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FIXED POINT ALGEBRAS OF UHF–ALGEBRAS II

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1. Introduction

In this note we consider a C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ of product type action, where $\mathfrak{A}$ is a UHF-algebra and G is a finite group. In [3], [6] and [7], the author, A. Kishimoto and N. J. Munch considered properties of the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$. In their results if G is *abelian*, then the space of tracial states on the fixed point algebra $\mathfrak{A}^\alpha$ is n-simplex (the number n is the cardinality of a subgroup K of G whose elements are weakly inner in the trace representation of $\mathfrak{A}$) and in particular some conditions for $\mathfrak{A}^\alpha$ to be UHF were obtained.

In this paper we show that the number of extremal tracial states on the fixed point algebra $\mathfrak{A}^\alpha$ is the cardinality of the orbit space $\widehat{K}/G$ where K is some normal subgroup of G and we get conditions under which $\mathfrak{A}^\alpha$ is a UHF-algebra when G is *non-abelian*.

2. Notations and preliminaries

Let G be a finite group and K_n, $n \in \mathbb{N}$ be matrix factors of rank $|K_n|$, that is, $|M_n(\mathbb{C})| = n$. Here $M_n(\mathbb{C})$ means always the factor of all $n \times n$ complex matrices. Consider unitary representations $\pi_n : G \to K_n$ and define the homomorphism α of G into the group of all *-automorphisms of $\mathfrak{A} = \bigotimes_{n=1}^{\infty} K_n$ by $\alpha_g = \bigotimes_{n=1}^{\infty} Ad\pi_n(g)$.

We assume throughout that the automorphisms α_g are not inner in $\mathfrak{A}$ except $g =$the unit e in G.

If G is a (*non-abelian*) finite group, the structure of ideals in $\mathfrak{A}^\alpha$ was investigated in [8] by N. Riedel. Let τ be the unique tracial state on a UHF-algebra $\mathfrak{A}$. Since the trace is α-invariant, we obtain a W^*-dynamical system $(\pi_\tau(\mathfrak{A})'', G, \widetilde{\alpha})$ which the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ is

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extended to (here π_τ is the G.N.S.-representation constructed by τ). We set $K = \{g \in G : \tilde{\alpha}_g$ is an inner automorphism of $\pi_\tau(\mathfrak{A})''\}$. Let $\widehat{K}$ be the dual object of K. Since K is a normal subgroup of G, we obtain a G-space $(G, \widehat{K})$ with the action $(g \cdot \pi)(k) = \pi(g^{-1}kg)$ for $k \in K$, $g \in G$ and $\pi \in \widehat{K}$. We give an equivalent relation $\sim$ by $\pi \sim \rho$ $(\pi, \rho \in \widehat{K})$ if and only if $g \cdot \pi = \rho$ for some $g \in G$. Then we obtain a G-space $\widehat{K}/\sim$ (denoted by $\widehat{K}/G$).

By [8, §3], we may assume that there exists an invariant set Ω in the dual object $\widehat{G}$ of G such that the set Ω is the family $J(\pi_n)$ of all irreducible subrepresentations of π_n $(n \geq 2)$. By [5, Proposition 2.7(vii)], there is a normal subgroup H of G such that $\Omega = \{\pi \in \widehat{G} : \pi|_H$ is trivial$\}$. By the above assumption and [2, Lemma 3.5], the invariant set Ω is the whole space of $\widehat{G}$. Since $J(\pi_3) = \widehat{G}$ and $\pi_1 \otimes \pi_2$ contains a trivial representation of G, we have $J(\pi_1 \otimes \pi_2 \otimes \pi_3) = \widehat{G}$. After "compressing" $K_1' = K_1 \otimes K_2 \otimes K_3$, we may assume that $J(\pi_n) = \widehat{G}$ for all $n \geq 1$. Then we show, by [8, Theorem 3.1], that the fixed point algebra $\mathfrak{A}^\alpha$ is simple.

