(x)| \leq \sum |f_k(x) - \lambda_{ki}| P_{ki}(x) < \varepsilon$. Put $F = (f_1, \ldots, f_n)$ and $F_k = (p_1, \ldots, p_k, f_{k+1}, \ldots, f_n)$. Then $\phi(F) = \phi(F_n) + \sum_{i=1}^n \{\phi(F_{k-1}) - \phi(F_k)\}$. Hence $\|\phi(F_k) - \phi(F_{k+1})\| < \varepsilon \|\phi\|$ and $\|\phi(F_n)\| \leq 1$, so that $\|\phi(F)\| \leq n\varepsilon \|\phi\| + 1$, and since ε is arbitrary, $\|\phi\| \leq 1$.

If a is an element of some unital C^*-algebra $\mathcal{B}$, with $\|a\| \leq 1$, then there is a unital homomorphism $\phi : C(T) \to \mathcal{B}$ with $\phi(p) = p(a)$ [consequence of 13, Theorem 2.6].

PROPOSITION 6. *Let $\mathcal{A}_k$ be a subalgebra of $\mathcal{B}_k$ with $1_k \in \mathcal{A}_k$, and let $\mathcal{S}_k = \mathcal{A}_k + \mathcal{A}_k^*$. If $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{C}$ is positive multilinear, then $\|\phi(a_1, \ldots, a_n)\| \leq \|\phi(1, \ldots, 1)\| \|a_1\| \cdots \|a_n\|$ for all a_k in $\mathcal{A}_k$* (cf. [13, Corollary 2.8]).

Proof. Let $A = (a_1, \ldots, a_n) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, $\|a_k\| \leq 1$ for $1 \leq k \leq n$. By the proof of [13, Corollary 2.8], there is a unital homomorphism $\psi_k : C(T) \to \bar{\mathcal{S}}_k$ with $\psi_k(p) = p(a_k)$. Since ϕ is positive, ϕ is bounded, so that ϕ is extended to a multilinear positive map of $\bar{\mathcal{S}}_1 \times \cdots \times \bar{\mathcal{S}}_n$. Define $\Psi = (\psi_1, \ldots, \psi_n) : [C(T)]^n \to \bar{\mathcal{S}}_1 \times \cdots \times \bar{\mathcal{S}}_n$ by $\Psi(x_1, \ldots, x_n) = (\psi_1(x_1), \ldots, \psi_n(x_n))$ for $x_k \in \bar{\mathcal{S}}_k$. Then $\phi \circ \Psi$ is a multilinear positive map, so that by Proposition 5, $\|\phi(a_1, \ldots, a_n)\| = \|(\phi \circ \Psi)(z, \ldots, z)\| \leq \|\phi \circ \Psi\| = \|\phi \circ \Psi(1, \ldots, 1)\| = \|\phi(1, \ldots, 1)\|$.

COROLLARY 7. *Let $\phi : \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \to \mathcal{C}$ be positive multilinear. Then $\|\phi\| = \|\phi(1, \ldots, 1)\|$.*

Proof. Apply Proposition 6.

REMARK 8. If $\phi : \mathcal{S} \to \mathcal{B}$ is a unital contraction, then ϕ is positive. But in case of a unital multilinear contraction, it is not true. For an example, define $\phi : M_2 \times M_2 \to \mathbf{C}$ by $\phi(a, b) = \frac{1}{2}[1, 1]ab[1, 1]^t$. Then ϕ is a unital contraction, but ϕ is not positive since $\phi(\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$ $= \frac{1}{2}(1 - i)$.

