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IDENTIFIABILITY OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATION

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1. Introduction

Let H and V be complex Hilbert spaces such that V is a dense subspace of H with continuous imbedding in H. The inner product and norm in H are denoted by (f,g) and $|f|$, and those in V are by $((u,v))$ and $||v||$. If X and Y are two Banach spaces, $B(X,Y)$ denotes the set of bounded linear mappings of X into Y, and $B(X) = B(X,X)$.

Let $a(u,v)$ be a bounded sesquilinear form defined on $V \times V$ and satisfying Gårding inequality

$$\operatorname{Re} a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0$$

for any $u, v \in V$. We define the operator A_0 as follows:

Given $u \in V$. If there exists an element f of H so that $a(u,v) = (f,v)$ for all $v \in V$, then $u \in D(A_0)$ and $Au = f$.

Using an element $f \in V^*$, we can extend the operator A_0 to an operator on V into V^*. This extension of A_0 is also denoted by the same letter A_0. It is well known that A_0 generates an analytic semigroup in both H and V^*. We may assume that $0 \in \rho(A_0)$ according to the Lax-Milgram theorem where $\rho(A_o)$ denotes the resolvent set of A_0.

The object of this paper is to construct some results on the identifiability for the following retarded functional differential equation of parabolic type

(1.1)
$$\frac{d}{dt}u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds, \ t \in (0,T],$$

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where the operators A_1 and A_2 are bounded liner operators from V to V^* and the real function $a(s)$ is assumed to be Hölder continuous in [-h,0]. The equation may be considered as an equation in both H and V^*. In [3,6] the fundamental results on the structural operator for the linear retarded functional differential equation was established. Recently, many authors have discussed the structural properties for retarded system (see $[1, \ldots, 6]$). In the paper G.Di Blasio, K.Kunisch and E.Sinestrari [1] they have developed an excellent state space theory for retarded system in the product space $F \times L^2(-h, 0; D(A)), h > 0$ (where $F = D_A(1/2, 2)$ is the Lions real interpolation space between D_A and H). The main theory is based on constructing a solution semigroup in $F \times L^2(-h, 0; D(A))$. Thus, in section 2 we define semigroup $S(t)$ in $Z = H \times L^2(-h, 0; V)$ and deal with the spectrum of the infinitesimal generator A of $S(t)$. We shall give that

$$\sigma(A) = \sigma_e(A) \cup \sigma_p(A)$$

where each nonzero point of $\sigma_e(A)$ is a cluster point of $\sigma(A)$ and $\sigma_p(A)$ consists only of discrete eigenvalues. In section 3 we study the problem of completeness of generalized eigenspaces of infinitesimal generator A. We obtain that the condition of the completeness of generalized eigenspaces of between the infinitesimal generators A and A_0 is the necessary and sufficient property. In order to obtain the condition for identifiability of the equation (1.1) we use the method which S. Nakagiri and M.Yamamoto [4] developed in the product space $X \times L^2(-h, 0; X)$. We establish the necessary and sufficient condition for identifiability is given as the so-called rank condition in terms of the multiplicity of eigenvalues.

2. Classification of spectrum

Consider the following linear retarded functional differential equation with initial values

$$\begin{aligned} (2.1) \qquad \frac{d}{dt}u(t) =& A_0 u(t) + A_1 u(t-h) \\ &+ \int_{-h}^{0} a(s) A_2 u(t+s) ds, \quad t \in (0, T], \\ (2.2) \qquad u(0) =& g^0, \quad u(s) = g^1(s) \quad s \in [-h, 0), \end{aligned}$$

where each operators is defined in section 1 and $g = (g^0, g^1) \in Z$. According to [5] the fundamental solution $W(t)$ of (2.1), (2.2) can be constructed. It is easily seen that the space

$$\{f \in V^* : \int_0^\infty \|A_0 \exp(tA_0)f\|_*^2 dt < \infty\}$$

considers with H, where $\|\cdot\|_*$ is the norm of V^*. Hence, in view of [1] the semigroup $S(t)$ in $Z = H \times L^2(-h, 0; V)$ is defined by