Put $W_g^{n,m} = \bigotimes_{i=n+1}^m \pi_i(g)$, $n < m$. Since $W^{n,m}$ is a unitary representation of G into $\bigotimes_{i=n+1}^m K_i$, we get an irreducible decomposition $W^{n,m} = \sum_{\pi \in \widehat{G}} \lambda_\pi^{n,m} \pi$ where $\lambda_\pi^{n,m}$ is the multiplicity of π in $W^{n,m}$. As [1, 2, 7, 8], $\mathfrak{A}^\alpha = \{x \in \mathfrak{A} : \alpha_g(x) = x$ for all $g \in G\}$ is equal to $\overline{\bigcup_{n=1}^\infty \mathfrak{A}_n^\alpha}$ where $\mathfrak{A}_n = \bigotimes_{i=1}^n K_i$ and — denotes the norm closure. Then the finite dimensional algebra $\mathfrak{A}_n^\alpha = \mathfrak{A}_n \bigcap \{W_g^{0,n} : g \in G\}'$ is isomorphic to $\bigoplus_{\pi \in \widehat{G}} \mathfrak{A}_\pi^n$ where $\mathfrak{A}_\pi^n$ is a non-zero factor of type $I_{\lambda_\pi^{0,n}}$ because of $J(\pi_i) = \widehat{G}$ for all $i \in \mathbb{N}$. Therefore $\mathfrak{A}^\alpha$ is an AF algebra and its structure is completely determined by the partial embedding $\mathfrak{A}_\pi^n \to \mathfrak{A}_\rho^{n+1}$ [1]. We define a positive operator $E_{\rho,\pi}^{n,m}$,

$$E_{\rho,\pi}^{n,m} = \int_G \overline{\chi_\rho(g)} \chi_\pi(g) W_g^{n,m} \, dg$$

where χ_π is the character of G associated with π and dg is a normalized Harr measure on G. The way how to prove the main theorem is essentially due to the one done in [7].

Let τ be the canonical trace on $\mathfrak{A}$, that is, $\tau = \bigotimes_{i=1}^\infty |K_i|^{-1} Tr$, where Tr is the usual trace on the matrix algebra K_i.

LEMMA 2.1 [7, LEMMA 2.1]. *The partial embedding* $\mathfrak{A}_\pi^n \to \mathfrak{A}_\rho^{n+1}$ (π, $\rho \in \widehat{G}$) *has a multiplicity* $|K_{n+1}|\tau(E_{\rho,\overline{\pi}}^{n,n+1})$.

Proof. Let $\pi \otimes \pi_{n+1} = \sum_{\omega \in \widehat{G}} \lambda_\omega \omega$ be the irreducible decomposition of $\pi \otimes \pi_{n+1}$ where λ_ω is the multiplicity of ω in $\pi \otimes \pi_{n+1}$. Then we obtain

$$\begin{aligned}|K_{n+1}|\tau(E_{\rho,\overline{\pi}}^{n,n+1}) &= \int_G \overline{\chi_\rho(g)}\chi_\pi(g) Tr(\pi_{n+1}(g))\, dg \\ &= \int_G \overline{\chi_\rho(g)}(Tr \otimes Tr)([\pi \otimes \pi_{n+1}](g))\, dg \\ &= \sum_{\omega \in \widehat{G}} \int_G \overline{\chi_\rho(g)} \lambda_\omega \chi_\omega(g)\, dg \\ &= \sum_{\omega \in \widehat{G}} \lambda_\omega \delta_{\rho,\omega} = \lambda_\rho\end{aligned}$$

where $\delta_{\rho,\omega}$ is Kronecker's delta.

REMARK 2.2. The partial embedding $\mathfrak{A}_\pi^n \to \mathfrak{A}_\rho^m$ ($n < m$) has multiplicity

$$|K_{n+1}||K_{n+2}| \cdots |K_m|\tau(E_{\rho,\overline{\pi}}^{n,m}).$$

By the quite same reason done at [7, §3], we require that $W_k^{n,\infty} = st - \lim_{m \to \infty} W_k^{n,m}$ exists for $k \in K$ and $n \in \mathbb{N}$. The restriction $\pi|_K$ to K of an irreducible representation π of G is $\sum_{\omega \in \widehat{K}} \beta_\omega \omega$ as an irreducible decomposition. Since K is a normal subgroup of G, the multiplicity β_ω is

$$\beta_\omega = \begin{cases} d_\pi > 0 & \text{when } \omega \in G\omega' \text{ for some } \omega' \in \widehat{K}, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this orbit $G\omega'$ by $s(\pi)$.

