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ON THE GAMETIC ALGEBRAS

SUK IM YOON

I. Introduction

Throughout this paper, K represents a commutative ring with unity, P an unitary K-module and $d : P \to K$ a surjective K-linear application whose extension is a K-derivation of degree -1, denoted also by d, on the symmetric algebra $S_K(P)$. On the K-module $S_K^m(P)$, we define a multiplication of K-algebra by $x * y = \binom{2m}{m}^{-1} \sum_{r+s=m} \frac{1}{r!s!} d^r(x) d^s(y)$ for all x and y in $S_K^m(P)$, where $d^r = d \circ d \circ \cdots \circ d$, r times and $d^r(x)d^s(y)$ is the multiplication in $S_K(P)$.

We suppose that K is a vector space over $\mathbb{Q}$. We denote this algebra by $S_K^m(P,d)$ and call it gametic algebra.

O. Reiersøl (cf [5]) introduced the multiplication of algebra by means of the derivation for the first time. A. Micali (cf [4]) has studied the derivation on the gametic algebra. In this paper, we give the categorical aspects of $S_K^m(P,d)$ and find a quotient algebra of $S_K^m(P,d)$ that is a commutative Jordan algebra. Furthermore, we are concerned with the automorphism of this quotient algebra.

II. The functorial properties of the gametic algebra

The construction of the gametic algebra permits us to study simultaneously the functorial properties of $S_K^m(P,d)$. In fact, consider the category whose objects are the pairs (P,d) where P is a K-module and $d : P \to K$ a surjective K-linear application. A morphism $f : (P,d) \to (P',d')$ is a K-linear application $f : P \to P'$ such that the diagram

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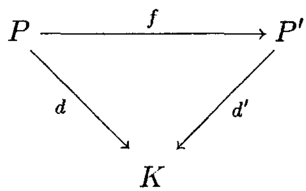

is commutative. For any morphism $f : (P, d) \to (P', d')$ and for all integers $r \geq 1$, we have the following commutative diagram:

$$\begin{array}{ccc} S_k^m(P) & \xrightarrow{d^r} & S_k^{m-r}(P) \\ {\scriptstyle S_k^m(f)}\downarrow & & \downarrow{\scriptstyle S_k^{m-r}(f)} \\ S_k^m(P') & \xrightarrow{d'^r} & S_k^{m-r}(P') \end{array}$$

where $S_k^m(f)$ is the extension of the morphism $f : P \to P'$. It is evident that $S_k^m(f)$ is a K-linear application. Furthermore it is a morphism of the algebra for the gametic multiplication. For any x and y in $S_k^m(P, d)$, we have

$$\begin{aligned} S_K^m(f)(x * y) &= \binom{2m}{m}^{-1} \sum_{r+s=m} \frac{1}{r!s!} S_K^{m-r}(f)(d^r(x)) S_K^{m-s}(f)(d^s(y)) \\ &= \binom{2m}{m}^{-1} \sum_{r+s=m} \frac{1}{r!s!} d'^r(S_K^m(f)(x)) d'^s(S_K^m(f)(y)) \\ &= S_K^m(f)(x) * S_K^m(f)(y). \end{aligned}$$

We remark that if $f : (P, d) \to (P', d')$ is a surjective morphism, then $S_k^m(f) : S_K^m(P, d) \to S_K^m(P', d')$ is also surjective morphism and the kernel of $S_K^m(f)$ is the ideal of $S_K^m(P, d)$ generated by the $\ker(f)$ i.e., $\ker(S_K^m(f)) = \ker(f) S_K(P) \cap S_K^m(P, d)$. But the injective morphism $g : (P, d) \to (P', d')$ does not induce the injective morphism $S_K^m(g)$ (cf [1]).