$$S(t)g = (u(t; g), u(t + \cdot; g)), \quad g = (g^0, g^1) \in Z$$

where $u(t; g)$ is the mild solution of (2.1),(2.2) satisfying the initial condition $u(0; g) = g^0, u(s; g) = g^1(s)$ for $s \in [-h, 0)$. Similarly, the semigroup $S_T(t)$ in the same space Z is defined for the adjoint equation

$$\begin{aligned} \text{(2.3)} \qquad \frac{d}{dt}v(t) =& A_0^* v(t) + A_1^*(t - h) \\ &+ \int_{-h}^0 a(s) A_2^* v(t + s) ds, \quad t \in (0, T], \\ \text{(2.4)} \qquad v(0) =& \phi^0, \quad v(s) = \phi^1(s) \quad s \in [-h, 0). \end{aligned}$$

Let A_T be the infinitesimal generator of the solution semigroup $S_T(t)$. In view of theorem 4.2 of [1] the infinitesimal generator A is characterized as following

LEMMA 2.1.

$$\begin{aligned} D(A) = \{(f^0, f^1) : f^1 \in W^{1,2}(-h, 0; V), f^0 = f^1(0), \\ A_0 f^0 + A_1 f^1(-h) + \int_{-h}^0 a(s) A_2 f^1(s) ds \in H\} \end{aligned}$$

$$A(f^0, f^1) = (A_0 f^0 + A_1 f^1(-h) + \int_{-h}^0 a(s) A_2 f^1(s) ds, f^1),$$

where $W^{1,2}(-h, 0; V)$ is the set of all functions whose derivatives in the distribution sense belong to $L^2(-h, 0; V)$.

For $\lambda \in C$ we define the densely defined closed linear operators by

$$\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h} A_1 - \int_{-h}^{0} e^{\lambda s} a(s) A_2 ds$$

$$\Delta_T(\lambda) = \lambda - A_0^* - e^{-\lambda h} A_1^* - \int_{-h}^{0} e^{\lambda s} a(s) A_2^* ds$$

LEMMA 2.2. *$(\lambda - A)f = \phi$ if and only if*

$$\begin{aligned}\Delta(\lambda) f^1(0) =& \phi^0 + \int_{-h}^{0} e^{-\lambda(h+r)} A_1 \phi^1(\gamma) d\gamma \\ &+ \int_{-h}^{0} a(s) \int_{s}^{0} e^{\lambda(s-r)} A_2 \phi^1(\gamma) d\gamma ds \\ f^1(s) =& e^{\lambda s} f^0 + \int_{s}^{0} e^{\lambda(s-\gamma)} \phi^1(\gamma) d\gamma\end{aligned}$$

LEMMA 2.3. *For $i = 1, 2, \ldots,$*

$$\begin{aligned}\mathrm{Ker}(\lambda - A)^k =& \Big\{ (\phi_0^0, e^{\lambda s} \sum_{i=0}^{k-1} (-s)^i \phi_i^0 / i!) : \sum_{i=j-1}^{k-1} (-1)^{i-j} \\ & \Delta^{(i-j+1)}(\lambda) \phi_i^0 / (i-j+1)! = 0, \quad j = 1, \ldots, k \Big\}.\end{aligned}$$

In what follows we assume that $A_1 = \gamma A_0, A_2 = A_0$ and the imbedding $V \subset H$ is compact. According to the Riesz-Schauder theorem A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_j; j = 1, 2, \ldots\}$$

which has no point of accumulation except possibly $\lambda = \infty$.

For $\lambda \in C$ then

$$\Delta(\lambda) = 1 - m(\lambda) A_0,$$

where

$$m(\lambda) = 1 + \gamma e^{-\lambda h} + \int_{-h}^{0} a(s) e^{\lambda s} a(s) ds.$$

It is easily seen that $m(\lambda)$ is an entire function and

$$m(\lambda) \to 1 \quad as \quad \mathrm{Re}\, \lambda \to \infty. \tag{2.5}$$

We assume that $m(0) \neq 0$ (see Theorem 2.2). The following Lemmas are proved as theorems 6.1 and 7.2 of S. Nakagiri[3].