LEMMA 2.3.

$$\lim_{m \to \infty} \tau(E_{\rho,\overline{\pi}}^{n,m}) = \int_K \overline{\chi_\rho(g)}\chi_\pi(g)\tau(W_g^{n,\infty})\, dg,$$

$$\lim_{n \to \infty}\left(\lim_{m \to \infty} \tau(E_{\rho,\overline{\pi}}^{n,m})\right) = \begin{cases} (|K|/|G|)d_\rho d_\pi |s(\pi)|, & \textit{if } s(\pi) = s(\rho) \\ 0 \quad , & \textit{otherwise} \end{cases}$$

where $|\cdot|$ *is the cardinality of a set.*

Proof. By [7, Lemma 2.2], we have

$$\begin{aligned}\lim_{m\to\infty}\tau(E^{n,m}_{\rho,\overline{\pi}}) &= \lim_{m\to\infty}\int_G \overline{\chi_\rho(g)}\chi_\pi(g)\tau\left(\otimes_{i=n+1}^m \pi_i(g)\right) dg\\ &= \lim_{m\to\infty}\int_G \overline{\chi_\rho(g)}\chi_\pi(g)\prod_{i=n+1}^m \tau(\pi_i(g))\, dg\\ &= \int_K \overline{\chi_\rho(g)}\chi_\pi(g)\tau(W_g^{n,\infty})\, dg.\end{aligned}$$

Since $\lim_{n\to\infty}\prod_{i=n}^\infty \tau(\pi_i(g)) = 1$ for $g\in K$, we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\tau(E^{n,m}_{\rho,\overline{\pi}})\right) = \int_K \overline{\chi_\rho(g)}\chi_\pi(g)\, dg.$$

By the orthogonality of characters of a compact group, we obtain

$$\int_K \overline{\chi_\rho(g)}\chi_\pi(g)\, dg = \begin{cases}(|K|/|G|)d_\rho d_\pi |s(\pi)| & \text{if } s(\pi)=s(\rho),\\ 0 & \text{otherwise.}\end{cases}$$

Let τ' be another normalized trace on $\mathfrak{A}^\alpha$. Then for a minimal projection $f^n_\pi \in \mathfrak{A}^n_\pi$ ($\pi\in\widehat{G}$), we put $\xi^n_\pi = \tau'(f^n_\pi)$. Then ξ^n_π is positive and is independent for the choice of f^n_π. By Lemma 2.1, the vectors $\xi^n = (\xi^n_\pi)_{\pi\in\widehat{G}}$ and $\xi^{n+1} = (\xi^{n+1}_\pi)_{\pi\in\widehat{G}}$ satisfy an equality

$$\xi^n_\pi = \sum_{\rho\in\widehat{G}} |K_{n+1}|\tau(E^{n,n+1}_{\rho,\overline{\pi}})\xi^{n+1}_\rho. \tag{1}$$

Then by setting $\eta^n_\pi = (\prod_{i=1}^n |K_i|)\xi^n_\pi$, we have

$$\eta^n_\pi = \sum_{\rho\in\widehat{G}} \tau(E^{n,n+1}_{\rho,\overline{\pi}})\eta^{n+1}_\rho,$$

that is, $\eta^n = \eta^{n+1}C(n,n+1)$ where $\eta^n = (\eta^n_\pi)_{\pi\in\widehat{G}}$ and the matrix $C(n,n+1) = (\tau(E^{n,n+1}_{\rho,\overline{\pi}}))_{\rho,\pi\in\widehat{G}}$.

REMARK 2.4. For $n < m < l$,

$$\eta^n = \eta^m C(n,m), \qquad C(m,l)C(n,m) = C(n,l), \tag{1–1}$$

where the matrix $C(n,m) = (\tau(E^{n,m}_{\rho,\overline{\pi}}))_{\rho,\pi\in\widehat{G}}$.

We compute

$$\begin{aligned}
|G|^{-1}\sum_{\pi\in\widehat{G}} \dim\pi\eta^n_\pi &= |G|^{-1}\sum_{\pi\in\widehat{G}}\dim\pi\left(\sum_{\rho\in\widehat{G}}\tau(E^{n,n+1}_{\rho,\overline{\pi}})\eta^{n+1}_\rho\right)\\
&= \sum_{\rho\in\widehat{G}}\left(|G|^{-1}\sum_{\pi\in\widehat{G}}\dim\pi\tau(E^{n,n+1}_{\rho,\overline{\pi}})\eta^{n+1}_\rho\right)\\
&= \sum_{\rho\in\widehat{G}}\left(\int_G\overline{\chi_\rho(g)}\left\{|G|^{-1}\sum_{\pi\in\widehat{G}}\dim\pi\chi_\pi(g)\right\}\tau(W^{n,n+1}_g)\,dg\right)\eta^{n+1}_\rho\\
&= \sum_{\rho\in\widehat{G}}\left(\int_G\overline{\chi_\rho(g)}\delta_{g,e}\tau(W^{n,n+1}_g)\,dg\right)\eta^{n+1}_\rho\\
&= \sum_{\rho\in\widehat{G}}|G|^{-1}\dim\rho\eta^{n+1}_\rho,
\end{aligned}$$