Let $\phi : \mathcal{S}_1 \times \cdots \mathcal{S}_n \to \mathcal{B}$ be positive multilinear. If we define $\phi_k : M_k(\mathcal{S}_1) \times \cdots \times M_k(\mathcal{S}_n) \to M_k(\mathcal{B})$ by

$$\phi_k([x_{ij}^1], [x_{ij}^2], \ldots, [x_{ij}^n]) = [\sum_{j_1, \ldots, j_{n-1}=1}^{k} \phi(x_{ij_1}^1, x_{j_1 j_2}^2, \ldots, x_{j_{n-1} j}^n],$$

then $[1,0]\ \phi_2(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & i1 \\ -i1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\ [1,0]^t = (1 - i)\phi(1, \ldots, 1)$. Hence ϕ_2 is not positive except trivial case. Thus in this paper, we define $\phi_k : M_k(\mathcal{S}_1) \times \cdots \times M_k(\mathcal{S}_n) \to M_k(\mathcal{B})$ by

$$\phi_k([x^1_{ij}], [x^2_{ij}], \ldots, [x^n_{ij}]) = [\phi(x^1_{ij}, x^2_{ij}, \ldots, x^n_{ij})].$$

We say ϕ is k-positive if and only if ϕ_k is positive, ϕ is completely positive if and only if ϕ_k is positive for $k \in N$ and ϕ is completely bounded if and only if $\|\phi\|_{cb} = sup\{\|\phi_k\| : k \in N\}$ is finite.

PROPOSITION 9. *Let $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to B$ be a unital 2-positive multilinear map. Then ϕ is contractive* (cf. [13, Proposition 3.2]).

Proof. Let $a_k \in \mathcal{S}_k, \|a_k\| \leq 1$ and $A_k = \begin{bmatrix} 1 & a_k \\ a_k^* & 1 \end{bmatrix}$ for $1 \leq k \leq n$. Then $\phi_2(A_1, \ldots, A_n) = \begin{bmatrix} 1 & \phi(a_1, \ldots, a_n) \\ \phi(a_1^*, \ldots, a_n^*) & 1 \end{bmatrix}$ is positive and hence $\|\phi(a_1, \ldots, a_n)\| \leq 1$.

REMARK 10. If $f : \mathcal{S} \to \mathbf{C}$ is positive, then f is completely positive. But in case of multilinear maps, it is not true. For an example, let $\mathcal{S}$ be the same as in Example 4 and define by $\phi : \mathcal{S} \times \mathcal{S} \to \mathbf{C}$ by $\phi(a_1 + b_1 z + c_1 \bar{z}, a_2 + b_2 z + c_2 \bar{z}) = a_1 a_2 + 2b_1 b_2 + 2c_1 c_2$. Then ϕ is unital positive and $\phi(z, z) = 2$. Hence ϕ is not 2-positive by Proposition 9.

PROPOSITION 11. *Let $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{B}$ be a completely positive multilinear map. Then ϕ is completely bounded and $\|\phi(1, \ldots, 1)\| = \|\phi\|$ $= \|\phi\|_{cb}$* (cf. [13, Proposition 3.5]).

Proof. Clearly we have that $\|\phi(1, \ldots, 1)\| \leq \|\phi\| \leq \|\phi\|_{cb}$, so it is sufficient to show $\|\phi\|_{cb} \leq \|\phi(1, \ldots, 1)\|$.

Let A_m be in $M_k(\mathcal{S}_m)$ with $\|A_m\| \leq 1$, and let I_{mk} be the unit of $M_k(\mathcal{S}_m)$ for $1 \leq m \leq n$. Then $\phi_{2k}(\begin{bmatrix} I_{1k} & A_1 \\ A_1^* & I_{1k} \end{bmatrix}, \ldots, \begin{bmatrix} I_{nk} & A_n \\ A_n^* & I_{nk} \end{bmatrix}) =$ $\begin{bmatrix} \phi_k(I_{1k}, \ldots, I_{nk}) & \phi_k(A_1, \ldots, A_n) \\ \phi_k(A_1^*, \ldots, A_n^*) & \phi_k(I_{1k}, \ldots, I_{nk}) \end{bmatrix})$ is positive for each positive integer k. Thus $\|\phi_k(A_1, \ldots, A_n)\| \leq \|\phi_k(I_{1k}, \ldots, I_{nk})\| = \|\phi(1, \ldots, 1)\|$, which completes the proof.

