Comm. Korean Math. Soc. 8 (1993), No. 2, pp. 329–333

THE PRINCIPAL COMPONENT OF AN H–STRUCTURE

KYU BUM HWANG

We consider structures consisting of a space X together with a multiplication $\mu : X \times X \to X$ that has a simple homotopy property. Such structures include, as special cases, both topological groups and path spaces. Then the loop spaces are topological spaces in which a continuous multiplication, the composition of two paths into one, is defined. A topological group is another example of a space in which a continuous composition law is defined.

In this paper, the principal component of the H-structure is defined and the relationships between H-structures and topological groups will be investigated. A path in X is an element of $X^I = \{f : I \to X \mid f :$ continuous map$\}$. The product of two paths f, g written $f * g$ is defined only in case $f(1) = g(0)$ and is the path

$$f * g = \begin{cases} f(2t) & 0 \le t \le 1/2, \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The inverse of a path $f \in X^I$ is the path $f^{-1} \in X^I$ defined by the rule $f^{-1}(t) = f(1-t)$, $0 \le t \le 1$. We take X^I with the compact open topology. Given $a, b \in X$, $\Omega(X; a, b)$ is the subspace of X^I consisting of all paths starting a and ending b. In case $a = b$, $\Omega(X; a, a)$ is written simply $\Omega(X; a)$ and is called the loop space of X bases at a. For any two paths, we denote $f \sim g$ if they belong to the same path component of X^I, that is, f is homotopic to g.

LEMMA 1. *The map $f \to f^{-1}$ of $\Omega(X; a, b) \to \Omega(X; b, a)$ is a homeomorphism. The mapping $(f, g) \to f * g$ of $\Omega(X; a, b) \times \Omega(X; b, c) \to \Omega(X; a, c)$ is continuous.*

Received February 11, 1992.

LEMMA 2. *The map* $f \to f * f^{-1}$ *of* $\Omega(X; a, b) \to \Omega(X; a)$ *is nullhomotopic and so also is the map* $f \to f^{-1} * f$, *where* $f^{-1} : I \to X$ *is defined* $f^{-1}(t) = f(1-t)$.

THEOREM 3. *For any two subsets of* A, B *of* X, *let* $\Omega(A, B) = \{f \in X^I \,|\, f(0) \in A,\ f(1) \in B\}$. *If* A *is contractible over* X *to* x_0, *then* $\Omega(A, B)$ *and* $A \times \Omega(x_0; B)$ *are of the same homotopy type.*

Proof. Define two maps

$$K : \Omega(A, B) \to A \times \Omega(x_0; B) \quad \text{by} \quad K(\omega) = (\omega(0), F * \omega),$$
$$L : A \times \Omega(x_0; B) \to \Omega(A, B) \quad \text{by} \quad L(a, \omega') = F_a * \omega'$$

where for $a \in A$, F_a is a path which will be defined next. We show that K and L are continuous. Since A is contractible over X to x_0, there is homotopy $F : X \times I \to X$ such that $F(x, 0) = x$, $F(x, 1) = x_0$ for every $a \in A$. Then the map $F' : y \to F_X(t)$ of $X \to X^I$ is continuous, where $F_X(t) = F(x, t)$ and also the map $\delta_0 : X^I \to X$ defined by $\delta_0(\omega) = \omega(0)$ is continuous. First, in order to show that K is continuous, it is sufficient to show that given projection map p, q, $p \cdot K$, $q \cdot K$ are continuous, where $p : A \times \Omega(x_0; B) \to A$ and $q : A \times \Omega(x_0; B) \to \Omega(x_0; B)$. Since $p \cdot K$ maps $\omega \in \Omega(A, B)$ to $\omega(0) \in A$ by the above statement, this map is continuous. While $q \cdot K$ is a composition map of the continuous maps, $\Omega(A; B) \to A \times \Omega(A; B) \to X^I \times \Omega(A; B) \to \Omega(x_0; B)$. Similarly, L is also a composition of continuous map, that is, $A \times \Omega(x_0; B) \to X^I \times \Omega(x_0; B) \to \Omega(A; B)$. Now we shall show that $L \cdot K \simeq 1_{\Omega(A;B)}$ and $K \cdot L \simeq 1_{A \times \Omega(x_0;B)}$, $L \cdot K$ maps $\omega \in \Omega(A; B)$ to $F_{\omega(0)} * {F_{\omega(0)}}^{-1} * \omega'$. But $F_{\omega(0)} * {F_{\omega(0)}}^{-1}$ is nullhomotopic. Hence $F_{\omega(0)} * {F_{\omega(0)}}^{-1} * \omega' \simeq \omega'$, that is $L \cdot K \simeq 1_{\Omega(A;B)}$. Also $K \cdot L$ maps $(a, \omega') \in A \times \Omega(x_0; B)$ to $(a, {F_a}^{-1} * F_a * \omega')$. Similarly ${F_a}^{-1} * F_a * \omega' \simeq \omega'$. So $K \cdot L \simeq 1_{A \times \Omega(x_0;B)}$.