THEOREM 2.1. *Let $\rho(A)$ be the resolvent set of the infinitesimal generator A of $S(t)$. Then*

$$\begin{aligned}\rho(A) &= \{\lambda : m(\lambda) \neq 0, \frac{\lambda}{m(\lambda)} \in \rho(A_0)\} \\ &= \{\lambda : \Delta(\lambda) \text{ is isomorphism from } V \text{ onto } V^*\}.\end{aligned}$$

Proof. If $m(\lambda) \neq 0$ and $\lambda/m(\lambda) \in \rho(A_0)$, then for all $\phi \in Z$, there exists $f = (f^0, f^1) \in D(A)$ such that Lemma 2.2 is satisfied. Hence $R(\lambda - A) = Z$ where $R(A)$ denotes the range of the operator A. Let $(\lambda - A)f = 0$. Then from Lemma 2.2 it follows that $\Delta(\lambda)f^1(0) = 0$. Therefore $f^1(0) = 0$ and hence $f^1(s) = 0$. We have proved that $\lambda \in \rho(A)$.

Conversely, if $m(\lambda) = 0$, then since $\Delta(\lambda) = \lambda I|V, \Delta(\lambda)$ is not mapping onto H. If $m(\lambda) \neq 0$ and $\lambda/m(\lambda) \in \sigma(A_0)$. Then the mapping $\Delta(\lambda) = m(\lambda)(\lambda/m(\lambda) - A_0)$ is not onto. Let $\phi = (\phi^0, 0)$ where $\phi^0 \in H \setminus \operatorname{Im} \Delta(\lambda)$. Then there is not $f^1(0)$ such that the relation in Lemma 2.2 is satisfied.

LEMMA 2.4. *Let $f : C \to C$ be analytic at z_0 and z_0 be a zero of f multiplicity $k \geq 1$. Then there exist a neighborhood V at zero and analytic function $\phi : V \to \operatorname{Dom} f$ such that $f(\phi(w)) = w^k$ where $\operatorname{Dom} f$ denotes the domain of f.*

Proof. There exists an analytic function g on neighborhood at z_0 such that $f(z) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$. Since $g(z) \neq 0$ on neighborhood at z_0 there exists an analytic function h such that $g(z) = h(z)^k$. Thus $(z - z_0)h(z)|_{z=z_0} = 0$ and

$$\begin{aligned}\frac{d}{dz}((z - z_0)h(z))|_{z=z_0} &= (h(z) + (z - z_0)\frac{d}{dz}h(z))|_{z=z_0} \\ &= h(z_0) \neq 0.\end{aligned}$$

Hence, from inverse mapping theorem it follows that there exist a neighborhood U at z_0 and a neighborhood V at zero such that the mapping $z \mapsto (z - z_0)h(z)$ is a homeomorphism from U onto V. If we denote by $\phi(w)$ the inverse of such mapping, then the function ϕ is analytic on V and for any $w \in V$ $(\phi(w) - z_0)h\phi(w) = w$ and $\phi(0) = z_0$. Therefore, it holds that

$$f(\phi(w)) = (\phi(w) - z_0)^k g(\phi(w)) = (\phi(w) - z_0)^k (h\phi(w))^k = w^k.$$

THEOREM 2.2. *If $\sigma(A)$ is the spectrum of the infinitesimal generator A of solution semigroup $S(t)$, then*

$$\sigma(A) = \sigma_e(A) \cup \sigma_p(A),$$

where $\sigma_e(A) = \{\lambda : m(\lambda) = 0\}$ and $\sigma_p(A) = \{\lambda : m(\lambda) \neq 0, \lambda/m(\lambda) \in \sigma(A_0)\}$. Each nonzero point of $\sigma_e(A)$ is not an eigenvalue of A but a cluster point of $\sigma(A)$. $\sigma_p(A)$ consists only of discrete eigenvalues.

