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ON THE COMMUTATIVITY OF CERTAIN SEMIPRIME RINGS

SUNG MI PARK, DONG JE KWAK AND YOUNG SOO PARK

1. Introduction

In all that follows R will represent an associative ring (may be without unity 1). For any pair x, y in R, we set as usual $[x,y] = xy - yx$.

In [1] Abdullah H. Moajil proved that, if R is a 2-torsion free semiprime ring such that $[xy, [xy, yx]] = 0$ for all x, y in R, then R is commutative. Giri and Dhoble proved that, if R is a 2-torsion free semiprime ring such that $[yxy, [xy, yx]] = 0$ for all x, y in R, then R is commutative.

Later in [3] Giri and Modi wished to generalize the result as follows: if R is a 2-torsion free semiprime ring such that $[(xy)^2, [xy, yx]] = 0$ for all x, y in R, then R is commutative. But there is a mistake in the paper. They proved the lemma that ring is p-torsion free if and only if $(p+1)$-torsion free. However we can easily show that it is false as taking counter examples, Z_2, Z_3 etc. So we modified the hypothesis, replacing 2-torsion free with 2 and 3 torsion free.

In this thesis, motivated by the above polynomial identities, we intend to prove the following results.

(1) Let R be a 2 and 3 torsion free prime ring such that $[(xy)^3, [xy, yx]] = 0$ for all x, y in R. Then R is commutative.

(2) Let R be a 2 and 3 torsion free semiprime ring such that $[(xy)^3, [xy, yx]] = 0$ for all x, y in R. Then R is commutative.

2. Preliminaries

We mention below three lemmas in which the first is trivial, next two are well known.

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LEMMA 2.1. *For all elements x, y, z in a ring R,*

$$[xy, z] = [x, z]y + x[y, z].$$

LEMMA 2.2. ([7]) *Let R be a prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation. Then one at least of d_1, d_2 is zero.*

LEMMA 2.3. ([5]) *Let R be a ring having no non-zero nil ideals in which for every x, y in R there exist integers $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $[x^m, y^n] = 0$. Then R is commutative.*

3. Main Results

We introduce our main theorems.

THEOREM 3.1. *Let R be a 2 and 3 torsion free prime ring such that $[(xy)^3, [xy, yx]] = 0$ for all x, y in R. Then R is commutative.*

Proof. Substituting $x + y$ for x in the given identity

$$(3.1) \qquad [(xy)^3, [xy, yx]] = 0.$$

Using it in subsequent expansion, we get

$$\begin{aligned}(3.2) \quad & [(xy)^3, [xy, y^2]] + [(xy)^3, [y^2, yx]] + [(xy)^2 y^2, [xy, yx]] \\ & + [(xy)^2 y^2, [xy, y^2]] + [(xy)^2 y^2, [y^2, yx]] + [xy^3 xy, [xy, yx]] \\ & + [xy^3 xy, [xy, y^2]] + [xy^3 xy, [y^2, yx]] + [xy^5, [xy, yx]] \\ & + [xy^5, [xy, y^2]] + [xy^5, [y^2, yx]] + [y^2 (xy)^2, [xy, yx]] \\ & + [y^2 (xy)^2, [xy, y^2]] + [y^2 (xy)^2, [y^2, yx]] + [y^2 xy^3, [xy, yx]] \\ & + [y^2 xy^3, [xy, y^2]] + [y^2 xy^3, [y^2, yx]] + [y^4 xy, [xy, yx]] \\ & + [y^4 xy, [xy, y^2]] + [y^4 xy, [y^2, yx]] + [y^6, [xy, yx]] \\ & + [y^6, [xy, y^2]] + [y^6, [y^2, yx]] = 0.\end{aligned}$$