PROPOSITION 12. *Let $\phi : C(\mathcal{X}_1) \times \cdots \times C(\mathcal{X}_n) \to \mathcal{B}$ be positive multilinear. Then ϕ is completely positive* (cf. [13, Theorem 3.10]).

Proof. Let P_k be positive in $M_m(C(\mathcal{X}_k))$ and let ε be given. Choose a finite open covering $\{O_{ki}\}_{i=1}^{l_k}$ of $\mathcal{X}_k$ and a finite subset $\{y_{ki}\}_{i=1}^{l_k}$ of $\mathcal{X}_k$

such that $\|P_k(x)-P_k(y_{ki})\| < \varepsilon$ for $x \in O_{ki}$, and let U_{ki} be a partition of unity subordinate to the covering. Then $\|P_k(x) - \sum_{i=1}^{l_k} U_{ki}(x)P_k(y_{ki})\| = \| \sum_{i=1}^{l_k} U_{ki}(x)(P_k(x) - P_k(y_{ki}))\| \leq \sum_{i=1}^{l_k} U_{ki}(x)\|P_k(x) - P_k(y_{ki})\| < \varepsilon$. But $\phi_m(U_{1i_1}P_1(y_{1i_1}), \ldots, U_{ni_n}P_n(y_{ni_n})) = [P_1(y_{1i_1}) * \cdots * P_n(y_{ni_n})] * \phi(U_{1i_1}, \ldots, U_{ni_n})$, where $A * B$ denotes the Schur product of A and B. Since $P_1(y_{1i_1}) * \cdots * P_n(y_{ni_n})$ is positive in M_m and $\phi(U_{1i_1}, \ldots, U_{ni_n})$ is positive, $[P_1(y_{1i_1}) * \cdots * P_n(y_{ni_n})] * \phi(U_{1i_1}, \ldots, U_{ni_n})$ is positive. Thus $\phi_m(P_1, \ldots, P_n)$, to within ε, is a sum of positive elements and hence is positive.

Let E_{ij} and F_{ij} denote the stand matrix units for M_m and M_n, respectively. For a matrix A, let $A_k = [A_{ij}]$ denote the $k \times k$ matrix with $A_{ij} = A$ for $1 \leq i, j \leq k$.

PROPOSITION 13. *Let $\phi : M_m \times M_n \to \mathcal{B}$ be bilinear. Then the following are equivalent* (cf. [13, Theorem 3.12]):

(1) *ϕ is completely positive.*
(2) *ϕ is mn-positive.*
(3) *$\phi_{mn}([E_{ij_n}], [F_{ij}]_m)$ is positive in $M_{mn}(\mathcal{B})$.*

Proof. (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). Since $[E_{ij_n}]$ is positive in M_{m^2n} and $[F_{ij}]_m$ is positive in M_{mn^2}, it is clear.

(3) $\Rightarrow$ (1). For this it is sufficient to assume that $\mathcal{B} = \mathcal{B}(\mathcal{H})$. Fix k and let $x_1, \ldots, x_k \in \mathcal{H}$, $A_1, \ldots, A_k \in M_m$ and $B_1, \ldots, B_k \in M_n$. It is sufficient to prove that $\sum_{ij=1}^{k}(\phi(A_i^*A_j, B_i^*B_j)x_j, x_i)$ is positive. Write $A_l = \sum_{r,s=1}^{m} a_{rsl}E_{rs}$ and $B_l = \sum_{e,f=1}^{n} b_{efl}F_{ef}$ so that $A_i^*A_j = \sum \bar{a}_{rsi}a_{rtj}E_{st}$ and $B_i^*B_j = \sum \bar{b}_{efi}b_{egj}F_{st}$. Set $X = (x_1, \ldots, x_k)^t$, $y_{retg} = \sum_{j=1}^{k} a_{rtj}b_{egj}x_j$, $y_{re} = (y_{re11}, \ldots, y_{re1n}, y_{re21}, \ldots, y_{remn})$. Then