since a left regular representation of G is $\sum_{\pi\in\widehat{G}}(\dim\pi)\pi$. Therefore we have

$$|G|^{-1}\dim\rho\eta^n_\rho \le \sum_{\rho\in\widehat{G}}|G|^{-1}\dim\rho\eta^n_\rho = \sum_{\rho\in\widehat{G}}|G|^{-1}\dim\rho\eta^1_\rho$$

and

$$\sup_{\rho\in\widehat{G}}\eta^n_\rho \le \sum_{\rho\in\widehat{G}}\dim\rho\eta^1_\rho$$

for all $n\in\mathbb{N}$. Hence we may take a subsequence $\{\eta^{n_q}\}$ of $\{\eta^n\}$ which converges to a vector $\eta = (\eta_\pi)_{\pi\in\widehat{G}}$. It follows from (1–1) that

$$\lim_{n_q\to\infty}(\eta^{n_p} - \eta^{n_q}) = \lim_{n_q\to\infty}\eta^{n_q}(C(n_p,n_q) - I)$$

where I is the identity matrix. By Lemma 2.3, we get

$$0 = \lim_{n_p \to \infty} \left(\lim_{n_q \to \infty} (\eta^{n_p} - \eta^{n_q}) \right) = \eta(C - I)$$

where the matrix C is equal to $\left((|K|/|G|)d_\rho d_\pi |s(\pi)| \delta_{s(\pi),s(\rho)}\right)_{\rho,\pi \in \widehat{G}}$. Then the vector η satisfies a relation

$$\eta_\pi = (|K|/|G|) \sum_{s(\pi)=s(\rho)} d_\rho d_\pi |s(\pi)| \eta_\rho.$$

We put

$$x_{s(\pi)} = \sum_{\rho \in \widehat{G}, s(\rho)=s(\pi)} d_\rho \eta_\rho.$$

Hence we obtain a vector $(x_{s(\pi)})_{s(\pi) \in \widehat{K}/G}$ such that

$$\eta_\pi = (|K|/|G|) d_\pi |s(\pi)| x_{s(\pi)}. \tag{2}$$

On the other hand, since $\eta^{n_p} = \eta^{n_q} C(n_p, n_q)$, $(n_p < n_q)$, we have

$$\eta^{n_p} = \lim_{n_q \to \infty} \eta^{n_q} C(n_p, n_q) = \eta C(n_p, \infty),$$

where $C(n, \infty) = \lim_{m \to \infty} C(n, m)$. Therefore, for all n, we have

$$\eta^n = \eta^{n_p} C(n, n_p) = \eta C(n_p, \infty) C(n, n_p) = \eta C(n, \infty). \tag{3}$$

3. Main results

THEOREM 3.1. *Let $(\mathfrak{A}, G, \alpha)$ and K be as in §2. Then the number of all extremal tracial states on the fixed point algebra $\mathfrak{A}^\alpha$ equals the cardinality of the orbit space $\widehat{K}/G$.*

Proof. For an orbit $s(\pi) \in \widehat{K}/G$, we set

$$x_{s(\pi)} = \begin{cases} x, & \text{if } s(\rho) = s(\pi) \\ 0, & \text{otherwise,} \end{cases}$$

and we define vectors

$$\eta_{s(\pi)} = (d_\pi |s(\pi)| \delta_{s(\pi),s(\rho)})_{\rho\in\widehat{G}} \quad \text{and} \quad \eta^n = \frac{x|K|}{|G|}\eta_{s(\pi)}C(n,\infty).$$

