Comm. Korean Math. Soc. 8 (1993), No. 2, pp. 315–328

# COMPLETE DIFFERENTIAL SYSTEMS FOR CERTAIN ISOMETRIC IMMERSIONS OF RIEMANNIAN MANIFOLDS

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## 0. Introduction

In this paper, we construct complete differential systems for certain rigid isometric immersions of smooth ($C^\infty$) Riemannian manifolds. A complete differential system for a given system of partial differential equations (PDE) is a Pfaffian system in a jet space of sufficiently high order such that every smooth solution of the PDE system corresponds to an integral manifold of the Pfaffian system. Thus by constructing a complete system one reduces the PDE problem including the existence and the regularity of solutions to an ODE problem. In §1, we present several examples of complete systems. In particular, we present a simplest example of E. Cartan's equivalence problem from which the authors learned of the idea of complete systems. Example 1.6 shows that an equivalence between Riemannian manifolds satisfies a complete differential system of order 2. Now we take up the question whether a locally rigid isometric immersion (Definition 2.1) of a Riemannian manifold into a higher dimensional Riemannian manifolds satisfies a complete system. Let $(M, g)$ and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds of dimension $n$ and $\tilde{n}$, respectively. Let $x$ and $\widetilde{x}$ be local coordinate systems of $M$ and $\widetilde{M}$, respectively. Let $f : M \to \widetilde{M}$ be a smooth mapping. In terms of coordinates $x$ and $\widetilde{x}$, write $f = (f^1, \dots, f^{\tilde{n}})$. Then $f$ is an isometric immersion if $f^1, \dots, f^{\tilde{n}}$ satisfies

$$(1) \qquad \sum_{\alpha,\beta=1}^{\tilde{n}} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \widetilde{g}_{\alpha\beta}(f(x)) = g_{ij}(x) \qquad \text{for each} \quad i, j = 1, \dots, n.$$

Received August 25, 1992.

Research partially supported by KOSEF K91012. Presented by the second author at the spring meeting 1992 of the Korean Mathematical Society

If a solution $f$ of (1) is rigid at $P \in M$ then the 1-jet at $P$ determines $f$. The problem is whether the 2-jet of $f$ depends on the 1-jet in a smooth manner if $f$ is locally rigid.

In this paper, we give affirmative answers to the question, firstly, in the case that $\widetilde{n} = n+1$, $\widetilde{M}$ is the Euclidean space and $f(M)$ has at least three nonzero principal curvatures and secondly, in the case that $M$ is a locally homogeneous space.

A complete differential system written in the form of PDE is an elliptic system. N. Tanaka, in [Ta], defined an isometric immersion $f : M^n \to \mathbf{R}^N$ to be elliptic if for every normal direction to $f(M)$ the second fundamental form has two eigenvalues of same sign. Let $f : M^n \to \mathbf{R}^N$ be a smooth isometric immersion. The implications among these notions are as follows : Let

i) $f$ satisfies a complete system of order 2,

ii) (1) prolongs to a second order system which is elliptic at $f$,

iii) $f$ is locally rigid,

iv) $f$ is elliptic in Tanaka's sense.

Then iii) $\Rightarrow$ i) is our conjecture of which Theorem 2.2 and Theorem 3.1 are proofs for special cases, i)$\Rightarrow$ iii) is shown in [CHY] in the case of codimension 1, i) $\Rightarrow$ ii) is trivial, iv) $\Rightarrow$ ii) is shown in [CH1] and iv) $\Rightarrow$ iii) is true under an additional condition that $f$ is infinitesimally rigid [Ta]. All manifolds in this paper are assumed to be smooth($C^\infty$).

## 1. Compatibility equations and Complete Systems

Let $f$ be a smooth ($C^\infty$) mapping from an open subset $X$ of $\mathbf{R}^n$ to an open subset $U$ of $\mathbf{R}^m$. Let $x = (x^1, \dots, x^n)$ and $u = (u^1, \dots, u^m)$ be the standard coordinates of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let $f(x) = (f^1(x), \dots, f^m(x))$. Let $U_k$ be the space of all the different $k$-th order partial derivatives of the component of $f$ at a point $x$. Set $U^{(q)} = U \times U_1 \times \cdots \times U_q$ be the Cartesian product space whose coordinates represent all the derivatives of a mapping $u = f(x)$ of all orders from 0 to $q$. A point in $U^{(q)}$ will be denoted by $u^{(q)}$. The space $J^q(X, U) = X \times U^{(q)}$ is called the $q$-th order jet space of the space $X \times U$. If $f : X \to U$ is smooth, $j^q f : X \to X \times U^{(q)}$ defines a smooth section. This smooth section $j^q f$ induced by $f$ is called by the $q$-graph of $f$. Consider a system of partial differential equations of order $q$ ($q \geq 1$) for unknown functions $u = (u^1, \ldots, u^m)$ of independent variables $x = (x^1, \ldots, x^n)$,