LEMMA 2.1. *Let K be a commutative ring with unity, P a K-module, $d : P \to K$ a surjective K-linear application and $\varphi : K \to K'$ a morphism of commutative ring with unity. Then there exist an isomorphism of K'-algebras $S_K^m(P,d) \otimes_K K' \approx S_K^m(P \otimes_K K', d')$.*

Proof. The K'-linear application $d : P \otimes_K K' \to K'$ defined by $x \otimes \lambda' \to d(x)\lambda'$ is surjective since $d'(e \otimes 1') = 1'$ where $1'$ is the unity element of K' and $e \in P$ such that $d(e) = 1$. If we define the multiplication of the algebra on the K'-module $S_K^m(P,d) \otimes_K K'$ by $(x \otimes \lambda') * (y \otimes \mu') = (x * y) \otimes (\lambda'\mu')$ for all x and y in $S_K^m(P,d)$ and λ' and μ' in K', then $S_K^m(P,d) \otimes_K K'$ is a gametic algebra.

LEMMA 2.2. *Let K' be the field of fractions of an integral domain K and $f : P \to P'$ an injective K-linear application. Then the K'-linear application $f \otimes id_{K'} : P \otimes_K K' \to P' \otimes_K K'$ is injective, also $S_K^m(f \otimes id_{K'}) : S_K^m(P \otimes_K K') \to S_K^m(P' \otimes_K K')$ is injective and $S_K^m(f \otimes id_{K'}) = S_K^m(f) \otimes id_{K'}$.*

PROPOSITION 2.3. *Let K' be the field of fractions of an integral domain K and P a K-module. Then there exist an isomorphism of $K' = S^{-1}K$-algebras $S^{-1}S_K^m(P,d) \approx S_{S_K^m K}(S^{-1}P, d')$ where $d' : S^{-1}P \to S^{-1}K$ is a surjective $S^{-1}K$-linear application defined by $(x/s) \to (d(x)/s)$, where $S = K - \{0\}$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} S^{-1}K \otimes_K (P,d) & \xrightarrow{f_{S_K^{-1}}} & S^m_{S^{-1}K}(S^{-1}K \otimes_K (P,d)) \\ {\scriptstyle id_{S^{-1}K} \otimes f_K} \big\downarrow & \overset{h}{\nearrow} & \\ S^{-1}K \otimes_K S_K^m(P,d) & & \end{array}$$

where $f_{S_K^{-1}}$ is the canonical $S^{-1}K$-linear application and $f_K : P \to S_K^m(P,d)$ is the canonical K-linear application. Lemma 2.1 and the universal property of the symmetric algebra imply that h is an isomorphism of $S^{-1}K$-algebras.

PROPOSITION 2.4. *Let K be a ring, S a multiplicative subset of K. Then there exist an isomorphism of $S^{-1}K$-algebras $S^{-1}K \otimes_K S_K^m(P,d) \approx S_{S^{-1}K}(S^{-1}P, d')$. The demonstration is the analogy of the Proposition 2.3.*

COROLLARY 2.5. *Let K be a ring, S a multiplicative subset of K. for any maximal ideal m of K, we have an isomorphism of K_m-algebras $(S_K^m(P,d))_m \approx S_{K_m}^m(P_m, d_m)$.*

PROPOSITION 2.6. *Let $((P_i, d_i), (f_{ji}, \lambda_{ji}),\ i \le j)$ be a system of inductive filters in the category of the pairs (P, d), $i \in I$. Then the morphisms $(f_{ji}, \lambda_{ji}) : (P_i, d_i) \to (P_i, d_i)$, $i \le j$, satisfies the following conditions:*

1) $(f_{ii}, \lambda_{ii}) = id(P_i, d_i)$, $\forall i \in I$;

2) $i \le j \le k \to (f_{kj}, \lambda_{kj}) \circ (f_{ji}, \lambda_{ji}) = (f_{ki}, \lambda_{ki})$