Proof. Let $\lambda_0 \neq 0$ be a zero of $m(\lambda)/\lambda$ of multiplicity $k \geq 1$. From the Lemma 2.4 it follows that there exists a analytic function ϕ on a neighborhood V at zero such that for any $\mu \in V$,

$$\frac{m(\phi(\mu))}{\phi(\mu)} = \mu^k \quad and \quad \phi(0) = \lambda_0.$$

Let us denote by λ_j a k-th root of $1/\mu_j$, then λ_j converges to zero as j tends to infinity. In fact, $\sigma(A_0) = \{\mu_j : j = 1, 2, \dots\}$ has no point of cluster point except for infinity point. If j is sufficiently large then $\lambda_j \in V$ and $\phi(\lambda_j)/m(\phi(\lambda_j)) = \mu_j \in \sigma(A_0)$. Hence, it holds that $\phi(\lambda_j) \in \sigma(A)$ and $\phi(\lambda_j)$ tends to $\phi(0) = \lambda_0$ as j tends to infinity.
We have proved nonzero point of $\sigma_e(A)$ is a cluster point of $\sigma(A)$.

Next, suppose that $m(\lambda_0) \neq 0, \lambda_0/m(\lambda_0) \in \sigma(A_0)$. If there exists a sequence $\{\lambda_j\}$ such that $\lambda_j/m(\lambda_j) \in \sigma(A_0)$. Since $\sigma(A_0)$ consists only of isolated points, we have $\lambda_j/m(\lambda_j) = \lambda_0/m(\lambda_0)$ for sufficiently large j. In view of the theorem of identity we have $m(\lambda) = \lambda_0\lambda/m(\lambda_0)$ which is contradictory to (2.5)

THEOREM 2.3. *Suppose $m(0) = 0$. Then zero is an eigenvalue of A with infinity multiplicity. The zero is an isolated point of $\sigma(A)$ if it is a simple zero of $m(\lambda)$ and a cluster point of $\sigma(A)$ if it is a multiple zero of $m(\lambda)$.*

Proof. If $m(0) = 0$, then for all $v \in Vf = (f^0, f^1)$ where $f^0 = v$ and $f^1(s) \equiv v$, $s \in [-h, 0)$ belongs to the eigenspace corresponding to zero of A. Thus the zero point is an eigenvalue of A with infinity multiplicity. The others of this theorem is obtained by similarly way in Theorem 2.1.

3. Completeness of generalized eigenspaces

Let λ be a pole of the resolvent of A of order k_λ and P_λ the spectral projection associated with λ

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda}(\mu - A)^{-1}d\mu$$

where Γ_λ is a small circle centered at λ such that it surrounds no point of $\sigma(A)$ except λ. Then the generalized eigenspace corresponding to λ is given by

$$Z_\lambda = \operatorname{Im} P_\lambda = \operatorname{Ker}(\lambda I - A)^{k\lambda}.$$

Defining the operator p_n by

$$p_n = \frac{1}{2\pi i}\int_{|\mu-\mu_n|=\epsilon_n}(\mu - A_0)^{-1}d\mu,$$

when the circle surrounds no point of $\sigma(A_0)$ except μ_n. Putting

$$H_n = p_n H = \{p_n u : u \in H\},$$

we have that from $p_n^2 = p_n$ and $H_n \subset V$ it follows that

$$p_n V = \{p_n u : u \in V\} = H_n.$$

It is well known that $\dim H_n < \infty$.

LEMMA 3.1. *Let $g = (g^0, g^1)$ belong to $H_n \times L^2(-h,0;H_n)$. Then for the solution u of* (2.1), (2.2) *we have $p_n u(t) = u(t)$.*

Proof. If we compose p_n on both sides of (2.1), (2.2), then $p_n u(t)$ is also a solution of (2.1), (2.2). From uniqueness of the solution the result follows.

