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# SMALL LIPSCHITZ MAPS OF CONTINUOUS FUNCTION SPACES

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## I. Introduction

For a locally compact Hausdorff space $X$ and a Banach space $G$ we denote by $C_0(X,G)$ the space of $G$-valued continuous functions on $X$ which vanish at infinity, provided with the supremum norm.

The classical Banach-Stone theorem states that if $C_0(X,\mathbb{C})$ and $C_0(Y,\mathbb{C})$ are isometrically isomorphic, then $X$ and $Y$ are homeomorphic. Amir [1] and Cambern [3] proved that this result is stable: if there is a linear homeomorphism $T$ from $C_0(X,\mathbb{C})$ onto $C_0(Y,\mathbb{C})$ with $\|T\|\|T^{-1}\| < 2$ then $C_0(X,\mathbb{C})$ and $C_0(Y,\mathbb{C})$ are actually isometrically isomorphic. Also Cambern [4], Jarosz [10] and many authors [2, 6, 7, 8, 9, 12] have considered problems of determining geometric properties of $G$ which allow generalizations of this theorem to the spaces of vector-valued functions $C_0(X,G)$. In [11], Jarosz proved a generalization dealing with scalar-valued functions, but replaced isometries by nonlinear isomorphisms.

In this paper, we consider a generalization dealing with the Banach algebra valued functions. Our result is similar to Theorem 1 in [11]. But it is a nonlinear generalization of the Banach-Stone theorem on spaces of a Banach algebra-valued continuous functions.

## II. Small Lipschitz maps of $C_0(X,G)$

Let $G$ be a unital Banach algebra satisfying the following conditions;

(1) There is $c>1$ such that for every $a,b$ in $G$ with $\|a\|,\|b\|\geq 1$,

$$\max\{\|a+b\|,\|a-b\|\}\geq c$$

(2) For every $a\in G$, $a$ is left invertible.

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For example, $\mathbb{R}^4$ has these conditions with a norm

$$\|(a,b,c,d)\| = \sqrt{a^2+b^2+c^2+d^2}$$

and a multiplication

$$\begin{aligned}(a_1,b_1,c_1,d_1)&(a_2,b_2,c_2,d_2)\\ &= (a_1a_2-b_1b_2-c_1c_2-d_1d_2, a_1b_2+b_1a_2+c_1d_2-c_2d_1,\\ &\qquad a_1c_2+c_1a_2-b_1d_2+d_1b_2, a_1d_2+d_1a_2+b_1c_2-c_1b_2)\end{aligned}$$

For a closed subspace $A$ of $C_0(X,G)$, we denote by $ChA$ the set of all points $x_0 \in X$ such that for any $\varepsilon > 0$ and any neighborhood $U$ of $x_0 \in X$ there is an $f$ in $A$ with $\|f\| = 1$, $f(x_0) = e$ ($e$ is unit in $G$) and $\|f(x)\| < \varepsilon$ for $x \in X - U$. $A$ is called an almost extremely regular subspace of $C_0(X,G)$ if $ChA$ is dense in $X$ and is called extremely regular if $ChA = X$.

For $x_0$ in $X$ a net $(f_\alpha)_{\alpha\in\Lambda} \subset C_0(X,G)$ is called peaking at $x_0$, if

(i) $\forall\alpha \in \Lambda$, $\|f_\alpha\| = 1$, $f_\alpha(x_0) = e$ and

(ii) $f_\alpha \to 0$ uniformly off any neighborhood of $x_0$

We denote by $P_A(x_0)$ the set of all nets $(f_\alpha)_{\alpha\in\Lambda}$ in $A$ such that $(f_\alpha)_{\alpha\in\Lambda}$ peaks at $x_0$. We define $(af)(x) = a\cdot f(x)$ for every $a$ in $G$ and $f$ in $A$. A map $T$ is called $\varepsilon$-bi-Lipschitz from a Banach space $E$ into a Banach space $F$ if

$$(1-\varepsilon)\|f-g\| \le \|Tf-Tg\| \le (1+\varepsilon)\|f-g\|.$$

To prove Theorem 1, we need the following proposition [5].

**Proposition A.** *Let $T$ be an $\varepsilon$-bi-Lipschitz map from a Banach space $E$ onto a Banach space $F$, with $\varepsilon < \frac{1}{3}$ and $T0 = 0$. Then for any $f, g$ in $E$ with $\|f\|, \|g\| \le 2$ we have*

$$\|(Tf+Ty)-T(f+g)\| \le \varepsilon',$$

$$\|f+g\| - \varepsilon' \le \|Tf+Tg\| \le \|f+g\| + \varepsilon'$$

*where $\varepsilon' = 100\varepsilon^{\frac{1}{10}}$.*

THEOREM 1. *Let* $X, Y$ *be locally compact Hausdorff spaces,* $A$ *an almost extremely regular subspace of* $C_0(X, G)$ *and* $B$ *an extremely regular subspace of* $C_0(Y, G)$.