Therefore we also set

$$\xi^n = \left(\prod_{i=1}^{n} |K_i|^{-1}\right)\eta^n.$$

Since $C(n+1,\infty)C(n,n+1) = C(n,\infty)$ by (1-1), we get

$$\begin{aligned}
\text{(4)} \qquad \xi^n &= \left(\prod_{i=1}^{n} |K_i|^{-1}\right)\frac{x|K|}{|G|}\eta_{s(\pi)}C(n,\infty) \\
&= \left(\prod_{i=1}^{n} |K_i|^{-1}\right)\frac{x|K|}{|G|}\eta_{s(\pi)}C(n+1,\infty)C(n,n+1) \\
&= \left(\prod_{i=1}^{n} |K_i|^{-1}\right)\eta^{n+1}C(n,n+1) \\
&= |K_{n+1}|\xi^{n+1}C(n,n+1),
\end{aligned}$$

which is the equality (1). If $\pi_1 = \sum_{\rho\in\widehat{G}} \lambda_\rho^{0,1}\rho$ is an irreducible decomposition, then

$$\bigoplus_{\rho\in\widehat{G}} \mathfrak{A}_\rho^1 = \bigoplus_{\rho\in\widehat{G}} (M_{\lambda_\rho^{0,1}}(\mathbf{C}) \otimes I_{\dim\rho}).$$

Since $|K_1|\xi^1 = \frac{x|K|}{|G|}\eta_{s(\pi)}C(1,\infty)$ and x is an arbitrary positive number, we can decide uniquely x such that $\sum_{\rho\in\widehat{G}} \xi_\rho^1\lambda_\rho^{0,1} = 1$. Since

$$|K_{n+1}|C(n,n+1)(\lambda_\rho^{0,n})_{\rho\in\widehat{G}} = (\lambda_\rho^{0,n+1})_{\rho\in\widehat{G}},$$

we have

$$\text{(5)} \qquad \sum_{\rho\in\widehat{G}} \xi_\rho^n\lambda_\rho^{0,n} = (\xi_\rho^n)_{\rho\in\widehat{G}}(\lambda_\rho^{0,n})_{\rho\in\widehat{G}}$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{n} |K_i|^{-1}\right) \frac{x|K|}{|G|}\eta_{s(\pi)}C(n,\infty)(\lambda_\rho^{0,n})_{\rho\in\widehat{G}} \\
&= \frac{x|K|}{|K_1||G|}\eta_{s(\pi)}C(n,\infty)C(1,n)(\lambda_\rho^{0,1})_{\rho\in\widehat{G}} \\
&= \frac{x|K|}{|K_1||G|}\eta_{s(\pi)}C(1,\infty)(\lambda_\rho^{0,1})_{\rho\in\widehat{G}} \\
&= (\xi_\rho^1)_{\rho\in\widehat{G}}(\lambda_\rho^{0,1})_{\rho\in\widehat{G}} = 1.
\end{aligned}
$$

Hence for each $\bigoplus_{\rho\in\widehat{G}}\mathfrak{A}_\pi^n$, we set a trace $\tau_{s(\pi)}^n = \sum_{\rho\in\widehat{G}}^{\oplus}\xi_\rho^n Tr$ where Tr are canonical traces on $M_{\lambda_\rho^{0,n}}(\mathbb{C})$ for all $\rho \in \widehat{G}$. Then $\{\tau_{s(\pi)}^n : n \in \mathbb{N}\}$ gives a tracial state (denoted by $\tau_{s(\pi)}$) on $\mathfrak{A}^\alpha$ by (4) and (5). Because of (2) and (3), the tracial states $\{\tau_{s(\pi)} : s(\pi) \in \widehat{K}/G\}$ are extremal on $\mathfrak{A}^\alpha$.

REMARK 3.2. Let $(\mathfrak{A}, G, \alpha)$ be as in §2. If G is abelian, the orbit space $\widehat{K}/G$ is equal to $\widehat{K}$. Since $|\widehat{K}| = |K|$, Theorem 4.2 in [7] follows from Theorem 3.1.

THEOREM 3.3. *The center of the fixed point algebra*

$$(\pi_\tau(\mathfrak{A})'')^{\widetilde{\alpha}} = \{x \in \pi_\tau(\mathfrak{A})'' : \widetilde{\alpha}_g(x) = x,\ g \in G\}$$

is $|\widehat{K}/G|$-dimensional.

Proof. At first, we must compute $\eta = (\eta_\pi)_{\pi\in\widehat{G}}$ in (2) for a restricted trace $\tau|_{\mathfrak{A}^\alpha}$ of the unique trace τ to $\mathfrak{A}^\alpha$. By an easy computation, we have

$$\xi_\pi^n = \dim\pi \prod_{i=1}^{n} |K_i|^{-1}, \quad \eta_\pi^n = \dim\pi,$$

therefore $\eta_\pi = \dim\pi$ for all $\pi \in \widehat{G}$. Then we may set $x_{s(\pi)}$ in (2) by $x_{s(\pi)} = \frac{|G|\dim\pi}{|K||s(\pi)|d_\pi}$ which depends only on the orbit $s(\pi)$. Hence the trace $\tau|_{\mathfrak{A}^\alpha}$ is of the form

$$\sum_{s(\pi)\in\widehat{K}/G} a_{s(\pi)}\tau_{s(\pi)}, \quad a_{s(\pi)} > 0, \qquad \sum_{s(\pi)\in\widehat{K}/G} a_{s(\pi)} = 1.$$

