Comm. Korean Math. Soc. 8 (1993), No. 2, pp. 255–265

A NEW PROOF OF KRALL'S THEOREM

KIL HYUN KWON AND BYEONG HOON YOO

1. Introduction

Let us consider a linear differential equation of order $N \geq 1$ of the form

$$\sum_{i=0}^{N} \ell_i(x) y^{(i)}(x) = \lambda y(x) \tag{1.1}$$

where $\ell_i(x)$, $i = 0, \ldots, N$, are real-valued smooth functions on R with $\ell_N(x) \not\equiv 0$ and λ is a real parameter, and ask : When does the differential equation (1.1) have an orthogonal polynomial set as solutions?

It is easy to see that if the differential equation (1.1) has polynomial solutions $P_n(x)$ of degree n for $n = 0, 1, \ldots, N$, then it must be of the form

$$L_N(y) = \sum_{i=0}^{N} \ell_i(x) y^{(i)}(x) = \sum_{i=0}^{N} \sum_{j=0}^{i} \ell_{ij} x^j y^{(i)}(x) = \lambda_n y(x) \tag{1.2}$$

where ℓ_{ij} are real constants and

$$\lambda_n = \ell_{00} + \ell_{11} n + \cdots + \ell_{NN} n(n-1) \cdots (n-N+1) \tag{1.3}$$

for $n = 0, 1, \ldots, N$.

In 1929, S. Bochner [1] (see also H. L. Krall and O. Frink [8]) proved that there are essentially (that is, up to a linear change of variable) only four distinct orthogonal polynomial sets satisfying the differential equation (1.2) for $N = 2$. They are now called the classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel. He also proposed

Received August 12, 1992. Revised January 25, 1993.

implicitly a problem of classifying all orthogonal polynomials satisfying the differential equation (1.2).

The classifying problem itself is not resolved yet in general except for $N = 2$ (due to S. Bochner [1]) and for $N = 4$ (due to H. L. Krall [7]). However, H. L. Krall [6] found a remarkable theorem (cf. Theorem 2.1) characterizing all differential equations of the form (1.2) which have an orthogonal polynomial set as solutions. Its proof in [6] is based on the notion of dual equation to the differential equation (1.2), which is developed by I. M Sheffer [15]. Later a second simpler proof using the generating functions of orthogonal polynomials was found by H. L. Krall and I. M. Sheffer [9].

Here we shall give a new third proof of the Krall's characterization theorem as well as some other equivalent characterizations. The idea of our proof is based on the observation that if the differential equation (1.2) has orthogonal polynomials as solutions, then it must be "symmetrizable on polynomials" (see section 2).

We believe that our new proof shed some light on the important problem of identifying the orthogonal polynomial solutions of the equation (1.2) as eigenfunctions of an operator which is self-adjoint on suitable Hilbert or Krein space (see [2, 3, 4, 5, 11]).

The first another is partially supported by Korea Sci. & Eng. Found. (90-08-00-02) and Global Analysis Research Center.

2. Main result

All polynomials in the followings are assumed to be real polynomials in one variable and we let $\mathcal{P}$ the space of all real polynomials. We shall call any liner functional σ on $\mathcal{P}$ a moment functional and

$$\sigma_n := \langle \sigma, x^n \rangle, \qquad n = 0, 1, \dots \tag{2.1}$$

the moments of σ. We denote the degree of a polynomial $\phi(x)$ by $\deg \phi$ with convention $\deg 0 = 0$. By a polynomial set, we mean a sequence of polynomials $\{\phi_n(x)\}_0^\infty$ with $\deg \phi_n = n$, $n = 0, 1, \dots$. Any polynomial set $\{\phi_n(x)\}_0^\infty$ determines a moment functional σ, called a canonical moment functional of $\{\phi_n(x)\}_0^\infty$ by requiring

$$\langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \qquad n = 1, 2, \dots. \tag{2.2}$$

Note that a canonical moment functional is uniquely determined by $\{\phi_n(x)\}_0^\infty$ up to a non-zero constant multiple.

DEFINITION 2.1. A polynomial set $\{P_n(x)\}_0^\infty$ is an orthogonal polynomial set (OPS in short) if there is a moment functional σ such that

$$\langle \sigma, P_m(x)P_n(x)\rangle = K_n\delta_{mn}, \qquad m \text{ and } n = 0,1,\dots \tag{2.3}$$

where K_n are non-zero real constants. If then, we call $\{P_n(x)\}_0^\infty$ an OPS relative to σ and $\{\sigma_n := \langle \sigma, x^n\rangle\}_0^\infty$ the moments of $\{P_n(x)\}_0^\infty$.

