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THE WEAK INVERTIBILITY FOR CHAINS IN A C^*–ALGEBRA

EUNG GI KIM, YONG BIN CHOI AND WOO YOUNG LEE

Throughout this paper suppose A is a C^*-algebra with unity. By a *chain* of two elements in A we mean a pair $(b, a) \in A^2$ for which

$$ba = 0. \tag{0.1}$$

Whether or not the chain condition (0.1) is satisfied, the pair $(b, a) \in A^2$ in a C^*-algebra A will be called *invertible* if there are v and u in A for which

$$vb + au = 1 \tag{0.2}$$

and will be called *weakly invertible* if there is implication, for arbitrary $c \in A$,

$$bc = c^*a = 0 \implies c = 0. \tag{0.3}$$

We shall call the pair $(b, a) \in A^2$ *regular* if there are a' and b' in A for which

$$a = aa'a \quad \text{and} \quad b = bb'b. \tag{0.4}$$

For the regularity, the reader is referred to [4],[6], or [7] and for the invertibility for a pair, to [5] or [7]. In this note we give the characterization of the weak invertibility for chains in a C^*-algebra.

Our first observation is elementary :

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LEMMA 1. *If $(b,a) \in A^2$ then there is implication*

$$(1.1) \qquad (b,a) \text{ invertible} \implies (b,a) \text{ weakly invertible}$$

and

$$(1.2) \qquad (b,a) \text{ invertible and a chain} \implies (b,a) \text{ regular.}$$

Proof. If (0.2) holds and $c \in A$ is arbitrary then

$$bc = c^*a = 0 \implies c^*c = c^*vbc + c^*auc = 0 \implies c = 0,$$

giving (1.1). Also multiplying on the left by b, and multiplying on the right by a, give

$$bvb = b \quad \text{and} \quad aua = a,$$

giving (1.2).

The weak invertibility for a pair can be tested by the zero divisor-ness of a single element :

THEOREM 2. *If $(b,a) \in A^2$ then*

$$(b,a) \text{ weakly invertibl} \iff b^*b + aa^* \text{ not a zero divisor.}$$

Proof. If (b,a) satisfies the condition (0.3) then

$$\begin{aligned}(b^*b + aa^*)c = 0 &\implies c^*(b^*b + aa^*)c = 0\\ &\implies (bc)^*bc + (c^*a)(c^*a)^* = 0\\ &\implies bc = c^*a = 0\\ &\implies c = 0,\end{aligned}$$

which shows that $b^*b + aa^*$ is not a left zero divisor ; being self-adjoint, it must be not a zero divisor. Conversely, if b^*b+aa^* is not a zero divisor then

$$bc = c^*a = 0 \implies (b^*b + aa^*)c = 0 \implies c = 0,$$

which says that (b,a) is weakly invertible.

For chains in a special C^*-algebra, the regularity and the weak invertibility imply the invertibility :

THEOREM 3. *Let $(b, a) \in A^2$ be a chain and suppose there exist a Hilbert space H and a faithful $*$-representation $\varphi : A \to B(H)$ for which $\varphi(A)$ is a closed $*$-ideal of $B(H)$. Then*

$$(3.1) \quad (b, a) \text{ regular and weakly invertible} \Longrightarrow (b, a) \text{ invertible.}$$

Proof. We begin by showing that if $w \in A$ is regular then there is *Moore-Penrose inverse* of w in A, say w', in the sense that

$$(3.2) \qquad w = ww'w, \;\; w' = w'ww', \;\; (w'w)^* = w'w, \;\; (ww')^* = ww'.$$