Since by Theorem 3.1, the center of $(\pi_\tau(\mathfrak{A})'')^{\tilde{\alpha}}$ is smaller than $|\widehat{K}/G|$-dimensional, it must be $|\widehat{K}/G|$- dimensional. Note that the minimal projections of its center are corresponded to $\{\tau_{s(\pi)}\}_{s(\pi)\in\widehat{K}/G}$.

We can obtain the next corollary directly from Theorem 3.3.

COROLLARY 3.4. *Let* $(\mathfrak{A}, G, \alpha)$ *be as in* §2. *The fixed point algebra* $(\pi_\tau(\mathfrak{A})'')^{\tilde{\alpha}}$ *is a factor if and only if the automorphism* $\tilde{\alpha}_g$ *is not inner in* $\pi_\tau(\mathfrak{A})''$ *for all* $g \neq e$.

Next we want get conditions under which the fixed point algebra $\mathfrak{A}^\alpha$ is UHF. We will follow to the line of proof investigated in [3].

Let $\mathcal{B}(l^2(G))$ be the algebra of all linear operators on $l^2(G)$. $\mathfrak{B}$ denotes the UHF-algebra $\bigotimes_{n=1}^\infty \mathcal{B}(l^2(G))$, i.e., the infinite tensor product of copies of $\mathcal{B}(l^2(G))$ with type $|G|^\infty$. We define a left regular representation λ of G on $l^2(G)$ by

$$(\lambda_g\xi)(h) = \xi(g^{-1}h) \quad \text{for } g,h \in G \text{ and } \xi \in l^2(G).$$

The action $Ad\lambda$ of G on $\mathcal{B}(l^2(G))$ is defined by

$$Ad\lambda_g(x) = \lambda_g x \lambda_g^* \quad \text{for } g \in G \text{ and } x \in \mathcal{B}(l^2(G)).$$

Also we define the action β of G on $\mathfrak{B}$ such that $\beta_g = \bigotimes_{n=1}^\infty Ad\lambda_g$ for all $g \in G$. Then we get a C^*-dynamical system $(\mathfrak{B}, G, \beta)$. Now we can obtain the following lemma and theorem as in [3].

LEMMA 3.5. *The fixed point algebra* $\mathfrak{B}^\beta$ *is* **-isomorphic to* $\mathfrak{B}$.

Proof. Only in this lemma, we use the same notation for $(\mathfrak{B}, G, \beta)$ as $(\mathfrak{A}, G, \alpha)$ in §2. By Lemma 2.1, we compute the multiplicity of partial embedding $\mathfrak{A}_\pi^n \to \mathfrak{A}_\rho^{n+1}$ $(\pi, \rho \in \widehat{G})$ as follows,

$$\begin{aligned} |\mathcal{B}(l^2(G))||G|^{-1}Tr(E_{\rho,\bar{\pi}}^{n,n+1}) &= Tr\left(\int_G \overline{\chi_\rho(g)}\chi_\pi(g)\lambda_g\, dg\right) \\ &= \int_G \overline{\chi_\rho(g)}\chi_\pi(g)Tr(\lambda_g)\, dg = \dim\pi \dim\rho \end{aligned}$$

because $Tr(\lambda_g) = |G|\delta_{g,e}$. Then the Bratteli diagram for $\mathfrak{B}^\beta$ is Figure 1;