Note that if $\{P_n(x)\}_0^\infty$ is an OPS relative to σ, then σ must be a canonical moment functional of $\{P_n(x)\}_0^\infty$.

The main goal of this paper is to provide a new simple and illuminating proof of the following theorem due to Krall [6].

THEOREM 2.1. *Let $\{P_n(x)\}_0^\infty$ be a polynomial set and $\{\sigma_n\}_0^\infty$ the moments of any canonical moment functional σ of $\{P_n(x)\}_0^\infty$. Then, $\{P_n(x)\}_0^\infty$ is an OPS satisfying the differential equation* (1.2) *for each $n = 0,1,\dots$ if and only if $\{\sigma_n\}_0^\infty$ satisfy*

(i) $\Delta_n := \det[\sigma_{i+j}]_{i,j=0}^n \neq 0$, $n = 0,1,\dots$
and

(ii) $S_k(m) := \sum\limits_{i=2k+1}^{N} \sum\limits_{j=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\ell_{i,i-j}\sigma_{m-j} = 0$

for $k = 0,1,\dots [\frac{N-1}{2}]$ and $m = 2k+1,\ 2k+2,\dots$, where

$$P(n,k) = \begin{cases} 0 & , \quad n = 0, \\ n(n-1)\cdots(n-k+1), & n = 1,2,\dots . \end{cases}$$

Furthermore, if then, N must be even, say, $N = 2r$ for some $r = 1,2,\dots$.

We begin with recalling a few well known facts on the symmetrizability of linear differential operators of the form

$$L := L(x,D) = \sum_0^N a_i(x)D^i \tag{2.4}$$

where $D = d/dx$, $a_i(x)$ are real-valued functions in $C^i(I)$, $a_N(x) \not\equiv 0$, and I is an open interval. The formal adjoint of L is a differential operator L^* defined by

$$L^*(y) = \sum_0^N (-1)^i (a_i y)^{(i)}, \ y(x) \text{ in } C^N(I). \tag{2.5}$$

The operator L is called symmetric if $L = L^*$. It is called symmetrizable if there is a real-valued function $s(x) \not\equiv 0$ in $C^N(I)$ such that sL is symmetric. Then we call $s(x)$ a symmetry factor of L.

LEMMA 2.2 (Littlejohn [12] and Littlejohn and Race [14]). *For any real-valued function $s(x) \not\equiv 0$ in $C^N(I)$, the followings are all equivalent:*

(i) *$s(x)$ satisfies $N+1$ equations*

$$\sum_{i=k}^{N}(-1)^i\binom{i}{k}(\ell_i s)^{(i-k)} = \ell_k s, \quad k = 0, 1, \ldots, N. \tag{2.6}$$

(ii) *$s(x)$ satisfies $r := [\frac{N+1}{2}]$ equations*

$$R_k(s) := \sum_{\ell=k}^{r}\sum_{j=0}^{2\ell-2k+1}\binom{2\ell}{2k-1}\binom{2\ell-2k+1}{j}\frac{2^{2\ell-2k+2}-1}{\ell-k+1}\cdot B_{2\ell-2k+2}a_{2\ell}^{(2\ell-2k+1-j)}s^{(j)} - a_{2k-1}s = 0, \quad k = 1, 2, \ldots, r. \tag{2.7}$$

where B_{2i} are Bernoulli numbers defined by

$$\frac{x}{\exp(x)-1} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty}\frac{B_{2i}x^{2i}}{(2i)!}.$$

(iii) *$s(x)$ satisfies $r := [\frac{N+1}{2}]$ equations*

$$\widetilde{R}_K(s) := \sum_{i=2k+1}^{N}(-1)^i\binom{i-k-1}{k}(a_i s)^{(i-2k-1)} = 0, \quad k = 0, 1, \ldots, r-1. \tag{2.8}$$

(iv) *For any two real-valued functions $y(x)$ and $z(x)$ in $C^N(I)$, one of which has a compact support in I*

$$\langle sLy, z\rangle := \int_I z(x)(sLy)(x)\,dx = \int_I y(x)(sLz)(x)\,dx := \langle y, sLz\rangle. \tag{2.9}$$

(v) *There are $r+1$ real-valued functions $f_i(x)$ in $C^{2i}(I)$, $i = 0, 1, \ldots,$ $r = [\frac{N+1}{2}]$, with $f_r(x) \not\equiv 0$, and*

$$(sLy)(x) = \sum_{0}^{r}[f_i(x)y^{(i)}(x)]^{(i)}, \quad y \in C^N(I). \tag{2.10}$$

Furthermore if any one of the above equivalent conditions holds, then $N = 2r$ *must be even.*

For any moment functional σ and any polynomial $\phi(x)$, we define two new moment functionals σ', derivative of σ, and $\phi\sigma$ by

$$\langle \sigma', \psi(x) \rangle = -\langle \sigma, \psi'(x) \rangle \tag{2.11}$$

and

$$\langle \phi\sigma, \psi(x) \rangle = \langle \sigma, \phi(x)\psi(x) \rangle \tag{2.12}$$

for ψ in $\mathcal{P}$. Then we have

$$(\phi\sigma)' = \phi'\sigma + \phi\sigma'. \tag{2.13}$$

Finally we need the following simple fact.

LEMMA 2.3. *Let* $\{P_n(x)\}_0^\infty$ *be an OPS relative to* σ*. Then we have :*

(i) *For any polynomial* $\phi(x)$, $\phi\sigma = 0$ *if and only if* $\phi(x) \equiv 0$ *;*

and

(ii) *For any moment functional* τ, $\langle \tau, P_n \rangle = 0$, $n \geq k+1$ *for some integer* $k \geq 0$ *if and only if* $\tau = \phi\sigma$ *for some polynomial* $\phi(x)$ *of degree* $\leq k$.

Proof. (i) Assume that $\phi\sigma \equiv 0$ but $\phi(x) \not\equiv 0$ and write $\phi(x)$ as $\phi(x) = \sum_0^n c_j P_j(x)$, $c_n \neq 0$ where $n = \deg \phi (\geq 0)$. Then by the orthogonality of $\{P_n(x)\}_0^\infty$ relative to σ, we have

$$0 = \langle \phi\sigma, P_n \rangle = \langle \sigma, \phi P_n \rangle = c_n \langle \sigma, P_n^2 \rangle$$

and so $c_n = 0$, which is a contradiction. The converse is trivial.

(ii) Consider a moment functional $\tilde{\tau}$ given by $\tilde{\tau} = \left(\sum_0^k c_j P_j(x) \right) \sigma$, where c_j are real constant to be determined. Then we have

$$\langle \tilde{\tau}, P_n \rangle = \sum_0^k c_j \langle \sigma, P_j P_n \rangle = \begin{cases} c_n \langle \sigma, P_n^2 \rangle, & n \leq k \\ 0 \quad , & n > k. \end{cases} \tag{2.14}$$

Assume $\langle \tau, P_n \rangle = 0$ for $n > k$. Then the equation (2.14) shows that if we take $c_j = \langle \tau, P_j \rangle \langle \sigma, P_j^2 \rangle^{-1}$, $j = 0, 1, \ldots, k$, $\langle \tau, P_n \rangle = \langle \tilde{\tau}, P_n \rangle$ for all $n \geq 0$ so that $\tau = \tilde{\tau}$ since $\{P_n(x)\}_0^\infty$ is a polynomial set. Conversely if $\tau = \phi\sigma$ for some polynomial $\phi(x)$ of degree $\leq k$, then $\langle \tau, P_n \rangle = \langle \sigma, \phi P_n \rangle = 0$ for $n > k$.

Now, we are ready to prove Theorem 2.1. In fact we shall prove the following which is equivalent to Theorem 2.1 and is of interest in itself (cf. [10]).

THEOREM 2.4. *Let $\{P_n(x)\}_0^\infty$ be an OPS, σ the canonical moment functional of $\{P_n(x)\}_0^\infty$, and $\{\sigma_n\}_0^\infty$ the moments of σ. Then, the following statements are all equivalent.*

(i) *For each $n = 0, 1, \ldots$, $P_n(x)$ satisfies the differential equation (1.2).*

(ii) *σL_N is symmetric on polynomials in the sense that*

$$\langle L_N(\phi)\sigma, \psi \rangle = \langle L_N(\psi)\sigma, \phi \rangle \tag{2.15}$$

for any polynomials $\phi(x)$ and $\psi(x)$.

(iii) *σ satisfies $r := [\frac{N+1}{2}]$ equations (with R_k as in (2.7))*

$$R_k\sigma = 0, \qquad k = 1, 2, \ldots, r. \tag{2.16}$$

(iv) *σ satisfies $r := [\frac{N+1}{2}]$ equations (with $\widetilde{R}_k$ as in (2.8))*

$$\widetilde{R}_k\sigma = 0, \qquad k = 0, 1, \ldots, r-1. \tag{2.17}$$

(v) *$\{\sigma_n\}_0^\infty$ satisfies $r := [\frac{N+1}{2}]$ recurrence relations $S_k(m) = 0$, $k = 0, 1, \ldots, r-1$ and $m = 2k+1, 2k+2, \ldots$.*

(vi) *$\{\sigma_n\}_0^\infty$ satisfies $r := [\frac{N+1}{2}]$ recurrence relations*

(2.18)

$$T_k(m) := \sum_{i=k}^{r} \sum_{j=0}^{2i} \binom{2i}{2k-1} P(m-2k+1, 2i-2k+1) \frac{2^{2i-2k+2}-1}{i-k+1} \cdot$$

$$B_{2i-2k+2}\ell_{2i,j}\sigma_{m-2i+j} + \sum_{j=0}^{2k-1} \ell_{2k-1,j}\sigma_{m-2k+1+j} = 0,$$

$k = 1, 2, \ldots, r$ and $m = 2k-1, 2k, \ldots$.

Furthermore if any one of the above equivalent conditions holds, then $N = 2r$ must be even.

Proof. For any polynomials $\phi(x)$ and $\psi(x)$, we have from (2.11), (2.12), and (2.13) that

$$\begin{aligned}\langle L_N(\phi)\sigma, \psi\rangle &= \left\langle \sum_{i=0}^{N} \ell_i \phi^{(i)}\sigma, \psi \right\rangle = \left\langle \sum_{i=0}^{N} (-1)^i (\psi \ell_i \sigma)^{(i)}, \phi \right\rangle \\ &= \left\langle \sum_{i=0}^{N} \sum_{k=0}^{i} (-1)^i \binom{i}{k} \psi^{(k)} (\ell_i \sigma)^{(i-k)}, \phi \right\rangle \\ &= \left\langle \sum_{k=0}^{N} \sum_{i=k}^{N} (-1)^i \binom{i}{k} (\ell_i \sigma)^{(i-k)} \psi^{(k)}, \phi \right\rangle.\end{aligned}$$

Hence, the condition (2.15) is equivalent to

$$\sum_{i=k}^{N} (-1)^i \binom{i}{k} (\ell_i \sigma)^{(i-k)} = \ell_k \sigma, \quad k = 0, 1, \ldots, N. \tag{2.19}$$

Therefore, the equivalence of the conditions (ii), (iii), and (iv) comes immediately from Lemma 2.2. Now, assume that the condition (ii) holds. Equivalently, it means that σ satisfies $N+1$ equations in (2.19). Since $L_N(P_n) = \sum_0^N \ell_i P_n^{(i)}$ is a polynomial of degree $\leq n$, we may write it as

$$L_N(P_n) = \sum_0^N \ell_i P_n^{(i)} = \sum_0^n c_j P_j$$

where c_j are constants depending on n. Then for $k = 0, 1, \ldots, n$, we have by (2.19)

$$\begin{aligned} c_k \langle \sigma, P_k^2 \rangle &= \left\langle \sigma, \sum_{i=0}^{N} \ell_i P_n^{(i)} P_k \right\rangle = \sum_{i=0}^{N} (-1)^i \left\langle (P_k \ell_i \sigma)^{(i)}, P_n \right\rangle \\ &= \sum_{j=0}^{N} \sum_{i=j}^{N} (-1)^i \binom{i}{j} \langle P_k^{(j)} (\ell_i \sigma)^{(i-j)}, P_n \rangle = \sum_{j=0}^{N} \langle P_k^{(j)} \ell_j \sigma, P_n \rangle \\ &= \sum_{j=0}^{N} \langle \sigma, P_k^{(j)} \ell_j P_n \rangle = \begin{cases} 0 & \text{if } k < n, \\ \lambda_n \langle \sigma, P_n^2 \rangle & \text{if } k = n. \end{cases}\end{aligned}$$

Hence, we have $c_k = 0$, $k < n$ and $c_n = \lambda_n$ so that $L_N(P_n) = \lambda_n P_n$.

Conversely, assume that the condition (i) holds. Multiplying $L_N P_n = \lambda_n P_n$ by P_k and applying σ we obtain

(2.20)
$$\left\langle \sigma, P_k \sum_0^N \ell_i P_n^{(i)} \right\rangle = \left\langle \sum_0^N (-1)^i (P_k \ell_i \sigma)^{(i)}, P_n \right\rangle$$
$$= \lambda_n \langle \sigma, P_k P_n \rangle = \begin{cases} 0 & \text{if } k \neq n\,, \\ \lambda_n \langle \sigma, P_n^2 \rangle & \text{if } k = n\,. \end{cases}$$

If we set $v_k := \sum_0^N (-1)^i (P_k \ell_i \sigma)^{(i)}$, then the equation (2.20) implies $\langle v_k, P_n \rangle = 0$ for $k > n$ so that by Lemma 2.3 we have

$$v_k = \sum_{j=0}^{k} \langle v_k, P_j \rangle \langle \sigma, P_j^2 \rangle^{-1} P_j(x)\sigma = \lambda_k P_k(x)\sigma, \qquad k = 0, 1, \ldots . \tag{2.21}$$

On the other hand, we have

(2.22)
$$v_k = \sum_{i=0}^{N} (-1)^i (P_k \ell_i \sigma)^{(i)} = \sum_{j=0}^{N} P_k^{(j)} \sum_{i=j}^{N} (-1)^i \binom{i}{j} (\ell_i \sigma)^{(i-j)} = \sum_{j=0}^{N} P_k^{(j)} u_j$$

where

$$u_j := \sum_{i=j}^{N} (-1)^i \binom{i}{j} (\ell_i \sigma)^{(i-j)}, \qquad j = 0, 1, \ldots, N.$$

Hence, we have from (2.21) and (2.22) that

$$v_k = \lambda_k P_k(x)\sigma = \sum_{j=0}^{N} P_k^{(j)} u_j = \sum_{j=0}^{k} P_k^{(j)} u_j, \qquad k = 0, 1, \ldots, N. \tag{2.23}$$

Finally we claim that $u_j = \ell_j(x)\sigma$, $j = 0, 1, \ldots, N$ so that the condition (2.19), i.e., (2.15) holds. For $j = 0$, $v_0 = \lambda P_0(x)\sigma = P_0 u_0$ and so

$u_0 = \lambda_0\sigma = \ell_0\sigma$. Assume that $u_j = \ell_j(x)\sigma$, $j = 0, 1, \ldots, k$ for some $k \leq N-1$. Then from (2.23) we have

$$v_{k+1} = \lambda_{k+1}P_{k+1}\sigma = \sum_{j=0}^{k} P_{k+1}^{(j)}\sigma + P_{k+1}^{(k+1)}u_{k+1}$$

and so

$$\begin{aligned} P_{k+1}^{(k+1)}u_{k+1} &= \lambda_{k+1}P_{k+1}\sigma - \sum_{j=0}^{k} P_{k+1}^{(j)}u_j \\ &= \Big(\lambda_{k+1}P_{k+1} - \sum_{j=0}^{k} P_{k+1}^{(j)}\ell_j\Big)\sigma \\ &= \Big(\sum_{j=0}^{N} \ell_j P_{k+1}^{(j)} - \sum_{j=0}^{k} P_{k+1}^{(j)}\ell_j\Big)\sigma \\ &= \ell_{k+1}P_{k+1}^{(k+1)}\sigma. \end{aligned}$$

Hence, $u_{k+1} = \ell_{k+1}(x)\sigma$ by Lemma 2.3.

Finally, the condition (v) (resp. (vi)) is just a restatement of the condition (iv) (resp. (iii)) in terms of the moments $\{\sigma_n\}_0^\infty$ of σ.

REMARK. The equivalence of two moment relations $S_k(m) = 0$ in (v) and $T_k(m) = 0$ in (vi) was first observed by L. L. Littlejohn [13] in which he gave the precise connection between them (see the equation (5.5) in [13]).

Now, Theorem 2.1 comes directly from Theorem 2.4 since a polynomial set $\{P_n(x)\}_0^\infty$ is an OPS if and only if the moments $\{\sigma_n\}_0^\infty$ of $\{P_n(x)\}_0^\infty$ satisfy the condition (i) in Theorem 2.2.

3. Applications

The condition (v) for the symmetry factor $s(x)$ in Lemma 2.2 has an analogue for the canonical moment functional σ of an OPS satisfying the differential equation (1.2). To be precise we have:

THEOREM 3.1. *Let $\{P_n(x)\}_0^\infty$ and σ be the same as in Theorem 2.4. Then any one of the equivalent conditions* (i) $\sim$ (vi) *in Theorem 2.4 is also equivalent to the following :*

(vii) *There are $r+1$ moment functionals $\{\tau_i\}_0^r$ such that $\tau_r \not\equiv 0$ and*

$$L_{2r}(\phi)\sigma = \sum_0^r (-1)^i [\phi^{(i)}\tau_i]^{(i)} \tag{3.1}$$

for every polynomial $\phi(x)$.

(viii) *There are r moment functionals $\{\tau_i\}_1^r$ such that $\tau_r \not\equiv 0$ and*

$$\sum_{i=1}^r \langle \tau_i, P_m^{(i)} P_n^{(i)} \rangle = 0 \tag{3.2}$$

for $m \neq n$, m and $n = 1, 2, \ldots$.

Furthermore, $\{P_n(x)\}_0^\infty$ is symmetric if and only if $\{\tau_i\}_0^r$ can be chosen to be symmetric.

The proof of Theorem 3.1 is straight forward application of Lemma 2.2 and Theorem 2.4 and we refer the details to [10].

In particular when $r = 1$, we get the following generalization of Hahn's characterization of classical orthogonal polynomials.

COROLLARY 3.2. *An OPS $\{P_n(x)\}_0^\infty$ is classical, that is, they satisfy the differential equation* (1.2) *with $N = 2$ if and only if $\{P_n'(x)\}_1^\infty$ is a weak orthogonal polynomial set in the sense that there is a nontrivial moment functional τ with*

$$\langle \tau, P_m' P_n' \rangle = 0 \tag{3.3}$$

for $m \neq n$, m and $n = 1, 2, \ldots$.

The rising interest in OPS's satisfying a differential equation (1.2) lies partly in the fact that they provide good examples of realizing the general Weyl-Titchmarsh theory of higher order differential equations (see [2, 3, 4, 5]). In this sense, the equivalence of conditions (i) and (ii) in Theorem 2.4 is quite interesting.

References

1. S. Bochner, *Über Strum-Liouvillesche Polynomsysteme*, Math. Z. **29** (1929), 730–736.
2. W. N. Everitt and L. L. Littlejohn, *Orthogonal polynomials and spectral theory: a survey*, Orthog. Polyn. & their Appl. IMACS Vol. 9, Baltzer (1991), 21–55.
3. A. M. Krall and L. L. Littlejohn, *A singular sixth order differential equation with orthogonal polynomial eigenfunctions*, Lecture Notes in Math., vol. 964, Springer-Verlag, 1983, pp. 435–444.
4. ______, *Orthogonal polynomials and singular Sturm-Liouville systems I*, Rocky Mountain J. Math. **16** (1986), 435–479.
5. ______, *Orthogonal polynomials and higher order singular Sturm-Liouville systems*, Acta Appl. Math. **17** (1989), 99–170.
6. H. L. Krall, *Certain differential equations for Tchebychev polynomials*, Duke Math. J. **4** (1938), 705–719.
7. ______, *On orthogonal polynomials satisfying a certain fourth order differential equation*, Penn. State College Studies, No. 6, 1940.
8. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1949), 100–115.
9. H. L. Krall and I. M. Sheffer, *A characterization of orthogonal polynomials*, J. Math. Anal. Appl. **8** (1964), 232–244.
10. K. H. Kown, J. K. Lee, L. L. Littlejohn, and B. H. Yoo, *Characterizations of Bochner-Krall orthogonal polynomials*, preprint.
11. K. H. Kwon and S. S. Han, *Spectral analysis of Bessel polynomials in Krein space*, Quaestiones Math. **14** (1991), 327–335.
12. L. L. Littlejohn, *Symmetry factors for differential equations*, Amer. Math. Monthly **7** (1983), 462–464.
13. ______, *On the classification of differential equations having orthogonal polynomial solutions*, Ann. Mat. Pura Appl. **4** (1984), 35–53.
14. L. L. Littlejohn and D. Race, *Symmetric and symmetrizable differential expressions*, Proc. London Math. Soc. **3** (1990), 344–364.
15. I. M. Sheffer, *On the properties of polynomials satisfying a linear differential equation* I, Trans. Amer. Math. Soc. **35** (1933), 184–214.

Department of Mathematics, KAIST
Taejon, Korea