An H-structure is a couple (X, μ) consisting of a space X and continuous map $\mu : X \times X \to X$ which has a following property. There exists a point $e \in X$ such that maps $x \to \mu(x, e)$ and $x \to \mu(e, x)$ are both homotopic to the identity.

LEMMA 4. *Let* $P = \{e \,|\, \text{both of the maps } x \to \mu(x, e) \text{ and } x \to \mu(e, x) \text{ are homotopic to } 1_X\}$. *Then* P *is a path component of* X *and is called the principal component of the* H*-structure* (X, μ).

Define $\mathrm{Comp}(X)$ to be the discrete space of path components of X and denote the path component containing $x \in X$ by $[x]$. Now we assume throughout that X is a topological group with the multiplication law $\mu : X \times X \to X$ and Y is a locally compact space. Note that X is an H-structure with the same multiplication law. We denote $\mu(x, y)$ by $x \cdot y$ for any $x, y \in X$.

LEMMA 5. *The space $X^Y = \{f : Y \to X \,|\, f$ is a continuous$\}$ with composition law $\mu' : X^Y \times X^Y \to X^Y$ defined by $\mu'(f, g) = \mu(f \times g)$ is a topological group.*

Proof. Since Y is a locally compact, $\delta : X^Y \times Y \to X$ defined by $\delta(f, y) = f(y)$ is a continuous. Hence the map $\alpha : X^Y \times X^Y \times Y \to X$ defined as the following composition map $X^Y \times X^Y \times Y \to X \times X \to X$ is continuous. Then the associated map $\alpha' : X^Y \times X^Y \to X^Y$ defined by $\alpha'(f, g)(y) = \alpha(f, g, y)$ for any $y \in Y$ is continuous. But $\alpha' \equiv \mu'$ that is $\alpha'(f, g) = \mu'(f, g)$ for any $f, g \in X^Y$. Hence μ' is continuous. It is obvious that the identity in X^Y is a constant map $e' : Y \to X$ defined by $e'(y) = e$, where e is an identity in X. We define f^{-1} for a given $f \in X^Y$ by $f^{-1}(y) = (f(y))^{-1}$. Then $f \cdot f^{-1}(y) = f(y)(f(y))^{-1} = e$, so $f \cdot f^{-1} = e'$ and $f^{-1} = (f)^{-1}$. We must prove that the map $f \to f^{-1}$ of $X^Y \to X^Y$ is continuous. Define $\beta : X^Y \times Y \to X$ by $\beta(f, y) = (f(y))^{-1}$, then β is continuous. Hence the associated map $\beta' : X^Y \to X^Y$ mapping f to f^{-1} is continuous. For any $f, g, h \in X^Y$,

$$\begin{aligned}((f \cdot g) \cdot h)(y) &= (f \cdot g)(y) \cdot h(y) = (f(y) \cdot g(y)) \cdot h(y)\\ &= f(y) \cdot (g(y) \cdot h(y)) = f(y)((g \cdot h)(y)) = (f \cdot (g \cdot h))(y)\end{aligned}$$

So $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

LEMMA 6. *$\mathrm{Comp}(X^Y)$ is also a topological group.*

Proof. Define the multiplication law as following $[f] \cdot [g] = [\mu'(f \cdot g)] = [f \cdot g]$. Since the multiplication μ in X^Y is continuous, this multiplication is well defined. Also the identity in $\mathrm{Comp}(X^Y)$ is an element $[e']$ where e' is the identity in X^Y. Then define $[f]^{-1} \equiv [f^{-1}]$, from the fact that $\mathrm{Comp}(X^Y)$ is given the discrete topology, the multiplication map is continuous.