$$\Delta_\lambda(x, u^{(q)}) = 0, \qquad \lambda = 1, \ldots, l, \tag{1.1}$$

where $\Delta_\lambda(x, u^{(q)})$ are smooth functions in their arguments. Then $\Delta = (\Delta_1, \ldots, \Delta_l)$ can be viewed as a smooth map from $X \times U^{(q)}$ into $\mathbf{R}^l$, so that the given system of partial differential equations describes the subset $\mathcal{S}_\Delta$ of zero set of $\Delta_\lambda$ in $X \times U^{(q)}$, called the solution subvariety of (1.1). From this point of view, a smooth solution of (1.1) is a smooth map $f : X \to U$ whose $q$-graph is contained in $\mathcal{S}_\Delta$.

A differential function $P(x, u^{(q)})$ of order $q$ defined on $X \times U^{(q)}$ is a smooth function of $x$, $u$, and derivatives of $u$ up to order $q$. The total derivatives of $P(x, u^{(q)})$ with respect to $x^i$ is the unique smooth function $D_i P(x, u^{(q+1)})$ defined by

$$D_i P(x, u^{(q+1)}) \equiv \frac{\partial P}{\partial x^i} + \sum_{a=1}^{m} \sum_J u^a_{J,i} \frac{\partial P}{\partial u^a_J},$$

where $J = (j_1, \ldots, j_n)$ is a multi-index such that $|J| \leq q$ and $J, i = (j_1, \ldots, j_i + 1, \ldots, j_n)$. For each nonnegative integer $r$, the $r$th-prolongation $\Delta^{(r)}$ of the system (1.1) is the system consisting of all the total derivatives of (1.1) of order up to $r$. Let $(\Delta^{(r)})$ be the ideal generated by $\Delta^{(r)}$ of the ring of differential functions on $X \times U^{(q+r)}$. If $\widetilde{\Delta} \in (\Delta^{(r)})$ for some $r$, the equation

$$\widetilde{\Delta}(x, u^{(q+r)}) = 0 \tag{1.2}$$

is called a compatibility equation for (1.1) in the sense that any smooth solution of (1.1) must satisfy (1.2). If $k$ is the order of the highest derivative involved in $\widetilde{\Delta}$, we call (1.2) a compatibility equation of order $k$.

We now define the complete system.

DEFINITION 1.1. We say that a $C^k$ $(k \geq q)$ solution $f$ of (1.1) satisfies a complete system of order $k$ if there exist compatibility equations of order $k$ of (1.1),

$$\widetilde{\Delta}_\nu(x, u^{(k)}) = 0, \quad \nu = 1, \ldots, N \tag{1.3}$$

which can be solved for all the $k$-th order partial derivatives as smooth functions of lower order terms at $f$, namely, for each $a = 1, \ldots, m$ and for each multi-index $J$ with $|J| = k$,

$$f^a_J = H^a_J(x, f^{(p)} : p < k) \tag{1.4}$$

for some function $H^a_J$ which is smooth in its arguments.

Any system of partial differential equations can be expressed as an exterior differential system with an independence condition (see [BCGGG]). Solving the given system of partial differential equations (1.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

$$du^a_I - \sum_{i=1}^n u^a_{I,i} dx^i = 0$$

for all multi-index $I$ with $|I| < q$ and $a = 1, \ldots, m$, with an independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on $S_\Delta$. If a solution of (1.1) satisfies a complete system of order $k$ and $S_k$, a solution variety of (1.3), is a submanifold of $k$-th order jet space $J^q(X, U)$ then we have a special form of Pfaffian system on $S_k$.