By using Lemma 2.1 twice successively, we obtain

$$(3.3) \quad [(xy)^3, [x, y^2]]y + [x, y^2][(xy)^3, y] + y[(xy)^3, [y^2, x]]$$

$$
\begin{aligned}
&+ [(xy)^3, y][y^2, x] + [(xy)^2y^2, [xy, yx]] + [(xy)^2y^2, [x, y^2]]y \\
&+ [x, y^2][(xy)^2y^2, y] + [(xy)^2y^2, y][y^2, x] + y[(xy)^2y^2, [y^2, x]] \\
&+ [xy^3xy, [xy, yx]] + [x, y^2][xy^3xy, y] + [xy^3xy, [x, y^2]]y \\
&+ [xy^3xy, y][y^2, x] + y[xy^3xy, [y^2, x]] + [xy^5, [xy, yx]] \\
&+ [x, y^2][xy^5, y] + [xy^5, [x, y^2]]y + [xy^5, y][y^2, x] \\
&+ y[xy^5, [y^2, x]] + [y^2(xy)^2, [xy, yx]] + [y^2(xy)^2, [x, y^2]]y \\
&+ [x, y^2][y^2(xy)^2, y] + y[y^2(xy)^2, [y^2, x]] + [y^2(xy)^2, y][y^2, x] \\
&+ [y^2xy^3, [xy, yx]] + [x, y^2][y^2xy^3, y] + [y^2xy^3, [x, y^2]]y \\
&+ [y^2xy^3, y][y^2, x] + y[y^2xy^3, [y^2, x]] + [y^4xy, [xy, yx]] \\
&+ [y^4xy, [x, y^2]]y + [x, y^2][y^4xy, y] + [y^4xy, y][y^2, x] \\
&+ y[y^4xy, [y^2, x]] + [y^6, [xy, yx]] + [y^6, [x, y^2]]y \\
&+ y[y^6, [y^2, x]] = 0.
\end{aligned}
$$

Substituting $x + y$ for x in (3.3) and using (3.3), we get

$$
\begin{aligned}
(3.4)\quad & [(xy)^2y^2, [x, y^2]]y + [xy^3xy, [x, y^2]]y + 2[xy^5, [x, y^2]]y \\
&+ [y^2(xy)^2, [x, y^2]]y + 3[y^2xy^3, [x, y^2]]y + 3[y^4xy, [x, y^2]]y \\
&+ 7[y^6, [x, y^2]]y + [x, y^2][(xy)^2y^2, y] + [x, y^2][xy^3xy, y] \\
&+ 3[x, y^2][xy^5, y] + [x, y^2][y^2(xy)^2, y] + 3[x, y^2][y^2xy^3, y] \\
&+ 3[x, y^2][y^4xy, y] + y[(xy)^2y^2, [y^2, x]] + y[xy^3xy, [y^2, x]] \\
&+ 3y[xy^5, [y^2, x]] + y[y^2(xy)^2, [y^2, x]] + 3y[y^2xy^3, [y^2, x]] \\
&+ 3y[y^4xy, [y^2, x]] + 7y[y^6, [y^2, x]] + [(xy)^2y^2, y][y^2, x] \\
&+ [xy^3xy, y][y^2, x] + 3[xy^5, y][y^2, x] + [y^2(xy)^2, y][y^2, x] \\
&+ 3[y^2xy^3, y][y^2, x] + 3[y^4xy, y][y^2, x] + [(xy)^2y^2, [xy, y^2]] \\
&+ [(xy)^2y^2, [y^2, yx]] + 2[xy^5, [xy, yx]] + 3[xy^5, [xy, y^2]] \\
&+ 3[xy^5, [y^2, yx]] + 2[y^2xy^3, [xy, yx]] + 3[y^2xy^3, [xy, y^2]] \\
&+ 3[y^2xy^3, [y^2, yx]] + 6[y^6, [xy, yx]] + 7[y^6, [xy, y^2]] \\
&+ 7[y^6, [y^2, yx]] + [xy^3xy, [xy, y^2]] + [xy^3xy, [y^2, yx]] \\
&+ 2[y^4xy, [xy, yx]] + 3[y^4xy, [xy, y^2]] + 3[y^4xy, [y^2, yx]] \\
&+ [xy^5, [x, y^2]]y + [y^2(xy)^2, [xy, y^2]] + [y^2(xy)^2, [y^2, yx]] \\
&= 0.
\end{aligned}
$$

By Lemma 2.1 and using the fact that R is a 2-torsion free, the equation (3.4) becomes

$$\begin{aligned}(3.5)\quad & [(xy)^2,[x,y^2]]y+[xy^3xy,[x,y^2]]y+3[xy^5,[x,y^2]]y \\ & +[y^2(xy)^2,[x,y^2]]y+3[y^2xy^3,[x,y^2]]y+3[y^4xy,[x,y^2]]y \\ & +7[y^6,[x,y^2]]y+[x,y^2][(xy)^2y^2,y]+[x,y^2][xy^3xy,y] \\ & +3[x,y^2][xy^5,y]+[x,y^2][y^2(xy)^2,y]+3[x,y^2][y^2xy^3,y] \\ & +3[x,y^2][y^4xy,y]+y[(xy)^2y^2,[y^2,x]]+y[xy^3xy,[y^2,x]] \\ & +3y[xy^5,[y^2,x]]+y[y^2(xy)^2,[y^2,x]]+3y[y^2xy^3,[y^2,x]] \\ & +3y[y^4xy,[y^2,x]]+7y[y^6,[y^2,x]]+[(xy)^2y^2,y][y^2,x] \\ & +[xy^3xy,y][y^2,x]+3[xy^5,y][y^2,x]+[y^2(xy)^2,y][y^2,x] \\ & +3[y^2xy^3,y][y^2,x]+3[y^4xy,y][y^2,x]+[(xy)^5,[xy,yx]] \\ & +[y^2xy^3,[xy,yx]]+3[y^6,[xy,yx]]+[y^4xy,[xy,yx]] \\ & =0.\end{aligned}$$