To do this, suppose that $\varphi : A \to B(H)$ is a faithful $*$- representation and that $w \in A$ is regular. Then $\varphi(w) \in B(H)$ is also regular and hence, by [7, Theorem 8.7.1], there exists Moore-Penrose inverse $s \in B(H)$ of $\varphi(w)$. If $\varphi(A)$ is a closed $*$- ideal of $B(H)$ then, since

$$s = s\varphi(w)s \; \in \; \varphi(A),$$

there exists $w' \in A$ for which $\varphi(w') = s$. In particular, we have

$$\varphi(ww'w) = \varphi(w), \;\; \varphi(w'ww') = \varphi(w'),$$
$$\varphi((w'w)^*) = \varphi(w'w), \;\; \varphi((ww')^*) = \varphi(ww'),$$

which, by the injectivity of φ , gives (3.2). Towards (3.1), assume that (b, a) is regular and weakly invertible. By (3.2), there are Moore-Penrose inverses v and u in A of, respectively, b and a. We now observe that

$$b(1 - (vb + au)) = b - bvb = 0$$

and

$$\begin{aligned}(1 - (vb + au))^* a &= (1 - (vb)^* - (au)^*)a \\ &= (1 - vb - au)a \\ &= a - aua \\ &= 0,\end{aligned}$$

which, by assumption, gives that $vb+au = 1$; therefore (b, a) is invertible.

We have been unable to decide whether or not the condition that $\varphi(A)$ is a closed $*$-ideal of $B(H)$ can be dropped from Theorem 3. Of course, if $A = B(H)$ then the implication (3.1) always holds (recall by [7, Theorem 8.7.1] that if $a \in B(H)$ is regular then a has Moore-Penrose inverse in $B(H)$). What is not so obvious, and looks like an interesting problem, is to show whether or not if A is a closed $*$-subalgebra of $B(H)$ and if $a \in A$ is regular then a has Moore-Penrose inverse in A. However the answer is not evident even in the case that A is a Von-Neumann algebra.

The characterization of weakly invertible operators between Hilbert spaces turns out to be delicate:

THEOREM 4. *If $A = B(H)$ for a Hilbert space H and $(S,T) \in A^2$ is a chain then*

$$(4.1) \qquad (S,T) \text{ weakly invertible} \iff S^{-1}(0) = \overline{T(H)}.$$

Proof. If (S,T) is weakly invertible then for arbitrary $R \in B(H)$,

$$SR = R^*T = 0 \implies R = 0,$$

which says that $S^{-1}(0) \cap (T^*)^{-1}(0) = \{0\}$, and hence $S^{-1}(0) \cap \overline{T(H)}^{\perp} = \{0\}$, which implies that, by the chain-ness of (S,T), $S^{-1}(0) = \overline{T(H)}$. Towards the backward implication, suppose $S^{-1}(0) = \overline{T(H)}$. If $SR = R^*T = 0$ then for arbitrary $x \in H$ we have $SRx = 0$, and hence $Rx \in \overline{T(H)}$, so that $R^*Rx = 0$, giving $R = 0$, which shows that (S,T) is weakly invertible.

We might remark that, remembering that if $T \in B(H)$ for a Hilbert space H then

$$T \text{ regular} \iff T(H) \text{ closed},$$

it clearly follows from (3.1) and (4.1) that if $(S,T) \in B(H)^2$ is a chain then

$$(S,T) \text{ invertible} \iff S^{-1}(0) = T(H)$$

The condition (0.3) extends to triples (c, b, a) and longer. To extend the condition we simply impose corresponding conditions on each of the pairs (c, b) and (b, a).

The weak invertibility condition for some long chain can be reproduced in the form (0.3) :

THEOREM 5. *If* $(b, a) \in A^2$, *and if* $(0, b, a, 0) \in A^4$ *then the followings are equivalent :*

$$(5.1) \qquad (0, b, a, 0) \textit{ is weakly invertible,}$$

$$(5.2) \qquad \left(\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \right) \textit{ is weakly invertible.}$$

Proof. If $(0, b, a, 0)$ is weakly invertible then for arbitrary $x, y, z \in A$,

$$(5.3) \qquad \begin{aligned} x^*b = 0 &\Longrightarrow x = 0, \\ by = y^*a = 0 &\Longrightarrow y = 0, \\ az = 0 &\Longrightarrow z = 0. \end{aligned}$$