$$\begin{cases} du^a - \sum_{j=1}^n u^a_j dx^j = 0, & \\ \quad \vdots & \\ du^a_I - \sum_{j=1}^n u^a_{I,j} dx^j = 0, & |I| = k-2, \\ du^a_I - \sum_{i=1}^n H^a_{I,i} dx^i = 0, & |I| = k-1. \end{cases} \tag{1.5}$$

with an independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $H^a_{I,i}$ are as in (1.4). Thus a solution $u = f(x)$ of (1.1) of class $C^k$ satisfies a complete system of order $k$ if and only if

$$(x) \mapsto (x, f(x), \partial_J f(x) : |J| \leq k-1)$$

is an integral manifold of the Pfaffian system (1.5). In particular, we have

PROPOSITION 1.2. *Let $f$ be a solution of (1.1) of class $C^k$. Suppose that $f$ satisfies a complete system (1.4), then $f$ is $C^\infty$. Furthermore, if (1.1) is real analytic and each $H^a_J$ is real analytic then $f$ is real analytic.*

We now give some examples of complete systems.

EXAMPLE 1.3. Any ordinary differential equation

$$F(x, y, y', \ldots, y^{(s)}) = 0$$

with $\partial F/\partial y^{(s)} \neq 0$, is a complete systems of order $s$.

EXAMPLE 1.4. Let $M$ be a manifold with an affine connection $\nabla$. A vector field $X$ is said to be parallel if

$$\nabla X = 0. \tag{1.6}$$

Then (1.6) is a complete system of order 1.

EXAMPLE 1.5. Let $M$ be a Riemannian manifold of dimension $n$ with metric $g$. A vector field $X$ is called an infinitesimal isometry if

$$L_X g = 0. \tag{1.7}$$

In [Han], it is shown that the first prolongation of (1.7) forms a complete system of order 2.

EXAMPLE 1.6. Let $(M, g)$ and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds of dimension two. The so-called equivalence problem in this case is determining whether an isometry exists between $M$ and $\tilde{M}$. To solve this, let $\{\omega^1, \omega^2\}$ and $\{\tilde{\omega}^1, \tilde{\omega}^2\}$ be orthonormal coframe on $M$ and $\tilde{M}$, respectively. There exist an isometry $f : M \to \tilde{M}$ if and only if

$$f^*\tilde{\omega}^1 = \cos\mu\,\omega^1 + \sin\mu\,\omega^2, \qquad f^*\tilde{\omega}^2 = -\sin\mu\,\omega^1 + \cos\mu\,\omega^2, \tag{1.8}$$

for some angle $\mu$. In terms of local coordinates $x$ and $\tilde{x}$ of $M$ and $\tilde{M}$, respectively, let $\omega^i = \sum_{j=1}^2 a^i_j dx^j$, $\tilde{\omega}^i = \sum_{j=1}^2 \tilde{a}^i_j d\tilde{x}^j$, for $i = 1, 2$.
Then (1.8) becomes
(1.9)

$$\begin{pmatrix} \tilde{a}^1_1 \circ f & \tilde{a}^1_2 \circ f \\ \tilde{a}^2_1 \circ f & \tilde{a}^2_2 \circ f \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} a^1_1(x) & a^1_2(x) \\ a^2_1(x) & a^2_2(x) \end{pmatrix}$$

which is a system of four equations for three unknowns $f^1, f^2$ and $\mu$. Finding such $(f, \mu)$ is the so-called equivalence problem of Riemannian structures. Here we present E.Cartan's method to this problem. Consider differential forms $\Omega^j, j = 1, 2, 3$ on $M \times S^1$ :

$$\Omega^1 \equiv \cos\theta\,\omega^1 + \sin\theta\,\omega^2,$$
$$\Omega^2 \equiv -\sin\theta\,\omega^1 + \cos\theta\,\omega^2,$$

and

$$\Omega^3 \equiv d\theta + a\omega^1 + b\omega^2,$$

where $\theta$ is the coordinate for $S^1$ and $a$ and $b$ are smooth functions on $M$ defined by $d\omega^1 = a\omega^1 \wedge \omega^2$ and $d\omega^2 = b\omega^1 \wedge \omega^2$. Then we have $d\Omega^1 = \Omega^3 \wedge \Omega^2$, $\Omega^2 = -\Omega^3 \wedge \Omega^1$ and $\Omega^3 = K\Omega^1 \wedge \Omega^2$ where $K$ is independent of $\theta$. We put tilde on the corresponds notions of $\tilde{M}$. If $(f, \mu)$ is a solution of (1.8), then $F : M \times S^1 \to \tilde{M} \times S^1$ defined by $F(p, \theta) = (f(p), \theta - \mu(p))$ satisfies

$$F^*\tilde{\Omega}^j = \Omega^j, \qquad j = 1, 2, 3. \tag{1.10}$$

Conversely, if a mapping $F : M \times S^1 \to \tilde{M} \times S^1$ satisfies (1.10) then $F$ is of the form $(f(p), \theta - \mu(p))$, where $f : M \to \tilde{M}$ satisfies (1.8) (cf. [JAC]).

To find a complete system for $f$ we substitute $\tilde{\Omega}^3 = d\tilde{\theta} + \tilde{a}\tilde{\omega}^1 + \tilde{b}\tilde{\omega}^2$ in $F^*\tilde{\Omega}^3 = \Omega^3$ to get

$$\begin{aligned} d\theta - d\mu + &(\tilde{a} \circ f)\, f^*\tilde{\omega}^1 + (\tilde{b} \circ f)\, f^*\tilde{\omega}^2 \\ &= d\theta + a(x)\omega^1 + b(x)\omega^2. \end{aligned}$$

Thus,

$$\begin{aligned} d\mu = \ & [(\tilde{a} \circ f)\cos\mu - (\tilde{b} \circ f)\sin\mu - a(x)]\omega^1 \\ + \ & [(\tilde{a} \circ f)\sin\mu - (\tilde{b} \circ f)\cos\mu - b(x)]\omega^2. \end{aligned} \tag{1.11}$$

Express (1.9) as

$$[\tilde{A} \circ f][\frac{\partial f^i}{\partial x^j}] = T(\mu)A(x).$$

Apply $d$ to (1.9) to get

$$d[\tilde{A} \circ f]\,[\frac{\partial f^i}{\partial x^j}] + [\tilde{A} \circ f]\, d[\frac{\partial f^i}{\partial x^j}] = T'(\mu)d\mu\, A(x) + T(\mu)\, dA(x). \tag{1.12}$$

In (1.12) substitute (1.11) for $d\mu$ and $[\tilde{A} \circ f][\frac{\partial f^i}{\partial x^j}][A(x)]^{-1}$ for $T(\mu)$ and solve for $d[\frac{\partial f^i}{\partial x^j}]$ to get a complete system of order 2 for $f$.

## 2. Complete Systems for Certain Rigid Isometric Immersions

In this section we give a partial answer to the question on the relationship between rigidity of isometric immersions and existence of a complete system.

DEFINITION 2.1. Let $M$ be a Riemannian manifold of dimension $n$ and $\tilde{M}$ be a Riemannian manifold of dimension $\tilde{n}$, $\tilde{n} \geq n$. Let $P$ be a point of $M$. An isometric immersion $f : M \longrightarrow \tilde{M}$ is said to be rigid at $P$ if for any open neighborhood $N$ of $P$ in $M$ there exists an open set $N'$ with $P \in N' \subset N$ having the following property : If $\bar{f} : N' \longrightarrow \tilde{M}$ is any isometric immersion of $N'$ into $\tilde{M}$, there exists an isometry $\tau$ of $\tilde{M}$ such that $\bar{f} = \tau \circ f$. $f$ is said to be locally rigid if $f$ is rigid at each point of $M$.

In the case that $\widetilde{M} = \mathbf{R}^{n+1}$, we show that if an isometric immersion $f$ has at least three nonzero principal curvatures then $f$ satisfies a complete system of order 2. It is well known that if $f$ has three nonzero principal curvatures then $f$ is rigid (cf. [Sp]). In this case, the Gauss equations and the first prolongation of local isometric immersion equations form a complete system.

Let $(M, g)$ be a Riemannian manifold of dimension $n$ with a coordinate system $(x^1, \ldots, x^n)$. An isometric immersion $f = (f^1, \ldots, f^{n+1}) : M \to \mathbf{R}^{n+1}$ is an immersion given locally by the functions $f^a(x), a = 1, \ldots, n+1$ satisfying

(2.1)
$$\Delta: \qquad \Delta_{ij}(x, f^{(1)}) \equiv \sum_{a=1}^{n+1} \frac{\partial f^a}{\partial x^i}\frac{\partial f^a}{\partial x^j} - g_{ij}(x) = 0 \qquad i,j = 1, \ldots, n.$$

The Gauss' equations are known as compatibility equations of order 2, which are obtained from the equations of the second prolongation of (2.1) and eliminating the third order terms (see [CH1]). We write these equations as follows :

$$(2.2) \quad (i,j,k,l) \equiv \sum_{a=1}^{n+1}\{f^a_{ik}(x)\cdot f^a_{jl}(x) - f^a_{il}(x)\cdot f^a_{jk}(x)\} - G_{ijkl}(x) = 0,$$

where

$$G_{ijkl}(x) \equiv \frac{1}{2}\{\frac{\partial^2 g_{il}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l} - \frac{\partial^2 g_{jl}}{\partial x_j \partial x_k}\}.$$

We observe that the number of non-trivial Gauss' equations is $\frac{n^2(n^2-1)}{12}$.

Our result is the following

THEOREM 2.2. *Let $(M, g)$ be a Riemannian manifold of dimension $n$ $(n \geq 3)$. If there is an isometric immersion $f : M \to \mathbf{R}^{n+1}$ of class $C^2$ such that $f(M)$ has at least three nonzero principal curvatures at each point, then the system of local isometric embedding equations* (2.1) *satisfies a complete system of order* 2. *Furthermore, $f \in C^\infty$.*

*Proof.* The problem is local, so we may assume that $M$ is an open set of $\mathbf{R}^n$ containing the origin. Let $f$ be an isometric immersion of class $C^2$ such that $f(M)$ has at least three nonzero principal curvatures. Without loss of generality, we may assume that three nonzero principal curvatures, $\lambda_1$, $\lambda_2$, and $\lambda_3$, correspond to principal directions $f_*\frac{\partial}{\partial x_1}$, $f_*\frac{\partial}{\partial x_2}$, and $f_*\frac{\partial}{\partial x_3}$. We consider the set of equations which consists of the first prolongation $\Delta^{(1)}$ of (2.1) and the Gauss' equations (2.2). Note that the number of second derivatives of $f$ is $(n+1)n(n+1)/2$ and the number of equations of the first prolongation of (2.1) is $nn(n+1)/2$. We add $n(n+1)/2$ equations from the Gauss equations to $\Delta^{(1)}$, as compatibility equations, denote the totality by $\bar{\Delta}$, and view $\bar{\Delta}$ as $(n+1)n(n+1)/2$ set of smooth functions on the 2-jet space $J^2(\Omega, \mathbf{R}^{n+1})$, namely,

$$\bar{\Delta} : U^{(1)} \times U_2 \to \mathbf{R}^{(n+1)n(n+1)/2}$$

where $U^{(1)}$ and $U_2$ are as in §1. Now we will express all the second derivatives of $f$ by applying the implicit function theorem to $\bar{\Delta}$. It is sufficient to show that the Jacobian matrix $J(\bar{\Delta})$ of $\bar{\Delta}$ with respect to the second order derivatives is nonsingular at $(0, f(0), f_i(0), f_{ij}(0))$. Since the principal directions are $f_*\frac{\partial}{\partial x_1}$, $f_*\frac{\partial}{\partial x_2}$, and $f_*\frac{\partial}{\partial x_3}$, we choose $n(n+1)/2$ equations from Gauss' equations (2.2) as a row vector as follows:

$$\begin{aligned}
&(2,3,2,3), \text{ and } (1,k,1,k) && \text{for } k = 2, \dots, n,\\
&(3,1,3,2), \text{ and } (2,1,2,k) && \text{for } k = 3, \dots, n,\\
&\qquad\qquad\qquad (1,j,1,k) && \text{for } j = 2, \dots, n-1 \text{ and } k = j+1, \dots, n.
\end{aligned}$$

Then at $(0, f(0), f_j(0), f_{ij}(0))$ we have

$$J(\bar{\Delta}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where the block $A$ corresponds to $\Delta^{(1)}$, the block $B$ does to the Gauss equations chosen above, and the size of $A$ is $n\frac{n(n+1)}{2} \times n\frac{n(n+1)}{2}$ and that of $B$ is $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$. The block $B$ is of the form

$$B = \begin{pmatrix} B_1 & & & & & \\ & B_2 & & & & \\ & & B_3 & & & \\ & & & B_4 & & \\ & & & & \ddots & \\ & & & & & B_n \end{pmatrix},$$

where all other elements are zero. The $n \times n$ matrix $B_1$ is obtained by the differentiation with respect to $u^a_{ij}$, second jet coordinates, of the first $n$ equations of the equations chosen above, $(n-1)\times(n-1)$ matrix $B_2$ is obtained by the same method from the next $n-1$ equations, and for each $k = 3, \ldots, n$, $(n-(k-1)) \times (n-(k-1))$ matrix $B_k$ is obtained by the same scheme from the next $n-k+1$ equations. Each block has following form, with all other elements are zero except $*$,

$$B_1 = \begin{pmatrix} 0 & \lambda_3 & \lambda_2 & & & \\ \lambda_2 & \lambda_1 & 0 & & & \\ \lambda_3 & 0 & \lambda_1 & & & \\ * & & & \lambda_1 & & \\ \vdots & & & & \ddots & \\ * & & & & & \lambda_1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \lambda_3 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_1 \end{pmatrix},$$

$$\vdots$$

$$B_n = (\,\lambda_1\,).$$

It is easy to see that the block $A$ is of rank $n\frac{n(n+1)}{2}$ since we can assume that $f^a_i(0) = \delta^a_i$ for $a = 1, \ldots, n$ and $B$ is nonsingular since there is at least three nonzero $\lambda_1, \lambda_2$, and $\lambda_3$ at the reference point. Therefore, for all $a = 1, \ldots, n+1$, $i, j = 1, \ldots, n$ we have

$$f^a_{ij} = H^a_{ij}(x, f^b_k : k = 1, \ldots, n, \quad b = 1, \ldots, n+1),$$

where each $H^a_{ij}$ is a smooth function in its arguments, which is a complete system of order 2. The last assertion comes from the Proposition 1.2.

## 3. Complete Systems for Rigid Immersions of Locally Homogeneous Spaces into Riemannian Manifolds

In this section we show that if a Riemannian manifold has a set of infinitesimal isometries which span the whole tangent space at each point, then a locally rigid isometric immersion satisfies a complete system of order 1.

THEOREM 3.1. *Suppose that $(M, g)$ is a Riemannian manifold of dimension $n$ and $f : M \longrightarrow \tilde{M}$ is a $C^1$ isometric immersion of $M$ into a Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $\tilde{n}$, $\tilde{n} \geq n$. Let $\mathcal{G}$ be the set of infinitesimal isometries of $M$ and let $P \in M$. Suppose that $\mathcal{G}$ span $TM$ and that $f$ is rigid at $P$. Then there exists an open neighborhood $\mathcal{O}$ of $P$ in $M$ such that the $C^1$ isometric immersions of $\mathcal{O}$ into $\tilde{M}$ satisfies a complete system of order 1.*

To prove the Theorem 3.1 we need the following

LEMMA 3.2. *Suppose that $(M, g)$ is a Riemannian manifold of dimension $n$ and $f : M \longrightarrow \tilde{M}$ is a $C^1$ isometric immersion of $M$ into a Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $\tilde{n}$, $\tilde{n} \geq n$. Suppose that $f$ is rigid at $P \in M$ and $X$ is an infinitesimal isometry of $M$. Then there is an open neighborhood $N$ of $P$ in $M$ and an infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ such that $\tilde{X}(f(m)) = f_*(X(m))$ at each $m \in N$.*

*Proof of Lemma 3.2.* Let $X_t$ denote the flow of $X$ in $M$. There exist an neighborhood $\bar{N}$ of $P$ in $M$ and a positive real number $t_0$ such that the flow $X_t$ is well defined for all $t \in (-t_0, t_0)$ at each $m \in \bar{N}$. For each $t \in (-t_0, t_0)$ the map

$$\tilde{f}_t := f \circ X_t : \bar{N} \longrightarrow \tilde{M}$$

is an isometric immersion of $\bar{N}$ into $\tilde{M}$. By the hypothesis that $f$ is rigid at $P$, there exists an isometry $\tau_t$ in $\tilde{M}$ such that $\tilde{f}_t = \tau_t \circ f$ on a neighborhood $N$ of $P$ with $N \subset \bar{N}$. Define a vector field $\tilde{X}$ on $\tilde{M}$ by

$$\tilde{X}(\tilde{m}) := \left.\frac{d}{dt}\right|_{t=0} \tau_t(\tilde{m}), \quad \tilde{m} \in \tilde{M}.$$

Then $\tilde{X}$ is an infinitesimal isometry of $\tilde{M}$ and

$$\begin{aligned}\tilde{X}(f(m)) &= \frac{d}{dt}\bigg|_{t=0} \tau_t(f(m)) = \frac{d}{dt}\bigg|_{t=0} \tilde{f}_t(m) \\ &= \frac{d}{dt}\bigg|_{t=0} (f \circ X_t)(m) = f_*(X(m))\end{aligned}$$

at each $m \in N$.

*Proof of Theorem 3.1.* Choose infinitesimal isometries $X_j$, $j = 1, \ldots, n$ on $M$ such that $X_j(P)$'s are linearly independent. There exists a neighborhood $N$ of $P$ in $M$ such that $X_j$'s are linearly independent at each $m \in N$. By the Lemma 3.2 there exist infinitesimal isometries $\tilde{X}_j$'s on $\tilde{M}$ such that

$$\tilde{X}_j(f(m)) = f_*(X_j(m)), \quad m \in N. \tag{3.1}$$

Choose coordinate systems $(x^1, \ldots, x^n)$ on a neighborhood $\mathcal{O}$ of $P$ contained in $N$ and $(\tilde{x}^1, \ldots, \tilde{x}^{\tilde{n}})$ on a neighborhood $\tilde{\mathcal{O}}$ of $f(P)$ in $\tilde{M}$. In these local coordinates, we set $f$, $X_j$, and $\tilde{X}_j$ as $f = (f^1, \ldots, f^{\tilde{n}})$, $X_j = \sum_{i=1}^{n} \xi_j^i(x) \frac{\partial}{\partial x^i}$, and $\tilde{X}_j = \sum_{\lambda=1}^{\tilde{n}} \tilde{\xi}_j^\lambda(\tilde{x}) \frac{\partial}{\partial \tilde{x}^\lambda}$. From (3.1) we have

$$\begin{aligned}\sum_{\lambda=1}^{\tilde{n}} \left\{\tilde{\xi}_j^\lambda(f(x))\right\} \frac{\partial}{\partial \tilde{x}^\lambda} &= \tilde{X}_j(f(x)) = f_*(X_j(x)) \\ &= f_*\left(\sum_{i=1}^{n} \xi_j^i(x) \frac{\partial}{\partial x^i}\right) \\ &= \sum_{\lambda=1}^{\tilde{n}} \left\{\sum_{i=1}^{n} \xi_j^i(x) \frac{\partial f^\lambda}{\partial x^i}(x)\right\} \frac{\partial}{\partial \tilde{x}^\lambda}\end{aligned}$$

on $f(\mathcal{O})$. Thus

$$\tilde{\xi}_j^\lambda(f(x)) = \sum_{i=1}^{n} \xi_j^i(x) \frac{\partial f^\lambda}{\partial x^i}(x), \quad \lambda = 1, \ldots, \tilde{n},\ j = 1, \ldots, n.$$

on $\mathcal{O}$. So we obtain a system of partial differential equations

$$\sum_{i=1}^{n} \xi_j^i(x) f_i^\lambda = \tilde{\xi}_j^\lambda(f), \quad \lambda = 1, \ldots, \tilde{n},\ j = 1, \ldots, n.$$

Since $X_j$'s are linearly independent on $\mathcal{O}$, the matrix $[\xi_j^i(x)]$ is invertible at each $x \in \mathcal{O}$. Therefore $f$ satisfies a complete system of order 1.

A locally homogeneous space $M$ is a Riemannian manifold which satisfies the following property : for all $x \in M$, there exists a neighborhood of $x$ in $M$ such that for each point $\bar{x}$ in that neighborhood, there exists an isometry of $M$ sending $x$ to $\bar{x}$. For example, symmetric spaces are locally homogeneous (see [Dub]). It is easy to see that the set of infinitesimal isometries of a locally homogeneous space $M$ span $TM$. Thus we have the following

COROLLARY 3.3. *Suppose that $(M, g)$ is a locally homogeneous space of dimension $n$ and $f : M \longrightarrow \tilde{M}$ is a $C^1$ isometric immersion of $M$ into a Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $\tilde{n}$, $\tilde{n} \geq n$. Suppose that $f$ is rigid at $P \in M$. Then $f$ is indeed smooth in a open neighborhood of $P$ in $M$.*

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