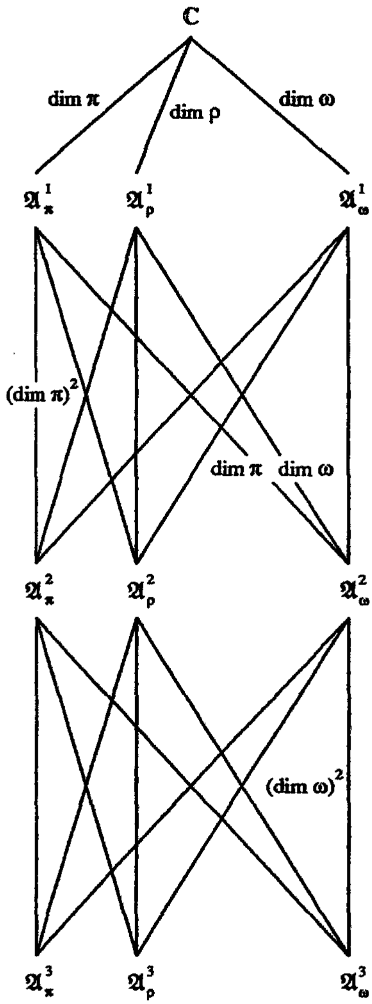

Figure 1

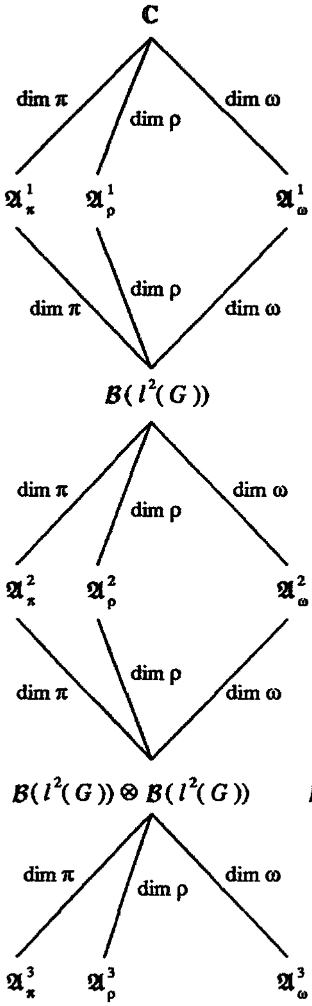

Figure 2

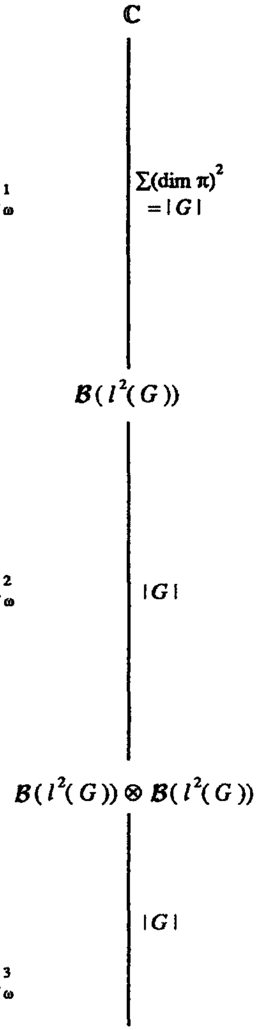

Figure 3

We transform Figure 1 to Figure 2 and Figure 3. Then Figure 3 is a Bratteli diagram of $\mathfrak{B}$, hence $\mathfrak{B}^\beta$ is *-isomorphic to $\mathfrak{B}$. (See [1].)

THEOREM 3.6. *Let* $(\mathfrak{A}, G, \alpha)$ *be as in* §2. *Then the followings are equivalent:*

(i) $\mathfrak{A}^\alpha$ *is a UHF-algebra.*

(ii) $\mathfrak{A}^\alpha$ *is* **-isomorphic to* $\mathfrak{A}$.

(iii) *There exists a UHF-algebra* $\mathfrak{C}$ *such that* $\mathfrak{A}$ *is* **-isomorphic to* $\mathfrak{C} \bigotimes \mathfrak{B}$ *and* α *is conjugate to* $\iota \otimes \beta$, *where* ι *is the identity automorphism of* $\mathfrak{C}$.

(iv) *There exists an increasing sequence* $\{ n_k : k \in \mathbb{N}\}$ *of non negative integers such that* $n_1 = 0$ *and*

$$C(n_k, n_{k+1}) = (|G|^{-1} \dim \rho \dim \pi)_{\rho,\pi \in \widehat{G}}$$

i.e., $\tau(E^{n_k, n_{k+1}}_{\rho,\overline{\pi}}) = |G|^{-1} \dim \rho \dim \pi$ *for all* $\rho, \pi \in \widehat{G}$ *and all* $k \in \mathbb{N}$.

Proof. By Lemma 3.5, (iii)⇒(ii)⇒(i) is evident.

