Comm. Korean Math. Soc. 8 (1993), No. 4, pp. 587–596

ADDITIVE FUNCTIONS ON ARITHMETIC PROGRESSIONS

JUNGSEOB LEE AND KWANG YOUNG LEE

A complex-valued function f, defined on the set of positive integers, is called additive if $f(mn) = f(m) + f(n)$ for any relatively prime integers m and n. For any integers a, d and real $x > 0$, we define

$$E_f(x; d, a) = \sum_{\substack{n \le x \\ n \equiv a \bmod d}} f(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \le x \\ (n,d)=1}} f(n).$$

This quantity measures the uniformity of the distribution of values of additive functions on arithmetic progressions. It is natural to expect that $E_f(x; d, a)$ is small for all reduced residue classes a modulo d and all integers d up to x.

In Chapter 7 of his book [4], Elliott has extensively investigated the behavior of additive functions on arithmetic progressions. Later Hildebrand [7], motivated by Elliott's work, proved the following.

For any given $c < \frac{1}{2}$ and all additive functions f,

$$\sum_{p^\alpha \le x^c} p^\alpha \max_{\substack{a \\ (a,p)=1}} |E_f(x; p^\alpha, a)|^2 \ll_c \frac{x^2 \log\log x}{\log x} \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}$$

where p^α runs through prime powers,and the implicit constant depends only on c.

One can find in Elliott [5] a similar inequality under more severe restrictions. Their results can legitimately be considered as counterparts in the theory of additive functions to the Bombieri-Vinogradov theorem in prime number theory. As in the case of the Bombieri-Vinogradov

Received May 6, 1993.

The work of the second author was supported by the Ajou University Research Fund.

theorem, it is desired to prove a theorem which allows the constant c in the above theorem to be as large as 1.

In this note, considering the mean square average of $E_f(x;d,a)$, we will prove, in a relatively simple manner, results which are useful for large moduli. The following theorem has been sought as an analogue of the result on primes in arithmetic progressions, due to Davenport-Halberstam [4] and Gallagher [6, Theorem 3].

THEOREM 1. *For any $x>0$, $Q>0$ and all additive functions f,*

$$\sum_{d\le Q}\sum_{\substack{a=1\\(a,d)=1}}^{d}|E_f(x;d,a)|^2 \ll B\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^\alpha}$$

where

$$B=B(x,Q)=\min\left(Qx\log\log x, Qx+\frac{Q^2\log\log x}{x^2}\right).$$

One finds that the above inequality holds with $B=Qx$ for $Q\le x^3/\log\log x$.

Confining the moduli to prime powers, we obtain an inequality in a slightly strengthened form.

THEOREM 2. *For any $x>0$, $Q>0$ and all additive functions f,*

$$\sum_{p^\alpha\le Q}\max_{y\le x}\sum_{\substack{a=1\\(a,p)=1}}^{p^\alpha}|E_f(y;p^\alpha,a)|^2 \ll B\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^\alpha}$$

where

$$B=B(x,Q)=\min\left((x+Q)x\log\log x, x^2+Qx+\frac{Q^2\log\log x}{x^2}\right).$$

The following theorem is concerning general moduli though it is meaningful when Q is significantly smaller than x.

THEOREM 3. *Let $\epsilon > 0$ be given. Then for any $x > 0$, $Q > 0$ and all additive functions f, we have*

$$\sum_{d\leq Q} d \max_{y\leq x} \max_{\substack{a \\ (a,d)=1}} |E_f(y;d,a)| \ll \frac{Q^{\frac{19}{9}+\epsilon}x}{\sqrt{\log x}} \left(\sum_{p^\alpha \leq x} \frac{|f(p^\alpha)|^2}{p^\alpha} \right)^{\frac{1}{2}}.$$

To prove the theorems, we will need several lemmas. The first one is the well-known maximal version of the large sieve inequality, due to Montgomery [8].

LEMMA 1. *For any real $x > 0$, $Q > 0$ and any complex numbers a_n,*

$$\sum_{d\leq Q} \frac{d}{\varphi(d)} \sideset{}{^*}\sum_{\chi \bmod d} \max_{y\leq x} |\sum_{n\leq y} a_n \chi(n)|^2 \ll (x+Q^2) \sum_{n\leq x} |a_n|^2$$

where $\sum^*_{\chi \bmod d}$ denotes the sum over all primitive characters modulo d.