$$
\begin{aligned}
&(\phi_k([A_i^*A_j], [B_i^*B_j])X, X) \\
&= \sum_{ij=1}^{k} \phi(A_i^*A_j, B_i^*B_j)x_j, x_i) \\
&= \sum_{rst=1}^{m} \sum_{efg=1}^{n} (\phi(E_{st}, F_{fg})(\sum_{ij=1}^{k} \bar{a}_{rsi}\bar{b}_{efi}a_{rtj}b_{egj}x_j, x_i)
\end{aligned}
$$

$$= \sum_{rst=1}^{m} \sum_{efg=1}^{n} (\phi(E_{st}, F_{fg}) y_{retg}, y_{resf})$$

$$= \sum_{r=1}^{m} \sum_{e=1}^{n} (\phi_{mn}([E_{ij_n}], [F_{ij}]_m) y_{re}, y_{re}).$$

Since $(\phi_{mn}([E_{ij_n}], [F_{ij}]_m)$ is positive, this last sum is positive for each r and e. Hence ϕ is completely positive.

In multilinear case, by similar way we get the following.

PROPOSITION 14. *Let $\phi : M_{k_1} \times \cdots \times M_{k_n} \to \mathcal{B}$ be a multilinear map, let $\{E_{ij}^{k_l}\}$ denote the standard matrix units for M_{k_l} and let $a = k_1 \cdots k_n$, $t_l = k_1 \cdots k_{l-1}$, $s_l = k_{l+1} \cdots k_n$, $t_1 = 1$, $s_n = 1$, $E_l = [E_{ijs_l}^{k_l}]_{t_l}$: Then the following are equivalent:*

(1) *ϕ is completely positive.*
(2) *ϕ is a-positive.*
(3) *$\phi_a(E_1, \ldots, E_n)$ is positive.*

LEMMA 15. *Let $a, b, c \in \mathcal{B}$, let a be positive, and let c be invertible and positive. If $a \geq b^* c^{-1} b$, then $\|c\| a \geq b^* b$.*

Proof. Since $c^{-\frac{1}{2}} \|c\| c^{-\frac{1}{2}} \geq I$, $\|c\| b^* c^{-1} b \geq b^* b$, so $\|c\| a \geq b^* b$.

PROPOSITION 16. *Let $\phi : \mathcal{B}_1 \times \cdots \times \mathcal{B}_r \to \mathcal{B}(\mathcal{H})$ be n-positive multilinear. Then for $a_{ki} \in \mathcal{B}_k$, $i = 1, 2, \ldots, n-1$ $k = 1, \ldots, r$, we have*

$$[\phi(a_{1i}^*, \ldots, a_{ri}^*) \phi(a_{1j}, \ldots, a_{rj})] \leq \|\phi\| [\phi(a_{1i}^* a_{1j}, \ldots, a_{ri}^* a_{rj})]$$

in $M_{n-1}(\mathcal{B}(\mathcal{H}))$ (cf. [6, Theorem 2].

Proof. For some fixed $a_k = a_{k0} \in \mathcal{B}_k$ and arbitrary $a_{ki} \in \mathcal{B}_k$, put $A_k = [a_k, a_{k1}, \ldots, a_{kn-1}]^* [a_k, a_{k1}, \ldots, a_{kn-1}]$, $A_{ij} = [a_{1i}^* a_{1j}, \ldots, a_{ri}^* a_{rj}]$, put $X = (x_0, \ldots, x_{n-1})^t$ for $x_0, \ldots, x_{n-1} \in \mathcal{H}$.