The rest of the proof is analogous to that in [3]. Suppose that (i) holds. Then, by [1, 2.5 and 2.6], there are increasing sequences $\{\mathfrak{B}(k)\}_{k=1}^\infty$ of matrix factors and $\{n_k\}_{k=1}^\infty$ of non negative integers ($n_1 = 0$ and $\mathfrak{A}_0^\alpha$) such that

$$\mathfrak{A}^\alpha_{n_k} = \bigoplus_{\pi \in \widehat{G}} \mathfrak{A}^{n_k}_\pi \subset \mathfrak{B}(k) \subset \mathfrak{A}^\alpha_{n_{k+1}} = \bigoplus_{\pi \in \widehat{G}} \mathfrak{A}^{n_{k+1}}_\pi .$$

Let a^k_π (resp. b^k_π) be the multiplicity of partial embedding of $\mathfrak{A}^{n_k}_\pi \to \mathfrak{B}(k)$ (resp. $\mathfrak{B}(k) \to \mathfrak{A}^{n_{k+1}}_\pi$). Then the multiplicity of partial embedding of $\mathfrak{A}^{n_k}_\pi \to \mathfrak{A}^{n_{k+1}}_\rho$ is by Lemma 2.1,

$$a^k_\pi b^k_\rho = \left(\prod_{i=n_k+1}^{n_{k+1}} |K_i| \right) \tau(E^{n_k, n_{k+1}}_{\rho,\overline{\pi}}). \tag{6}$$

Now we have

$$\sum_{\pi \in \widehat{G}} \dim \pi a^k_\pi b^k_\rho$$

(7)

$$
\begin{aligned}
&= \left(\prod_{i=n_k+1}^{n_{k+1}} |K_i|\right) \int_G \overline{\chi_\rho(g)} \left(\sum_{\pi\in\widehat{G}} \dim \pi \chi_\pi(g)\right) \tau(W_g^{n_k,n_{k+1}})\, dg \\
&= \left(\prod_{i=n_k+1}^{n_{k+1}} |K_i|\right) \int_G \overline{\chi_\rho(g)} |G| \delta_{g,e} \tau(W_g^{n_k,n_{k+1}})\, dg \\
&= \left(\prod_{i=n_k+1}^{n_{k+1}} |K_i|\right) \dim \rho.
\end{aligned}
$$

Similarly $\sum_{\rho\in\widehat{G}} \dim \rho a_\pi^k b_\rho^k = (\prod_{i=n_k}^{n_{k+1}} |K_i|) \dim \pi$. Hence we can write as

$$
\begin{aligned}
a_\pi^k &= a_k \dim \pi && \text{for all } \pi \in \widehat{G}\ (a_k : \text{independent for } \pi), \\
b_\rho^k &= b_k \dim \rho && \text{for all } \rho \in \widehat{G}\ (b_k : \text{independent for } \rho).
\end{aligned}
$$

Since $\sum_{\pi\in\widehat{G}} (\dim \pi)^2 = |G|$, we get

$$
a_k b_k = |G|^{-1} \prod_{i=n_k+1}^{n_{k+1}} |K_i|,
$$

by substituting a_k and b_k into (7). Therefore we have $\tau(E_{\rho,\overline{\pi}}^{n_k,n_{k+1}}) = |G|^{-1} \dim \pi \dim \rho$ by (6) which means that (iv) holds.

Next we will prove (iv) $\Longrightarrow$ (iii). Considering the trivial representation of G, we have

$$
|G|^{-1} \dim \rho = \int_G \overline{\chi_\rho(g)} \tau(W_g^{n_k,n_{k+1}})\, dg,
$$

which implies that the representation $W^{n_k,n_{k+1}}$ of G is equivalent to $(\prod_{i=n_k+1}^{n_{k+1}} |K_i|)|G|^{-1}$-multiple of the left regular representation λ. Hence there is a matrix factor C_k such that

$$
\bigotimes_{i=n_k+1}^{n_{k+1}} K_i = C_k \bigotimes B(l^2(G))
$$

and $Ad\,W^{n_k,n_{k+1}}$ is conjugate to $\iota \otimes Ad\lambda$ for all k where ι is the identity map on C_k. Put $\mathfrak{C} = \bigotimes_{k=1}^{\infty} C_k$. Therefore $\mathfrak{A}$ is *-isomorphic to $\mathfrak{C} \bigotimes \mathfrak{B}$ (identifying $\mathfrak{C} \bigotimes \mathfrak{B}$ and $\bigotimes_{k=1}^{\infty}(C_k \bigotimes B(l^2(G)))$ and α is conjugate to $\iota \otimes \beta$ where ι is the identity map on $\mathfrak{C}$, which implies (iii).

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