Put $A_{on} = A_0|_{H_n}$. For any $g \in H_n \times L^2(-h,0;H_n)$ the solution $u(t)$ of (2.1), (2.2) is the solution satisfied the following

$$\frac{d}{dt}u(t) = A_{0n}u(t) + \gamma A_{0n}u(t-h) + \int_{-h}^{0} a(s)A_{0n}u(t+s)\,ds \tag{3.1}$$

$$u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h, 0). \tag{3.2}$$

If we denotes the solution semigroup of the equation (3.1), (3.2) with A_{on} in place of A_0 by $S_n(t) = \exp(tA_n)$, then we have that

$$S_n(t) = S(t)|_{H_n \times L^2(-h,0;H_n)},$$
$$A_n = A|_{D(A_n)},$$
$$D(A_n) = \{(g^0, g^1); g^1 \in W^{1,2}(-h,0;H_n),\ g^0 = g^1(0)\}.$$

LEMMA 3.2. *The adjoint operator of p_n is represented by*

$$p_n^* = \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon_n} (\mu - A_0^*)^{-1} d\mu.$$

Proof. If $\mu \in \rho(A_0)$, then p_n is a bounded linear operator from V^* into V because $(\mu - A_0)^{-1}$ is an isomorphism from V^* into V. For any $\phi^0, \psi \in V^*$, from $(\phi^0, (\overline{\mu} - A_0^*)^{-1}\psi^0) = ((\mu - A_0)^{-1}\phi^0, \psi^0)$, we have

$$\begin{aligned}(p_n^*\phi^0, \psi^0) &= \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon} ((\mu - A_0^*)^{-1}\phi^0, \psi^0)\, d\mu \\ &= \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon} (\phi^0, (\overline{\mu} - A_0)^{-1}\psi^0 d\mu) \\ &= (\phi^0, \frac{1}{2\pi i}\int_{|\mu-\mu_n|=\epsilon} (\mu - A_0)^{-1}\psi^0 d\mu) \\ &= (\phi^0, p_n\psi^0).\end{aligned}$$

Let $\lambda_{ni}/m(\lambda_{ni}) = \mu_n, n = 1, 2, \ldots$, then

$$p_{ni} = \frac{1}{2\pi i}\int_{|\lambda-\lambda_{ni}|=\epsilon_{ni}} (\lambda - A)^{-1} d\lambda.$$

Set $Z_{ni} = \operatorname{Im} p_{ni}$.

LEMMA 3.3. *$\phi \in Z_{ni}$ if and only if there exists an integer k such that $(\lambda_{ni} - A_n)^k\phi = 0$.*

Proof. If $(\lambda_{ni} - A)^k\phi = 0$ where $\phi = (\phi^0, \phi^1)$, then from $\Delta(\lambda_{ni})^k\phi^0 = 0$ and $\Delta(\lambda_{ni})^k\phi^1(s) \equiv 0$ it follows that

$$(\mu_n - A_0)^k\phi^0 = 0, \quad (\mu_n - A_0)^k\phi^1(s) \equiv 0.$$

Hence, since $\phi^0 = p_n\phi^0 \in H_n$ and $\phi^1(s) = p_n\phi^1(s) \in H_n$ we have $(\lambda_{ni} - A_n)^k\phi = 0$. In view of the Lemma 3.1 $(\lambda_{ni} - A_n)^k\phi = 0$ implies $(\lambda_{ni} - A)^k\phi = 0$. Thus Lemma is proved.

THEOREM 3.1. *Suppose that $m(0) \neq 0, \gamma \neq 0$ and the generalizes eigenspaces of A_0 are complete in H. Then the generalized eigenspaces of A are complete in Z.*

Proof. From the corresponding result of A. Manitius([2]; Theorem 5.1 and 5.4(ii)) in the case a finite dimensional space, the generalized eigenspaces of A_n are complete in $H_n \times L^2(-h, 0; H_n)$. In view of Lemma 3.3 the generalized eigenspace of A_n are $\bigcup_{n=1}^{\infty} Z_{ni}$(We remark that in the case of a finite dimensional case the complex number λ satisfied with $m(0) = 0$ belongs to the resolvent set). Suppose that $(f, Z_{ni}) = 0$ for any n and any i where $f = (f^0, f^1) \in H \times L^2(-h, 0; V^*)$. Then in view of Lemma 3.3 we have that for all $\phi = (\phi^0, \phi^1) \in Z_{ni}$

$$\begin{aligned}
((p_n^* f^0, p_n^* f^1), (\phi^0, \phi^1)) &= (p_n^* f^0, \phi^0) + \int_{-h}^{0} (p_n^* f^1(s), \phi^1(s)) ds \\
&= (f^0, p_n \phi^0) + \int_{-h}^{0} (f^1(s), p_n \phi^1(s)) ds \\
&= (f^0, \phi^0) + \int_{-h}^{0} (f^1(s), \phi^1(s)) ds \\
&= ((f^0, f^1), (\phi^0, \phi^1)) = 0.
\end{aligned}$$

Thus $((p_n^* f^0, p_n^* f^1), Z_{ni}) = 0$ for any $i = 1, 2, \ldots$. Hence the element $(p_n^* f^0, p_n^* f^1)$ is orthogonal to $H_n \times L^2(-h, 0; H_n)$, and hence $p_n^* f^0 = 0$ and $p_n^* f^1(s) \equiv 0$. Since n is arbitrary number we have that $f^0 = 0$ and $f^1 \equiv 0$. We have proved that the generalized eigenspaces of A which is the set $\bigcup_{n,i} Z_{ni}$ are complete in $Z = H \times L^2(-h, 0; V)$.

4. Identifiability of linear retarded system

We denote the model system by the equation (2.1), (2.2) with A_0, γ, a replaced by A_0^m, γ^m, a^m respectively. The mild solution of (2.1), (2.2) is denoted by $u^m(t; g)$, and the solution semigroup for model system by $S^m(t) = \exp(tA^m)$. We assume that A_0^m and a^m satisfy the same type of assumptions as A_0 and a. The conclusions in section 2 holds also for A_0^m. We shall say that A_0, γ, a are identifiable if

$$A_0 = A_0^m, \quad \gamma = \gamma^m, \quad a(s) \equiv a^m(s)$$

follows from

$$u(t; g_i) \equiv u^m(t; g_i), \quad i = 1, \dots, q,$$

where $g_i = (g_i^0, g_i^1) \in Z, i = 1, \dots, q$, is a finite set of initial values.

Let $\{\mu_n^m : n = 1, 2, \dots\}$ be the set of eigenvalues of A_0^m, and by $\{\psi_{n1}^0, \dots, \psi_{nd_n}^0\}$ a base of $\mathrm{Ker}(\overline{\mu_n^m} - (A_0^m)^*)$, whose $d_n = \dim \mathrm{Ker}(\mu_n^m - A_0^m)$. Let $\{\lambda_{nj}^m : j = 1, 2, \dots\}$ be the totality of the complex numbers λ satisfying $\lambda / m^m(\lambda) = \mu_n^m$. Let A_T^m be the infinitesimal generator of the solution semigroup associated with the model equation with A_0^m replaced by its adjoint $(A_0^m)^*$. If we set $\psi_{nj}^k = ((\psi_{nj}^k)^0, \exp(\overline{\lambda_{nj}^m}(\psi_{nj}^k)^0))$, $\{(\psi_{nj}^k)^0 : k = 1, \dots, d_n\}$ is a base of $\mathrm{Ker}(\overline{\lambda_{nj}^m} - A_T^m)$. We denoted by $\{\phi_{nj}^k : k = 1, \dots, d_n\}$ a base of $\mathrm{Ker}(\lambda_{nj}^m - A^m)$. The structual operator F is defined by

$$\begin{aligned} Fg &= ([Fg]^0, [Fg]^1), \\ [Fg]^0 &= g^0, \\ [Fg]^1(s) &= \gamma A_0 g^1(-h - s) + \int_{-h}^{s} a(\gamma) A_0 g^1(\gamma - s) d\gamma. \end{aligned}$$

for $g = (g^0, g^1) \in Z$. It is easily to see that $F \in B(Z, Z^*)$. As is easily seen in [3; Theorem 8.4] the projection $P_{\lambda_{ni}}^m$ has the following equivalent representation

$$P_{\lambda_{ni}}^m g = \sum_{i=1}^{d_n} < Fg, \psi_{ni}^m >z \quad \phi_{ni}^m, \quad g \in Z.$$

Throughout this section we shall assume following:

RANK CONDITION: For the initial values $\{g_1, \dots, g_q\}$

$$\mathrm{rank}((F^m g_i, \psi_{nj}^k) Z : i \to 1, \dots, q, \quad k \downarrow 1, \dots, d_n) = d_n$$

for $n = 1, 2, \dots$.