Then

$$(\phi_n(A_1, \ldots, A_r)X, X)$$
$$= \sum_{ij=1}^{n-1} (\phi(A_{ij}) x_j, x_i) + \sum_{i=1}^{n-1} (\phi(A_{i0}) x_0, x_i)$$
$$+ \sum_{i=1}^{n-1} (\phi(A_{0i}) x_i, x_0) + (\phi(A_{00}) x_0, x_0)$$

is positive. We now fix $x_1, \dots, x_{n-1}$ and given $\varepsilon > 0$, put $A_\varepsilon = [\phi(A_{00}) + \varepsilon]^{-1}$, $x_0 = -[\phi(A_{00}) + \varepsilon]^{-1} \sum_{i=1}^{n-1} (\phi(A_{0i})x_i$. Then

$$\begin{aligned}
&(\phi_n(A_1, \dots, A_r)X, X) \\
&= \sum_{ij=1}^{n-1} (\phi(A_{ij})x_j, x_i) - 2\sum_{ij=1}^{n-1} (\phi(A_{i0})A_\varepsilon \phi(A_{0j})x_j, x_i) \\
&\quad + \sum_{ij=1}^{n} (\phi(A_{i0})A_\varepsilon \phi(A_{00})A_\varepsilon \phi(A_{0j})x_j, x_i)
\end{aligned}$$

is positive. But $2A_\varepsilon - A_\varepsilon^2 \phi(A_{00}) = A_\varepsilon + \varepsilon A_\varepsilon^2$. Hence

$$\begin{aligned}
&\sum_{ij=1}^{n-1} (\phi(A_{i0})A_\varepsilon \phi(A_{0j})x_j, x_i) \\
&\le \sum_{ij=1}^{n-1} (\phi(A_{i0})\{A_\varepsilon + \varepsilon A_\varepsilon^2\}\phi(A_{0j})x_j, x_i) \\
&\le \sum_{ij=1}^{n-1} (\phi(A_{ij})x_j, x_i).
\end{aligned}$$

So $[\phi(A_{10}), \dots, \phi(A_{n-10})]^t A_\varepsilon [\phi(A_{01}), \dots, \phi(A_{0n-1})] \le [\phi(A_{ij})]$.

Note that $[\phi(A_{01}), \dots, \phi(A_{0n-1}]^* = [\phi(A_{10}), \dots, \phi(A_{n-10}]^t$. So by Lemma 15,

$$[\phi(A_{01}), \dots, \phi(A_{0n-1})]^* [\phi(A_{01}), \dots, \phi(A_{0n-1})] \le \|\phi(A_{00}) + \varepsilon\| [\phi(A_{ij})].$$

Since ε is arbitrary and each A_k has approximate unit,

$$[\phi(a_{1i}^*, \dots, a_{ri}^*)(\phi(a_{1j}), \dots, a_{rj})] \le \|\phi\| [\phi(a_{1i}^* a_{1j}, \dots, a_{ri}^* a_{rj})].$$

In following, we set $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $V_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$, $U_k = \begin{bmatrix} x_k & y_k \\ z_k & w_k \end{bmatrix}$, $W_k = \begin{bmatrix} e_k & f_k \\ g_k & h_k \end{bmatrix} \in M_2$, set $\phi : M_2 \times M_2 \to M_2$ be bilinear, set $V * U$ the Schur product of A and B, and set

$$E = \begin{bmatrix} \phi(E_{11},E_{11}) & \phi(E_{11},E_{12}) & \phi(E_{12},E_{11}) & \phi(E_{12},E_{12}) \\ \phi(E_{11},E_{21}) & \phi(E_{11},E_{22}) & \phi(E_{12},E_{21}) & \phi(E_{12},E_{22}) \\ \phi(E_{21},E_{11}) & \phi(E_{21},E_{12}) & \phi(E_{22},E_{11}) & \phi(E_{22},E_{12}) \\ \phi(E_{21},E_{21}) & \phi(E_{21},E_{22}) & \phi(E_{22},E_{12}) & \phi(E_{22},E_{22}) \end{bmatrix}.$$

By elementary calculation,we get the following Lemmas.