LEMMA 2. *For any real number $Q > 0$, and any complex numbers a_n,*

$$\sum_{d\leq Q} \frac{1}{\varphi(d)} \sideset{}{^*}\sum_{\chi \bmod d} \max_{y\leq x} |\sum_{n\leq y} a_n \chi(n)|^2 \ll (x+Q) \sum_{n\leq x} |a_n|^2.$$

Proof. We see that for any $U > 0$,

$$\sum_{U<d\leq 2U} \frac{1}{\varphi(d)} \sideset{}{^*}\sum_{\chi \bmod d} \max_{y\leq x} |\sum_{n\leq x} a_n \chi(n)|^2$$
$$\ll \frac{1}{U} \sum_{U<d\leq 2U} \frac{d}{\varphi(d)} \sideset{}{^*}\sum_{\chi \bmod d} \max_{y\leq x} |\sum_{n\leq x} a_n \chi(n)|^2.$$

By Lemma 1, this is $\ll (xU^{-1}+U)\sum_{n\leq x}|a_n|^2$. Summing over $U = 2^k$, $1 < U \leq Q$, we complete the proof.

LEMMA 3. *For any integer d and real $x > 0$,*

$$\sum_{\substack{a=1 \\ (a,d)=1}}^{d} |E_f(x;d,a)|^2 = \frac{1}{\varphi(d)} \sideset{}{'}\sum_{\chi \bmod d} |\sum_{n\leq x} f(n)\chi(n)|^2$$

where $\sum'_{\chi \bmod d}$ *denotes the sum over all non-principal characters modulo* d.

Proof. We recall the orthogonal properties of Dirichlet characters. If $(a,d)=1$, then

$$\sum_{\chi \bmod d} \overline{\chi}(a)\chi(n) = \begin{cases} \varphi(d) & \text{if } a \equiv n \bmod d, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For any two characters χ and ξ modulo d,

$$\sum_{a=1}^{d} \chi(a)\overline{\xi}(a) = \begin{cases} \varphi(d) & \text{if } \chi = \xi, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

By the relation (1), we find that for any a with $(a,d)=1$,

$$E_f(x;d,a) = \frac{1}{\varphi(d)} \sum_{\chi \bmod d}{}' \overline{\chi}(a) \sum_{n \le x} f(n)\chi(n). \tag{3}$$

On squaring this out and summing over all reduced residue classes a modulo d,

$$\sum_{\substack{a=1 \\ (a,d)=1}}^{d} |E_f(x;d,a)|^2$$

$$= \frac{1}{\varphi(d)^2} \sum_{\chi}{}' \sum_{\xi}{}' \sum_{n \le x} f(n)\chi(n) \sum_{m \le x} \overline{f}(m)\overline{\xi}(m) \sum_{a=1}^{d} \chi(a)\overline{\xi}(a).$$

Applying the relation (2), we finish the proof.

LEMMA 4. *Let f be an additive function. Then for any $x > 0$,*

$$\sum_{n \le x} |f(n)|^2 \ll x \log\log x \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}.$$

Proof. Using the additivity of f and changing the order of summations, we find that the left hand is

$$\sum_{n\le x}|\sum_{p^\alpha\|n} f(p^\alpha)|^2 \le \sum_{p^\alpha\le x}\sum_{q^\beta\le x}|f(p^\alpha)f(q^\beta)| \sum_{\substack{n\le x\\ p^\alpha\|n,q^\beta\|n}} 1,$$

where $p^\alpha\|n$ means p^α divides n but $p^{\alpha+1}$ does not. First we estimate the contribution of those terms with $p\ne q$. In this case the innermost sum is not greater than $x/p^\alpha q^\beta$. Thus by Cauchy's inequality we find that the last sum with the restriction $p\ne q$ is

$$\le x\left(\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|}{p^\alpha}\right)^2 \le x\sum_{p^\alpha\le x}\frac{1}{p^\alpha}\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^\alpha}.$$

Since $\sum_{p^\alpha\le x}p^{-\alpha}\ll\log\log x$, we obtain the wanted bound.

The contribution of those terms with $p=q$ to the sum is bounded by

$$x\sum_{\substack{p,\alpha\\ p^\alpha\le x}}\sum_{\beta\le\alpha}\frac{|f(p^\alpha)|}{p^\alpha}|f(p^\beta)| = x\sum_{\substack{p,\alpha\\ p^\alpha\le x}}\sum_{\beta\le\alpha}\frac{|f(p^\alpha)|}{p^{3\alpha/4}}p^{\beta/4}\frac{|f(p^\beta)|}{p^{\alpha/4}}p^{-\beta/4}.$$

By the inequality of arithmetic and geometric means, this is

$$\begin{aligned}&\le x\sum_{\substack{p,\alpha\\ p^\alpha\le x}}\sum_{\beta\le\alpha}\left(\frac{|f(p^\alpha)|^2}{p^{3\alpha/2}}p^{\beta/2}+\frac{|f(p^\beta)|^2}{p^{\alpha/2}}p^{-\beta/2}\right)\\ &= x\left(\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^{3\alpha/2}}\sum_{\beta\le\alpha}p^{\beta/2}+\sum_{p^\beta\le x}\frac{|f(p^\beta)|^2}{p^{\beta/2}}\sum_{\alpha\ge\beta}p^{-\alpha/2}\right)\\ &\ll x\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^\alpha}.\end{aligned}$$

This is less than the wanted bound.