Replacing x by $x+y$ in (3.5), using (3.5) and Lemma 2.1, we get

$$\begin{aligned}(3.6)\quad & 3[xy^5,[x,y^2]]y+3[y^2xy^3,[x,y^2]]y+18[y^6,[x,y^2]]y \\ & +3[y^4xy,[x,y^2]]y+3[x,y^2][xy^5,y]+3[x,y^2][y^2xy^3,y] \\ & +3[x,y^2][y^4xy,y]+3y[xy^5,[y^2,x]]+3y[y^2xy^3,[y^2,x]] \\ & +18y[y^6,[y^2,x]]+3y[y^4xy,[y^2,x]]+3[xy^5,y][y^2,x] \\ & +3[y^2xy^3,y][y^2,x]+3[y^4xy,y][y^2,x]+3[y^6,[xy,yx]] \\ & =0.\end{aligned}$$

Since R is 3-torsion free, so (3.6) yields

$$\begin{aligned}(3.7)\quad & [xy^5,[x,y^2]]y+[y^2xy^3,[x,y^2]]y+6[y^6,[x,y^2]]y \\ & +[y^4xy,[x,y^2]]y+[x,y^2][xy^5,y]+[x,y^2][y^2xy^3,y] \\ & +[x,y^2][y^4xy,y]+y[xy^5,[y^2,x]]+y[y^2xy^3,[y^2,x]] \\ & +6y[y^6,[y^2,x]]+y[y^4xy,[y^2,x]]+[xy^5,y][y^2,x] \\ & +[y^2xy^3,y][y^2,x]+[y^4xy,y][y^2,x]+[y^6,[xy,yx]] \\ & =0.\end{aligned}$$

Again replacing x by $x+y$ in (3.7) and using (3.7), we obtain

$$4[y^6,[x,y^2]]y+4y[y^6,[y^2,x]]=0. \tag{3.8}$$

Since R is 2-torsion free, (3.8) becomes

$$\begin{aligned} &[y^6,[x,y^2]]y+y[y^6,[y^2,x]]=0 \\ \text{or}\quad &y[y^6,[y^2,x]]-[y^6,[y^2,x]]y=0. \end{aligned} \tag{3.9}$$

Let I_r denote the inner derivation with respect to r. i.e., $I_r : x \longrightarrow [r,x]$, then we obtain

$$I_yI_{y^6}I_{y^2}(x)=0. \tag{3.10}$$

Using Lemma 2.2 either $I_yI_{y^6}=0$ or $I_{y^2}=0$, where $I_yI_{y^6}$ and I_{y^2} are inner derivations.

Case 1. $I_yI_{y^6}=0$ implies $I_yI_{y^6}(x)=0$ for any $x\in R$. Thus by Lemma 2.2 either $I_y=0$ or $I_{y^6}=0$. In former case R is commutative. In latter case $[y^6,x]=0$ for all x,y in R, which by Lemma 2.3 yields that R is commutative.

Case 2. $I_{y^2}=0$ implies $[y^2,x]=0$ for all x,y in R which by Lemma 2.3 gives that R is commutative.

THEOREM 3.2. *Let R be a 2 and 3 torsion free semiprime ring such that $[(xy)^3,[xy,yx]]=0$ for all x,y in R. Then R is commutative.*

Proof. Since R is semiprime ring, it is isomorphic to a subdirect sum of prime rings R_α each of which as a homomorphic image of R, satisfies the hypothesis places on R. But by Theorem 3.1 R_α are commutative. Hence R is commutative.

EXAMPLE 3.3. The following example shows that condition of semiprime ring is essential.

Let

$$R=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d\in Z_5\right\}.$$

The above ring satisfies the identity $[(xy)^3,[xy,yx]]=0$ for all x,y in R and is 2 and 3 torsion free, yet R is not commutative.

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Kyungpook National University
Taegu 702-701, Korea