LEMMA 17. *If $\phi(A,B) = (V^*AV) * (U^*BU)$, then $E = X^*X$, where $X = (ax, by, az, bw, cx, dy, cz, dw)$.*

LEMMA 17′. *If $\phi(A,B) = V^*(A * B)V$, then $E = X^*X$, where $X = (a, b, 0, 0, 0, 0, c, d)$.*

LEMMA 18. *If $E = X^*X$, where $X = (x_1, \ldots, x_8)$ with $x_1x_7 = x_3x_5$, $x_2x_8 = x_4x_6$, then there exist V, U such that $\phi(A,B) = (V^*AV) * (U^*BU)$.*

LEMMA 18′. *If $E = X^*X$, where $X = (a, b, 0, 0, 0, 0, c, d)$, then there exists V such that $\phi(A,B) = V^*(A * B)V$.*

LEMMA 19. *If $\phi(A,B) = \sum_k (V_k^*AV_k) * (U_k^*BU_k)$, then $E = \sum_k X_k^*X_k$, where $X_k = (a_kx_k, b_ky_k, a_kz_k, b_kw_k, c_kx_k, d_ky_k, c_kz_k, d_kw_k)$.*

LEMMA 19′. *If $\phi(A,B) = \sum_k V_k^*(A * B)V_k)$, then $E = \sum_k X_k^*X_k$, where $X_k = (a_k, b_k, 0, 0, 0, 0, c_k, d_k)$.*

LEMMA 20. *If $E = \sum_k X_k^*X_k$, where $X_k = (x_{k1}, \ldots, x_{k8})$ with $x_{k1}x_{k7} = x_{k3}x_{k5}$, $x_{k2}x_{k8} = x_{k4}x_{k6}$, then there exist V_k, U_k such that $\phi(A,B) = \sum_k (V_k^*AV_k) * (U_k^*BU_k)$.*

LEMMA 20′. *If $E = \sum_{i=1}^n X_k^*X_k$, where $X_k = (a_k, b_k, 0, 0, 0, 0, c_k, d_k)$, then there exists V_k such that $\phi(A,B) = \sum_{k=1}^n V_k^*(A * BVk)$.*

PROPOSITION 21. *$\phi(A,B) = \sum_k (V_k^*AV_k) * (U_k^*BU_k)$ for some V_k, U_k in M_2 if and only if there exist $X_k = (x_{k1}, \ldots, x_{k8})$ such that $E = \sum_k X_k^*X_k$ with $x_{k1}x_{k7} = x_{k3}x_{k5}, x_{k2}x_{k8} = x_{k4}x_{k6}$.*

Proof. By Lemma 19 and Lemma 20, it is clear.

PROPOSITION 21′. *$\phi(A,B) = \sum_k V_k^*(A * B)V_k$ for some $V_k \in M_2$ if and only if there exist $X_k = (a_k, b_k, 0, 0, 0, 0, c_k, d_k)$ such that $E = \sum_k X_k^* X_k$.*

Proof. By Lemma 19′ and Lemma 20′, it is clear.

PROPOSITION 22. *$\phi(A,B) = \sum_k\{(V_k^* A V_k)*(U_k^* B U_k) + W_k^*(A*B)W_k\}$ for some $V_k, U_k, W_k \in M_2$ if and only if there exist $X_k = (x_{k1}, \ldots, x_{k8})$ such that $E = \sum_{i=1}^n X_k^* X_k$ with $x_{k1}x_{k7} = x_{k3}x_{k5}, x_{k2}x_{k8} = x_{k4}x_{k6}$ or, $x_3 = x_4 = x_5 = x_6 = 0$.*

Proof. By Proposition 21 and Proposition 21′, it is clear.

REMARK 23. A linear map ϕ from M_n to M_m is completely positive if and only if it admits an expression $\phi(A) = \sum_i V_i^* A V_i$ where V_i are $n \times m$ matrices. But in multilinear case it is not true. For an example, Let $\phi : M_2 \times M_2 \to M_2$ be a bilinear map with $E = (1,0,1,0,1,0,0,0)^t(1,0,1,0,1,0,0,0)$. Then ϕ is completely positive, but $\phi(A,B) \neq \sum_k(V_k^* A V_k)*(U_k^* B U_k) + \sum_k W_k^*(A*B)W_k$ for any V_k, U_k, W_k in M_2 by [2, Remark 4].

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