Theorem 7. *The principal component P of X^Y is a normal subgroup.*

Proof. Since the multiplication map $\mu' : X^Y \times X^Y \to X$ is continuous, $\mu'(pxp)$ is path connected. But $\mu'(e', e') = e' \in P$ and so $\mu'(pxp) \subset P$. Similarly P is also closed under the inverse operation. Hence P is a subgroup. For any $f \in X^Y$, define continuous mapping $r_f, 1_f : X^Y \to X^Y$ by

$$r_f(g) = \mu'(g, f) = g \cdot f,$$
$$1_f(g) = \mu'(f, g) = f \cdot g \quad \text{for all} \quad g \in X^Y.$$

Then we must show that $g \cdot f \cdot g^{-1} \in P$ for each $f \in P, g \in X^Y$ i.e. $r_{g \cdot f \cdot g^{-1}}$ and $1_{g \cdot f \cdot g^{-1}}$ are homotopic to the identity mapping, there is a homotopy $F : X^Y \times I \to X^Y$ such that $F(g, 0) = g$, $F(g, 1) = r_f(g) = g \cdot f$ for each $g \in X^Y$.

Now we define a new homotopy $G : X^Y \times I \to X^Y$ between $r_{g \cdot f \cdot g^{-1}}$ and the identity mapping as the following composition mapping

$$X^Y \times I \to X^Y \times \{g\} \times I \to X^Y \times X^Y \to X^Y \to X^Y \times \{g^{-1}\} \to X^Y.$$

That is $G(h, t) = h \cdot F(g, t) \cdot g^{-1}$. Then

$$G(h, 0) = h \cdot F(g, 0) \cdot g^{-1} = h \cdot g \cdot g^{-1} = h$$
$$G(h, 1) = h \cdot (g \cdot f) \cdot g^{-1} = h \cdot (g \cdot f \cdot g^{-1}) = r_{g \cdot f \cdot g^{-1}}(h).$$

Similarly we can show that $1_{g \cdot f \cdot g^{-1}}$ is homotopic to the identity mapping. Therefore $g \cdot f \cdot g^{-1} \in P$.

Theorem 8. *$\mathrm{Comp}(X^Y)$ is isomorphic to X^Y/P.*

Proof. Define $\varphi : X^Y/P \to \mathrm{Comp}(X^Y)$ by $\varphi(f \cdot p) = [f]$. Suppose that for any $f, g \in X^Y$, $f \cdot p = g \cdot p$ i.e., $f^{-1} \cdot g \in P$. Then we must show that $[f] = \varphi(f \cdot p) = \varphi(g \cdot p) = [g]$. Since $f^{-1} \cdot g \in P$, $r_{f^{-1} \cdot g} \sim$ identity. Let $F : X^Y \times I \to X^Y$ be a homotopy between $r_{f-1,g}$ and the identity such that $F(h, 0) = h$, $F(h, 1) = r_{f^{-1} \cdot g}(h) = h \cdot (f^{-1} \cdot g)$. Then $F(f, 0) = f$, $F(f, 1) = f \cdot (f^{-1} \cdot g) = g$. Hence $F(f, 0) : I \to X^Y$ is a path in X^Y consisting f and g i.e., $f \simeq g$. Therefore $[f] = [g]$,

$\varphi((f\cdot p)\cdot(g\cdot p)) = \varphi(((f\cdot g)\cdot g)\cdot p) = [f\cdot g] = [f]\cdot[g] = (\varphi(f\cdot p))\cdot(\varphi(g\cdot p))$. So φ is also a group homomorphism. φ is obviously surjective. Suppose that for any $f, g \in X^Y$, $\varphi(f\cdot p) = [e']$ i.e., $[f] = [e']$. Then there is a homotopy $F : Y \times I \to x$ such that $F(y,0) = f(y)$, $F(y,1) = e'(y) = e$. The associated map $F' : I \to X^Y$ defined by $(F'(t))(y) = F(y,t)$ is a continuous path in X^Y from f to e'. Now define $G : X^Y \times I \to X^Y$ by $G(h,t) = \mu'(g\cdot F'(t)) = g\cdot F'(t)$. Then

$$G(h,0) = f\cdot F'(0) = g\cdot f = r_f(g),$$
$$G(h,1) = g\cdot F'(1) = g\cdot e' = g.$$

Hence $r_f \simeq$ identity. Similarly we can show that $1_f \simeq$ identity. Therefore φ is injective.

References

1. B. Gray, *Homotopy Theory*, Academic Press, New York, 1975.
2. J. Dugundji, *Topology*, Allyn and Bacon. Boston, 1970.
3. G. E. Bredon, *Introduction to compact transformation group*, Academic Press, New York, 1972.
4. E. Spanier, *Algebraic Topology*, MacGraw-Hill, New York, 1966.
5. J. Siegel, *G-spaces, H-spaces and W-spaces*, Pacific J. Math. **31** (1969), 209–214.

Department of Mathematics
Korea Military Academy
Seoul 139–799, Korea