Proof of Theorem 1. Squaring out $E_f(x;d,a)$ using the definition, we have

$$|E_f(x;d,a)|^2 = |\sum_{\substack{n\le x\\ n\equiv a \bmod d}} f(n)|^2 - \frac{1}{\varphi(d)} \sum_{\substack{n\le x\\ n\equiv a \bmod d}} f(n) \sum_{\substack{n\le x\\ (n,d)=1}} \overline{f}(n)$$
$$- \frac{1}{\varphi(d)} \sum_{\substack{n\le x\\ n\equiv a \bmod d}} \overline{f}(n) \sum_{\substack{n\le x\\ (n,d)=1}} f(n) + \frac{1}{\varphi(d)^2} |\sum_{\substack{n\le x\\ (n,d)=1}} f(n)|^2.$$

Summing over all reduced residue classes a modulo d and interchanging the summations, we find that

$$\sum_{\substack{a=1\\ (a,d)=1}}^{d} |E_f(x;d,a)|^2 = \sum_{\substack{n\le x\\ (n,d)=1}} |f(n)|^2 - \frac{1}{\varphi(d)} |\sum_{\substack{n\le x\\ (n,d)=1}} f(n)|^2$$
$$\le \sum_{n\le x} |f(n)|^2.$$

Using Lemma 4, we see that the summation of the above over all positive integers $d \le Q$ is bounded by

$$Qx \log\log x \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}.$$

To get the second bound, we define $A = A(f,x) = x^{-1}\sum_{p^\alpha\le x} f(p^\alpha)/p^\alpha$ and an arithmetic function $g(n) = f(n) - A$. Then a straightforward calculation leads to

$$E_f(x;d,a) = E_g(x;d,a) + A\left(\sum_{\substack{n\le x\\ n\equiv a \bmod d}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n\le x\\ (n,d)=1}} 1\right)$$
$$= E_g(x;d,a) + O(A).$$

We use the same argument as above to get

$$\sum_{\substack{a=1\\ (a,d)=1}}^{d} |E_g(x;d,a)|^2 \le \sum_{n\le x} |g(n)|^2 = \sum_{n\le x} |f(n) - A|^2.$$

By the Turán-Kubilius inequality, this is

$$\ll x \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}.$$

Summing over all positive integers $d \le Q$,

$$\sum_{d \le Q} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} |E_g(x; d, a)|^2 \ll Qx \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}. \tag{4}$$

Now we estimate the contribution of $O(A)$ term. First we find by Cauchy's inequality that

$$|A|^2 \le \frac{1}{x^2} \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha} \sum_{p^\alpha \le x} \frac{1}{p^\alpha} \ll \frac{\log\log x}{x^2} \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}.$$

Summing over all reduced residue classes a modulo d and all integers $d \le Q$, we get

$$\sum_{d \le Q} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} |A|^2 \ll \frac{Q^2 \log\log x}{x^2} \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}. \tag{5}$$

Combining (4) and (5), we obtain the second bound and complete the proof.

Proof of Theorem 2. By Lemma 3, we find that

(6)
$$\sum_{p^\alpha \le Q} \max_{y \le x} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^\alpha} |E_f(y; p^\alpha, a)|^2 = \sum_{p^\alpha \le Q} \frac{1}{\varphi(p^\alpha)} \max_{y \le x} \sum_{\chi \bmod p^\alpha}{}' \Big|\sum_{n \le y} f(n)\chi(n)\Big|^2.$$

Since any non-principal character χ modulo p^α is induced by some primitive character χ_1 modulo p^β, $1 \le \beta \le \alpha$, and $\chi(n) = \chi_1(n)$ for every integer n, the above is

$$\begin{aligned}&\sum_{p^\alpha\le Q}\frac{1}{\varphi(p^\alpha)}\max_{y\le x}\sum_{\beta=1}^{\alpha}\sum_{\chi \bmod p^\beta}{}^{*}\,|\sum_{n\le y}f(n)\chi(n)|^2\\ \le&\sum_{p^\beta\le Q}\frac{1}{\varphi(p^\beta)}\sum_{t\ge1}\frac{1}{p^t}\max_{y\le x}\sum_{\chi \bmod p^\beta}{}^{*}\,|\sum_{n\le y}f(n)\chi(n)|^2\\ \ll&\sum_{p^\beta\le Q}\frac{1}{\varphi(p^\beta)}\max_{y\le x}\sum_{\chi \bmod p^\beta}{}^{*}\,|\sum_{n\le y}f(n)\chi(n)|^2\\ \ll&(x+Q)\sum_{n\le x}|f(n)|^2\end{aligned}$$