Suppose that for arbitrary $c_{ij} \in A$ $(i,j=1,2,3)$

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} c_{11}^* & c_{21}^* & c_{31}^* \\ c_{12}^* & c_{22}^* & c_{32}^* \\ c_{13}^* & c_{23}^* & c_{33}^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;$$

and thus

$$(5.4) \qquad \begin{aligned} ac_{11} = ac_{12} = ac_{13} = 0, \\ bc_{21} = bc_{22} = bc_{23} = 0, \\ c_{21}^*a = c_{22}^*a = c_{23}^*a = 0, \\ c_{31}^*b = c_{32}^*b = c_{33}^*b = 0, \end{aligned}$$

which together with (5.3) gives $c_{ij} = 0$ (i,j=1,2,3), giving (5.2). Conversely, if (5.2) holds then for arbitrary $u, v, w \in A$,

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} u & 0 & 0 \\ v & 0 & 0 \\ w & 0 & 0 \end{bmatrix} = \begin{bmatrix} u^* & v^* & w^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \quad u = v = w = 0 \ ;$$

and thus

$$\left.\begin{aligned} &au = 0 \\ &bv = v^*a = 0 \\ &w^*b = 0 \end{aligned}\right\} \quad \Longrightarrow u = v = w = 0,$$

giving (5.1).

APPLICATION 6. Let $a = (a_1, a_2)$ be a commuting pair of elements in a C^*- algebra A and let $\Lambda(a)$ be the *Koszul complex* determined by a (cf. [2], [3], [5], [7]). Then $\Lambda(a)$ can be represented by the operator matrices

$$\Lambda(a) \ : \ 0 \longrightarrow A \xrightarrow{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} \begin{bmatrix} A \\ A \end{bmatrix} \xrightarrow{[-a_2 \ a_1]} A \longrightarrow 0\,.$$

If the Koszul complex $\Lambda(a)$ is weakly invertible (exact, resp.) at every stage then a is called *weakly Taylor invertible* (*Taylor invertible*, resp.). By a slight extension of Theorem 5, $a = (a_1, a_2)$ is weakly Taylor invertible if and only if

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \end{bmatrix} \right) \text{ is weakly invertible ;}$$

in turn, by Theorem 2,

$$\text{(6.1)} \qquad a = (a_1, a_2) \text{ weakly Taylor invertible}$$

$$\Longleftrightarrow \begin{bmatrix} a_1^*a_1 + a_2^*a_2 & 0 & 0 & 0 \\ 0 & a_1a_1^* + a_2^*a_2 & a_1a_2^* - a_2^*a_1 & 0 \\ 0 & a_2a_1^* - a_1^*a_2 & a_1^*a_1 + a_2a_2^* & 0 \\ 0 & 0 & 0 & a_1a_1^* + a_2a_2^* \end{bmatrix}$$

not a zero divisor.

As a special case of (6.1), if $T = (T_1, T_2)$ is a doubly commuting ($[T_i, T_j^*] = 0$ for all $i \neq j$) pair of operators acting on a Hilbert space H then

$$\text{(6.2)} \quad T \text{ weakly Taylor invertible} \Longleftrightarrow \left.\begin{matrix} T_1^*T_1 + T_2^*T_2 \\ T_1T_1^* + T_2T_2^* \\ T_1T_1^* + T_2^*T_2 \\ T_1^*T_1 + T_2T_2^* \end{matrix}\right\} \text{ one-one };$$

because a diagonal matrix is not a zero-divisor if and only if each diagonal entry is not a zero divisor (cf. [8, Theorem 1.3]) (also, note that four single operators are all self-adjoint and that $T \in B(H)$ is not a left zero divisor if and only if T is one-one (cf.[1, Theorem 57.1])). In particular, it clearly follows from Fuglede-Putnam theorem that if T_1 and T_2 are normal and they commute then (6.2) holds. We would like to remark that (6.2) is parallel to [2, Corollary 3.7] and that (6.1) and (6.2) can be extended to n-tuple of elements.

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Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea