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GENERIC LOCAL BIFURCATION OF HAMILTONIAN SYSTEMS WITH DOUBLE ZERO EIGENVALUES

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1. Preliminaries

Consider a 1-parameter family of Hamiltonian systems on $\mathbf{R}^2$

$$\dot{z} = J\nabla H^{\mu}(z),\ z = (x, y) \in \mathbf{R}^2,\ \mu \in \mathbf{R} \tag{1.1}$$

with standard symplectic form $\omega = dx \wedge dy$, where

$$H^{\mu} = \sum_{j=2}^{\infty} H_j^{\mu}$$

is of class $C^{\infty}(\mathbf{R}^2, \mathbf{R})$ and $H_j^{\mu}(x, y)$ is a homogeneous polynomial in x and y of degree j for each $\mu \in \mathbf{R}$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let

$$A^{\mu} = D^2 H^{\mu}(0) = \begin{pmatrix} a_{\mu} & b_{\mu} \\ b_{\mu} & c_{\mu} \end{pmatrix}.$$

Then (1.1) becomes

$$\dot{z} = JA^{\mu}z + O(|z|^2), \tag{1.2}$$

where $JA^{\mu} \in sp(1, \mathbf{R}) = sl(2, \mathbf{R})$. Here $sp(1, \mathbf{R})$ is the space of infinitesimally symplectic linear maps on $\mathbf{R}^2$ and $sl(2, \mathbf{R})$ is the special linear algebra of linear maps on $\mathbf{R}^2$ with trace zero.

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First, we consider the linearized system of (1.2) at $z = 0$ and $\mu = 0$:

$$\dot{z} = JA^0 z, \ z = (x, y) \in \mathbf{R}^2. \tag{1.3}$$

The characteristic polynomial of JA^0 is given by

$$\lambda^2 + \det(JA^0) = \lambda^2 - (b_0^2 - a_0 c_0).$$

Now we have the following 4 cases at $\mu = 0$:

(i) hyperbolic case ($b_0^2 - a_0 c_0 > 0$)

$$JA^0 \text{ has normal form } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad H_2^0 = xy.$$

(ii) elliptic case ($b_0^2 - a_0 c_0 < 0$)

$$JA^0 \text{ has normal form } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad H_2^0 = \frac{1}{2}(x^2 + y^2).$$

(iii) parabolic case ($b_0^2 - a_0 c_0 = 0$)

$$JA^0 \text{ has normal form } \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad H_2^0 = \pm\frac{1}{2}y^2.$$

(iv) double degenerate case ($a_0 = b_0 = c_0 = 0$)

$$JA^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad H_2^0 = 0.$$

Note that if we identify $sp(1, \mathbf{R})$ with $\mathbf{R}^3 = \{(a, b, c) | a, b, c \in \mathbf{R}\}$ then the pabolic case $b_0^2 = a_0 c_0$ forms the surface of the double cone and the hyperbolic case lies outside the cone and the elliptic case lies inside the cone in (a, b, c)-space and the degenerate case $a_0 = b_0 = c_0$ falls into the origin.

Hence it is clear from the stratification above that for $|\mu| > 0$ sufficiently small, the hyperbolic and elliptic cases are structurally stable and the parabolic case with codim 1 in $sp(1, \mathbf{R})$ and the degenerate case (iv) with codim 3 in $sp(1, \mathbf{R})$ are structurally unstable.

Next, we consider the nonlinear equation (1.2) with $\mu \neq 0$.

In the hyperbolic or elliptic case (at $\mu = 0$), since $z = 0$ is a nondegenerate critical point of H^0, by the Morse Lemma (See [4]), $H^\mu(x, y)$ can be reduced, by a local diffeomorphism depending smoothly on μ, to the form

$$(1.4) \qquad \begin{aligned} H^\mu(x,y) &= x^2 - y^2 \quad \text{(in the hyperbolic case)}, \\ H^\mu(x,y) &= x^2 + y^2 \quad \text{(in the elliptic case)} \end{aligned}$$

independently of μ in a small neighborhood of (0,0). Hence, even if the local diffeomorphisms are not symplectic, the nonlinear level curves of the original $H^\mu(x, y)$ are still locally diffeomorphic to the linear level curves of the new $H^\mu(x, y)$ given in (1.4) near $z = 0$ except the time directions of the flows.

However, in the parabolic case (iii), $z = 0$ is now a degenerate critical point with $H_2^0 = \pm\frac{1}{2}y^2$ and has $corank D^2 H_2^0(0) = 1$ and hence by the parametrized Splitting Lemma (See [4]), $H^\mu(x, y)$ can be reduced to the form

$$H^\mu(x,y) = \frac{1}{2}y^2 + V(x,\mu)$$

with $V^\mu(x) = \mathcal{O}(|x|^3)$, by means of a local right-left morphism $(\phi(x,\mu), \psi(\mu))$ near $z = 0$. Moreover, since the codimension of $J\nabla H_2^0$ is 1 in $sp(1, \mathbf{R})$, by the unfolding theory (See [4]), $H^\mu(x, y)$ is equivalent to the form

$$(1.5) \qquad H^\mu(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 - \mu x.$$

This represents the *fold catastrophe* in the degenerate x-direction (i.e., the direction transversal to the surface of the double cone) up to the parabolic increase in the y-direction and the corresponding flows for each μ exhibit the *Hamiltonian saddle-node bifurcation* (See [7]).

In the double-degenerate case (iv) $a_0 = b_0 = c_0 = 0$, our Hamiltonian becomes at $\mu = 0$

$$H^0(x,y) = H_3^0(x,y) + \mathcal{O}((|x| + |y|)^4).$$

Then, since $H_2^0 = 0$ has codim 3 in $\mathbf{R}^3$ and $H^0(x,y)$ is 3-determined by the Mather's theorem (See [4]), $H^0(x,y)$ is equivalent to

$$H^0(x,y) = x^2y + \frac{1}{3}y^3 \quad \text{or} \quad x^2y - \frac{1}{3}y^3. \tag{1.6}$$

Now, by Thom's theorem [4], $H^0(x,y)$ has one of the following universal unfoldings

(1.7)

$$H^{abc} = x^2y - \frac{1}{3}y^3 + a(x^2+y^2) + bx + cy \quad \text{(elliptic umbilic)},$$

$$H^{abc} = x^2y + \frac{1}{3}y^3 + a(y^2-x^2) + bx + cy \quad \text{(hyperbolic umbilic)}.$$

Note that the case (iii), i.e., the 1-parameter family (1.5), is just a subfamily of the above 3-parameter family of umbilic H^{abc}. Thus, the parameter μ is related to the parameters a, b, c. The bifurcation set and the corresponding qualitative type of the Hamiltonian flows in each case of (1.7) can be seen in [4].

2. Hamiltonian normal forms and stratifications

Consider the Hamiltonian system $(\mathbf{R}^4, \omega = \sum_{i=1}^2 dx_i \wedge dy_i, H)$ with C^∞ Hamiltonian

$$H(x,y) = \sum_{i=2}^{\infty} H_i(x,y), \quad (x,y) \in \mathbf{R}^4,$$

where the linear Hamiltonian vector field

$$X_{H_2}(x,y) = J\nabla H_2(x,y), \quad J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues $0, 0, \pm i\alpha$ and $X_{H_2} \in sp(2,\mathbf{R})$. Then, a normal form for H_2 is given by

$$H_2(x,y) = \pm\frac{1}{2}(x_1^2 + \alpha^2 y_1^2) \overset{\text{set}}{=} S_2 \tag{2.1}$$

for the semisimple case and

$$H_2(x,y) = \pm\frac{1}{2}(x_1^2 + \alpha^2 y_1^2) \pm \frac{1}{2}x_2^2 \overset{\text{set}}{=} S_2 + N_2 \tag{2.2}$$

for the non-semisimple case (See [1], [2], [3] and [5]). Hence, X_{S_2} is semisimple and X_{N_2} is nilpotent in $sp(2,\mathbf{R})$.

Now we introduce the definition of a normal form for H with respect to H_2 (See [1], [2]).

DEFINITION 1. Let $\mathcal{P}_j$ be the space of homogeneous polynomials in x_1, x_2, y_1 and y_2 of degree j. Define the linear map

$$ad_{H_2} : C^\infty(\mathbf{R}^4, \mathbf{R}) \longrightarrow C^\infty(\mathbf{R}^4, \mathbf{R})$$

by

$$ad_{H_2} = \sum_{j=1}^{2} \left\{ \frac{\partial H_2}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H_2}{\partial x_j}\frac{\partial}{\partial y_j} \right\}. \tag{2.3}$$

Let $ad_{H_2}^{(j)}$ be the restriction of ad_{H_2} to $\mathcal{P}_j$. Then we say that $H = \sum_{j=2}^{\infty} H_j$, $(H_j \in \mathcal{P}_j)$ is in *normal form with respect to* H_2 if $H_j \in C_j$ for $j \geq 3$.

From the theory of Hamiltonian normal forms (See [1],[2]), we see that if $H_2 = S_2$ then we may take

$$Cj = \operatorname{Ker} ad_{S_2}^{(j)} \tag{2.4}$$

and if $H_2 = S_2 + N_2$ then

$$Cj = \text{complement of } \operatorname{Im} ad_{N_2}^{(j)} \cap \operatorname{Ker} ad_{S_2}^{(j)}. \tag{2.5}$$

In the semisimple case, (2.3) becomes

$$ad_{H_2} = \alpha^2 y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1}. \tag{2.6}$$

Setting $z_1 = x_1 + i\alpha y_1$, $z_2 = x_2 + iy_2$, (3.4) becomes in $(z, \bar{z})$ coordinates,

$$ad_{\tilde{H}} = -i\alpha\left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}\right). \tag{2.7}$$

Let $\tilde{\mathcal{P}}_m$ be the space of homogeneous Hermitian polynomials in z and $\bar{z}$ of degree m and let $P_m \in \tilde{\mathcal{P}}_m$. Then we can write

$$P_m(z, \bar{z}) = \sum_{|k|+|l|=m} c_{kl} z^k \bar{z}^l, \tag{2.8}$$

where $c_{kl} = \bar{c}_{lk}$ and $z^k \bar{z}^l = z_1^{k_1} z_2^{k_2} \bar{z}_1^{l_1} \bar{z}_2^{l_2}$. Then, we have

$$ad_{\tilde{H}_2}(z^k \bar{z}^l) = -i\alpha(k_1 - l_1)(z^k \bar{z}^l).$$

Hence, we see that $z^k \bar{z}^l \in \operatorname{Ker} ad_{\tilde{H}_2}^{(m)}$ if and only if

$$k_1 = l_1 \quad \text{and} \quad k_1 + k_2 + l_1 + l_2 = m, \tag{2.9}$$

where $ad_{\tilde{H}_2}^{(m)}$ is the restriction of $ad_{\tilde{H}_2}$ to $\tilde{\mathcal{P}}_m$. After a simple calculation, we see that

$$\operatorname{Ker} ad_{\tilde{H}_2}^{(2)} = \langle |z_1|^2, |z_2|^2, z_2^2, \bar{z}_2^2 \rangle$$

or, in the real space $\mathcal{P}_2$, we have

$$\operatorname{Ker} ad_{H_2}^{(2)} = \langle x_1^2 + \alpha^2 y_1^2, x_2^2 + y_2^2, x_2^2 - y_2^2, x_2 y_2 \rangle. \tag{2.10}$$

Now, by the definition of normal form for $H = \sum_{j=2}^{\infty} H_j$ with respect to the semisimple $H_2 = S_2$, we must have

$$H_j \in \operatorname{Ker} ad_{H_2}^{(j)} \quad \text{for} \quad j \geq 3.$$

But, if $H \in \operatorname{Ker} ad_{S_2}$, then $\{S_2, H\} = 0$ and so

$$\exp\{t ad_{S_2}\} \cdot H = H.$$

That is, $H(x, y)$ is invariant under the S^1-action of the one-parameter group of symplectic diffeomorphisms generated by the linear Hamiltonian vector field X_{S_2} which has the eigenvalues $0, 0, \pm i\alpha$. Hence, by the Hilbert theorem (See [6]), we have

$$H \in \operatorname{Ker} ad_{S_2} \quad \text{iff} \quad H = F(S_2, x_2, y_2), \tag{2.11}$$

where

$$F \in \mathcal{C}^\infty(\mathbf{R}^3, \mathbf{R}) \quad \text{and} \quad S_2 = \frac{1}{2}(x_1^2 + \alpha^2 y_1^2).$$

Therefore, we can state the following lemma.

LEMMA 1. *A normal form for H with respect to $H_2 = S_2$ can be written as*

$$
\begin{aligned}
H(x, y) &= F(S_2, x_2, y_2) \\
&= S_2 + P_1(x_2, y_2)S_2 + P_3(x_2, y_2) \\
&\qquad + cS_2^2 + P_2(x_2, y_2)S_2 + P_4(x_2, y_2) + \cdots ,
\end{aligned}
\tag{2.12}
$$

where $P_j(x_2, y_2)$ is a homogeneous polynomial in x_2 and y_2 of degree j and c is an arbitrary constant.

Next, we consider the non-semisimple case (2.2). In this case, we also have

$$ad_{H_2} = ad_{S_2} + ad_{N_2},$$

where

$$ad_{N_2} = \sum_{j=1}^{2} \left\{ \frac{\partial N_2}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial N_2}{\partial x_j}\frac{\partial}{\partial y_j}. \right\} = -x_2 \frac{\partial}{\partial y_2}.$$

Again, by setting $z_2 = x_2 + iy_2$, we have

$$ad_{\tilde{N}_2} = -\frac{i}{2}\left\{ (z_2\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}) - (z_2\frac{\partial}{\partial \bar{z}_2} - \bar{z}_2\frac{\partial}{\partial z_2}) \right\}. \tag{2.13}$$

Now, to find the codimension of $H_2 = S_2 + N_2$ in $\mathcal{P}_2 \backsimeq \mathbf{R}^{10}$, we first compute

$$\operatorname{Im} ad^{(2)}_{\tilde{N}_2} | \operatorname{Ker} ad^{(2)}_{\tilde{S}_2}$$

where $\operatorname{Ker} ad^{(2)}_{\tilde{S}_2}$ is spanned by

$$e_1 = |z_1|^2,\ e_2 = |z_2|^2,\ e_3 = z_2^2,\ e_4 = \bar{z}_2^2.$$

Then, the action of $ad^{(2)}_{\tilde{N}_2}$ on the basis $\langle e_1, e_2, e_3, e_4 \rangle$ of $\operatorname{Ker} ad^{(2)}_{\tilde{S}_2}$ is given by

$$ad^{(2)}_{\tilde{N}_2} \cdot (e_1, e_2, e_3, e_4) = (0, \frac{i}{2}(e_3 - e_4), -i(e_2 + e_3), i(e_2 + e_4)).$$

Hence, we see from (2.5) that

$$C_2 = e_1 = |z|^2 = x_1^2 + \alpha^2 y_1^2. \tag{2.14}$$

Now, we consider the stratification of S_2 and $S_2 + N_2$ in the space $\mathcal{P}_2$ of homogeneous quadratic polynomials in $\mathbf{R}^4$. Notice that $\mathcal{P}_2(\mathbf{R}^4, \mathbf{R}) \simeq sp(2, \mathbf{R})$ by the isomorphism

$$H_2 \longrightarrow X_{H_2} \in sp(2, \mathbf{R}).$$

Since $\operatorname{Im} ad_{H_2}^{(2)}$ is the tangent space at H_2 to the orbit of the action of the Lie group of symplectic diffeomorphisms, we see that

$$cod(H_2) = cod(\operatorname{Im} ad_{H_2}^{(2)}) \quad \text{in} \quad \mathcal{P}_2.$$

Hence, from (2.10) and (2.14), we have

$$cod(S_2) = 4, \qquad cod(S_2 + N_2) = 1.$$

However, $S_2 = \frac{1}{2}(x_1^2 + \alpha^2 y_1^2)$ contains a parameter α and as α varies the orbit of S_2, that is, $A^{-1} S_2 A$ under the action of symplectic matrices A also varies transversally since the eigenvalues $\pm i\alpha$ varies with α. Thus, taking this role of parameter α into account, we see that

$$cod\Big(\bigcup_{\alpha \in \mathbf{R}} \operatorname{orbit} S_2^{\alpha}\Big) = 3$$

in $\mathcal{P}_2$ if we think of the union of the orbit S_2^{α} as a manifold in $sp(2, \mathbf{R}) \simeq \mathbf{R}^{10}$. Therefore, we have the following conclusion.

LEMMA 2. *Let S_2 and N_2 be given as in* (2.1) *and* (2.2). *Then the stratification of S_2 and $S_2 + N_2$ has a double-cone structure in $\mathcal{P}_2 \simeq \mathbf{R}^{10}$ with*

$$cod(S_2^{\alpha}) = 3 \quad \text{and} \quad cod(S_2 + N_2) = 1.$$

Hence, when we unfold $S_2^{\alpha} = \frac{1}{2}(x_1^2 + \alpha y_1^2)$, we need 3 extra parameters λ, μ and ν as in $\mathbf{R}^2$ case except α and the 1-parameter unfolding of $S_2 + N_2$ will be just a subfamily of the 3-parameter unfolding of S_2^{α} and so all the lower order catastrophes such as fold surfaces of line of cusps will correspond to $S_2 + N_2$. Therefore, from now on, we concentrate only on the catastrophes of S_2^{α} and try to find the lower order catastrophes from them.

3. Bifurcation Analysis

Now, we consider again the semisimple quadratic Hamiltonian

$$H_2 = S_2 = \frac{1}{2}(x_1^2 + \alpha^2 y_1^2).$$

The S_2-normal form for $H = \sum_{j=2} \infty H_j$ is given by (2.12). Introducing the action-angle coordinates (I, ϕ) with $I = S_2$

$$x_1 = \sqrt{2I}\cos\phi, \quad y_1 = \frac{1}{\alpha}\sqrt{2I}\sin\phi, \tag{3.1}$$

then the symplectic form becomes

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dI \wedge d\phi + dx_2 \wedge dy_2.$$

Also, in terms of the action-angle coordinates (I, ϕ, x_2, y_2), the Hamiltonian in (2.12) becomes

$$H(x,y) = \tilde{H}(I, \phi, x_2, y_2) = F(I, x_2, y_2)$$

and our Hamiltonian system reduces to the following equations

$$\begin{aligned} \dot{I} &= \frac{\partial F}{\partial \phi} = 0 & \dot{\phi} &= -\frac{\partial F}{\partial I} \\ \dot{x}_2 &= \frac{\partial F}{\partial y_2} & \dot{y}_2 &= -\frac{\partial F}{\partial x_2} \end{aligned} \tag{3.2}$$

Since $\dot{I} = 0$, $I = \text{constant}geq0$ and hence $H(x,y) = F(I, x_2, y_2)$ can be viewed as a 1-parameter family of Hamiltonians on $(\mathbf{R}^2, \omega = dx_2 \wedge dy_2)$ with $I = \frac{1}{2}(x_1^2 + \alpha^2 y_1^2)$ as a parameter. Thus, our problem to study the Hamiltonian vector fields on $(\mathbf{R}^4, \omega = \sum_{i=1}^2 dx_i \wedge dy_i)$ has been reduced to examining the Hamiltonian vector fields on the (x_2, y_2)-plane, for each value of $Igeq0$, with the Hamiltonian $F^I(x_2, y_2) = F(I, x_2, y_2)$.

Now, we can apply the general theory in Section 1 by identifying $F^I(x_2, y_2)$ with $H^\mu(x, y)$ there. Near the origin, i.e., near $I = 0$ and $x_2 = y_2 = 0$, the local type of the critical point $(x_2, y_2) = 0$ is determined by the quadratic term $P_2(x_2, y_2)$ when $Ineq0$ and by the cubic term $P_3(x_2, y_2)$ when $I = 0$. Writing

$$I \cdot P_2(x, y) = a_I x^2 + 2b_I xy + c_I y^2,$$

we have the same stratification as in Section 1. Since $a_0 = b_0 = c_0 = 0$ when $I = 0$, this corresponds to the non-generic double-degenerate case (iv) in Section 1 and so $F^I(x_2, y_2)$ becomes, with setting $x = x_2$ and $y = y_2$,

$$F^0(x, y) = P_3(x, y) + \mathcal{O}((|x| + |y|)^4).$$

Hence, $F^0(x, y)$ is equivalent to the form either $x^3 - 3xy^2$ or $x^3 + y^3$ by a local diffeomorphism and so $F^0(x, y)$ has the standard unfolding either the elliptic umbilic of the hyperbolic umbilic as in (1.7), where one of a, b, c may be taken as I. In either case, the bifurcation set and the corresponding Hamiltonian phase portrait can be obtained which exhibits a very complicated codimension 3 bifurcation.

On the other hand, in the generic case $I > 0$, the Hamiltonian vector fields with Hamiltonian $F^I(x, y)$ undergoes a saddle-node bifurcation as already mentioned in Section 1 and therefore we can state the following conclusion.

THEOREM 1. *Let* $(\mathbf{R}^4, \omega = \sum_{i=1}^2 dx_i \wedge dy_i, H)$ *be a Hamiltonian system with double zero and purely imaginary eigenvalues.*
Then the 4-dimensional Hamiltonian flow undergoes generically an elliptic saddle-node bifurcation due to the rotation in (x_1, y_1)*-plane.*

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