We have also the system of inductive filters $(S_K^m(P_i, d_i), S_K(f_{ji}, \lambda_{ji}),\ i \le j)$ and there exist naturally an isomorphism of K-algebras $\varinjlim S_K^m(P_i, d_i) \approx S_K^m(\varinjlim(P_i, d_i))$, that is, we have the following two commutatives diagram:

$$\begin{array}{ccc} (P_i, d_i) & \xrightarrow{\ \imath\ } & S_K^m(P_i, d_i) \\ \downarrow & & \downarrow \\ \varinjlim(P_i, d_i) & \xrightarrow{\ \alpha\ } & \varinjlim S_K^m(P_i, d_i) \end{array}$$

where the morphism α is uniquely determined by a morphism $\imath$ and

$$\begin{array}{ccc} \varinjlim(P_i, d_i) & \xrightarrow{\ \alpha\ } & \varinjlim S_K^m(P_i, d_i) \\ \downarrow & \overset{f}{\nearrow} \ \underset{g}{\swarrow} & \\ S_K^m(\varinjlim(P_i, d_i)) & & \end{array}$$

where f and g are homomorphisms such that

$$f \circ g = I_{S_K^m(\varinjlim(P_i, d_i))} \quad \text{and} \quad g \circ f = I_{\varinjlim S_K^m(P_i, d_i)}$$

III. Quotient algebra of $S_K^m(P, d)$

LEMMA 3.1. *Let e be an element of P such that $d(e) = 1$. Then every element x in $S_K^m(P, d)$ can be written uniquely in the form $x = \sum_{i=0}^m x_i e^{m-i}$, where $x_i \in S_K^i(\ker(d))$ $(i = 0, \ldots, m)$.*

Proof. Since $P = Ke \oplus \ker(d)$, we have

$$S_K^m(P) \approx \oplus_{i+j=m} S_K^i(Ke) \otimes_K S_K^j(\ker(d)).$$

THEOREM 3.1. *Let $S_K^m(P,d)$ be the gametic algebra of the K-module (P,d) and let e be an element of P such that $d(e) = 1$. If $M = \ker(d)S_K(P) \cap S_K^m(P,d)$ is the ideal of $S_K^m(P,d)$ generated by the K-module $\ker(d)$. Then $S_K^m(P,d)/M^2$ is a commutative Jordan algebra.*

Proof. Since the elements of the form $x_i e^{m-i}$ are in M^2 for $i \geq 2$, there is an isomorphism of the K-algebras $S_K^m(P,d)/M^2 \approx K \oplus \ker(d)$ given by $\bar{x} \mapsto (x_0, x_1)$, where $x = \sum_{i=0}^m x_i e^{m-i}$. the structure of K-algebra on $K \oplus \ker(d)$ is given by $(\lambda, x)(\mu, y) = (\lambda_\mu, \frac{1}{2}(\lambda y + \mu x))$ for all λ and μ in K and x and y in $\ker(d)$, where we consider K as a vector space over $\mathbb{Q}$.

THEOREM 3.2. *The group of automorphisms of the $S_K^m(P,d)/M^2$ is isomorphic to the affine group of* $\ker(d)$.

Proof. We know that $S_K^m(P,d)/M^2 \approx K + \ker(d)$ and the idempotent elements of $K + \ker(d)$ are of the form $(1,x)$ with x in $\ker(d)$ and $(0,0)$. Since the image of an idempotent element of an automorphism is also an idempotent element and $(\lambda,x) = \lambda(1,0) + (0,x)$ for all element (λ,x) in $K + \ker(d)$, for any automorphism $\delta \in \mathrm{Aut}(K + \ker(d))$

$$\delta(\lambda,x) = \lambda\delta(1,0) + \delta(0,x) = \lambda(1,x_\delta) + (0,u(x)) = (\lambda, \lambda x_\delta, u(x)),$$

where $\delta(1,0) = (1,x_\delta)$, $x_\delta \in \ker(d)$ and u is the restriction of δ to $\ker(d)$. Hence the application $\mathrm{Aut}(K \oplus \ker(d)) \to \mathrm{Aff}(\ker(d))$ defined by $\delta \mapsto (x_\delta, u)$ is an isomorphism. In fact, it is enough to show that this application is a morphism, i.e.,

$$\delta'\delta \mapsto (x_{\delta'} + u'(x_\delta), u'u) = (x_{\delta'}, u')(x_\delta, u).$$

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Department of Mathematics
Duk Sung Women's University
Seoul 132-714, Korea