The assumption of rank condition is satisfied if and only if

$$Span\{P_{\lambda_{nj}}^m g_1, \dots, P_{\lambda_{nd_j}}^m g_q\} = Z_{\lambda_{nj}}^m .$$

LEMMA 4.1. $\sigma_e(A^m) \subset \sigma_e(A)$, $\sigma_p(A^m) \subset \sigma_p(A)$.

Proof. Let $\lambda_0 \in \sigma_p(A^m)$. Then from Theorem 2.2 it follows that $m^m(\lambda_0) \neq 0$ and $\lambda_0/m^m(\lambda_0) \in \sigma(A_0^m)$. Because $A_0^m : V \to V^*$ is an isomorphism we have $\lambda_0 \neq 0$. Suppose that $\lambda_0 \in \rho(A)$, then there exists a positive number ϵ such that

$$\{\lambda : 0 < |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(A^m), \quad \{\lambda : |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(A).$$

Thus since

$$\begin{aligned} P_{\lambda_0}^m &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - A^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - A)^{-1} g_i d\lambda \\ &= 0 \end{aligned}$$

it is contradict to the rank condition. Hence $\lambda_0 \in \sigma(A)$. Suppose $\lambda_0 \in \sigma_e(A)$. Then $m(\lambda_0) \neq 0$ and since $\lambda_0 \neq 0$, λ is not eigenvalue. There exists a positive number $\epsilon > 0$ such that

$$\{\lambda : 0 < |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(A^m), \quad \{\lambda : |\lambda - \lambda_0| = \epsilon\} \subset \rho(A).$$

Since

$$\begin{aligned} P_\lambda^m g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - A^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - A)^{-1} g_i d\lambda \end{aligned}$$

we have

$$\begin{aligned}
AP^m_{\lambda_0}g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} A(\lambda-A)^{-1}g_i d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \{\lambda-(\lambda-A)\}(\lambda-A)^{-1}g_i d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-A)^{-1}g_i d\lambda - \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} g_i d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-A)^{-1}g_i d\lambda \\
&= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-A^m)^{-1}g_i d\lambda \\
&= A^m P^m_{\lambda_0}g_i.
\end{aligned}$$

By the similarly way we conclude that

$$A^k P^m_{\lambda_0}g_i = (A^m)^k P^m_{\lambda_0}g_i, \quad k = 0, 1, \ldots,$$

and we have

$$(\lambda_0 - A)^k P^m_{\lambda_0}g_i = (\lambda_0 - A^m)^k P^m_{\lambda_0}g_i = 0.$$

In view of $p^m_{\lambda_0}g_i \neq 0$ for some i it is contradiction that λ_0 is not eigenvalue. Therefore we have proved that $\lambda_0 \in \sigma_p(A)$, that is, $\sigma_p(A^m) \subset \sigma_p(A)$.

Next, let $\lambda_0 \in \sigma_e(A^m)$, then $m^m(\lambda_0) = 0$ and hence $\lambda \neq 0$. Therefore exists a sequence $\{\lambda_j\} \subset \sigma_p(A^m)$ such that λ_j converges to λ_0. Hence from $\lambda_j \in \sigma_p(A^m) \subset \sigma_p(A)$ it follows that λ_0 is a cluster point of $\sigma(A)$ and hence $\lambda_0 \in \sigma_e(A)$.

THEOREM 4.1. *Suppose that the generalized eigenspaces of A_0^m are complete in H and the rank condition is satisfied. Then A_0, γ, a are identifiable.*

We can proved this theorem following the proof of proposition 3.1 and theorem 3.1 of [4] by showing Lemma 4.1 instead of $\sigma(A^m) \subset \sigma(A)$ to start with.

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