by Lemma 2. Applying Lemma 4, we obtain the first bound. In order to get the second bound, we define A and g as in the proof of Theorem 1, and see that

$$E_f(y;d,a)=E_g(y;d,a)+O(A).$$

On successively applying Lemma 4 and the Turán-Kubilius inequality, we find that the contribution of $E_g(y;d,a)$ to the sum on the left hand of (6) is

$$\ll(x+Q)x\sum_{p^\alpha\le x}\frac{|f(p^\alpha)|^2}{p^\alpha}.$$

We treat the contribution of $O(A)$ term as in the proof of Theorem 1.

Proof of Theorem 3. By the identity (3), we have

$$\begin{aligned}\max_{\substack{a\\(a,d)=1}}|E_f(y;d,a)|&=\max_{\substack{a\\(a,d)=1}}|\frac{1}{\varphi(d)}\sum_{\chi \bmod d}{}'\,\overline{\chi}(a)\sum_{n\le y}f(n)\chi(n)|\\&\le\frac{1}{\varphi(d)}\sum_{\chi \bmod d}{}'\,|\sum_{n\le y}f(n)\chi(n)|.\end{aligned}$$

Since f is additive,

$$\sum_{n\le y}f(n)\chi(n)=\sum_{n\le y}\sum_{p^\alpha\|n}f(p^\alpha)\chi(n)=\sum_{p^\alpha\le y}f(p^\alpha)\chi(p^\alpha)\sum_{\substack{m\le y/p^\alpha\\p\nmid m}}\chi(m).$$

It is known that $\sum_{m\leq y}\chi(m)\ll y^{\frac{2}{3}}d^{\frac{1}{9}+\epsilon}$ for any non-principal character χ modulo d(See Burgess [2]). Thus the above is

$$\ll y^{\frac{2}{3}}d^{\frac{1}{9}+\epsilon}\sum_{p^\alpha\leq y}\frac{|f(p^\alpha)|}{p^{2\alpha/3}}.$$

Hence

$$\begin{aligned}&\sum_{d\leq Q}d\max_{y\leq x}\max_{\substack{a\\(a,d)=1}}|E_f(y;d,a)|\\ &\ll x^{\frac{2}{3}}\sum_{d\leq Q}d^{\frac{10}{9}+\epsilon}\sum_{p^\alpha\leq x}\frac{|f(p^\alpha)|}{p^{2\alpha/3}}\\ &\leq x^{\frac{2}{3}}Q^{\frac{19}{9}+\epsilon}\sum_{p^\alpha\leq x}\frac{|f(p^\alpha)|}{p^{2\alpha/3}}.\end{aligned}$$

By Cauchy's inequality the last sum above is not greater than

$$\left(\sum_{p^\alpha\leq x}\frac{1}{p^{\alpha/3}}\right)^{\frac{1}{2}}\left(\sum_{p^\alpha\leq x}\frac{|f(p^\alpha)|^2}{p^\alpha}\right)^{\frac{1}{2}}\ll\frac{x^{\frac{1}{3}}}{\sqrt{\log x}}\left(\sum_{p^\alpha\leq x}\frac{|f(p^\alpha)|^2}{p^\alpha}\right)^{\frac{1}{2}}.$$

This completes the proof.

References

1. E. Bombieri, *Le grand crible dans la théorie des nombres*, Astérique **18** (1974).
2. D. A. Burgess, *The character sum estimate with* $r=3$, J. London Math. Soc. **33** (1986), 219–226.
3. H. Davenport and H. Halberstam, *Primes in arithmetic progressions*, Michigan Math. J. **13** (1966), 485–489.
4. P. D. T. A. Elliott, *Arithmetic function and integer products*, Springer, New York, 1985.
5. P. D. T. A. Elliott, *Additive arithmetic functions on arithmetic progression*, Proc. London Math. Soc. **54** (1987), 15–37.
6. P. X. Gallagher, *The large sieve*, Mathematika **14** (1967), 14–20.

7. A. Hildebrand, *Additive functions on arithmetic progressions*, J. London Math. Soc. **34** (1986), 394–402.
8. H. L. Montgomery, *Maximal variants of the large sieve*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1982), 805–812.

Department of Mathematics
Ajou University
Suwon 441-749, Korea