*Assume there is an* $\varepsilon$*-bi-Lipschitz map* $T$ *from* $A$ *onto* $B$ *with* $\varepsilon \leq \varepsilon_0$ *such that* $T(af) = a(Tf)$ *for* $f \in C_0(X, G)$ *and* $a \in G$. *Then there is a homeomorphism* $\phi$ *from* $X$ *onto* $Y$ *and*

$$| \, \|Tf(\phi(x))\| - \|f(x)\| \, | \leq c(\varepsilon)\|f\|, \quad (f \in A, \quad x \in X)$$

*where* $\varepsilon_0$ *is an absolute constant and* $c(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

*Proof.* We divide the proof into a number of simple steps and at various points of the proof we use inequalities involving $\varepsilon$ which are valid only if $\varepsilon$ is sufficiently small. In these circumstances we will merely assume that $\varepsilon$ is near zero. These assumptions are the source of $\varepsilon_0$ and $\varepsilon' = 100\varepsilon^{\frac{1}{10}}$. Fix $M \geq 0$. For any $x_0 \in X$ we set

$$S_{x_0} = \{y \in Y : \exists (f_\alpha)_{\alpha \in \Lambda} \in P_A(x_0), \quad \exists \, (y_\alpha)_{\alpha \in \Lambda} \subset Y, \quad y_\alpha \to y \text{ and } \quad \forall \alpha \in \Lambda, \quad \|Tf_\alpha(y_\alpha)\| \geq M\}.$$

STEP 1. *If* $M \leq 1 - \varepsilon - \varepsilon'$, *then for any* $x_0 \in ChA$ *we have* $S_{x_0} \neq \emptyset$.

*Proof of Step 1.* For any $\varepsilon > 0$ and any neighborhood $U_\beta$ of $x_0$ there is $f_\beta \in A$ with $\|f_\beta\| = 1$, $f_\beta(x_0) = e$ and $\|f_\beta(x)\| < \varepsilon$ for $x \in X - U_\beta$. By Proposition A,

$$M \leq 1 - \varepsilon - \varepsilon' \leq \|f_\beta\| - \varepsilon' \leq \|Tf_\beta\| = \sup_{y \in Y} \|Tf_\beta(y)\|.$$

If $Y$ is compact, there is $y_\beta \in Y$ such that $\|Tf_\beta(y_\beta)\| \geq M$. Then $(y_\beta)$ has a convergent subnet. Thus $S_{x_0} \neq \emptyset$.

To consider the general case, let $(f_\alpha)_{\alpha \in \Lambda} \in P_A(x_0)$. Fix $\alpha_0 \in \Lambda$. Then for any $\alpha \in \Lambda$,

$$\|f_\alpha + f_{\alpha_0}\| = 2, \quad 2 - \varepsilon' \leq \|Tf_\alpha + Tf_{\alpha_0}\|, \quad \|Tf_\alpha\| \leq 1 + \varepsilon.$$

Thus for any $y$ in $Y$ with $\|Tf_\alpha(b)\| < 1 - \varepsilon - \varepsilon'$

$$\|Tf_\alpha(y) + Tf_{\alpha_0}(y)\| < 2 - \varepsilon'$$

and so

$$\sup\{\|Tf_\alpha(y)\| : y \in Y,\ \|Tf_{\alpha_0}(b)\| \geq 1-\varepsilon-\varepsilon'\} \geq 1-\varepsilon-\varepsilon' \geq M.$$

Since $\{y \in Y \| \|Tf_{\alpha_0}(y)\| \geq 1-\varepsilon-\varepsilon'\}$ is compact, we can choose a net $(y_\alpha)_{\alpha\in\Lambda}$ in $Y$ such that $y_\alpha \to y$ and for any $\alpha \in \Lambda$, $\|Tf_\alpha(y)\| \geq M$. Therefore $S_{x_0} \neq \emptyset$.

STEP 2. *If $M \geq \frac{1}{2} + 2\varepsilon'$, then for any $x_0 \in ChA$ the set $S_{x_0}$ has at most one point.*

*Proof of Step 2.* Assume $y^1, y^2$ are two distinct points of $S_{x_0}$ and let $(f^i_\alpha)_{\alpha\in\Lambda_i} \in P_A(x_0)$, $(y^i_\alpha)_{\alpha\in\Lambda_i} \subset Y$ $(i = 1,2)$ be the corresponding nets given by the definition of $S_{x_0}$. Without loss of generality we can assume that

$$(Tf^i_\alpha)(y^i_\alpha) \xrightarrow{\alpha\in\Lambda_i} a^i \in G$$

with $\|a^i\| \geq M$ for $i = 1,2$. Since $B$ is extremely regular there are $g_1, g_2 \in B$ such that

$$g_i(y^i) = \frac{a^i}{\|a^i\|} \text{ for } i = 1,2, \qquad \sup_{y\in Y} |\ \|g_1(y)\| + \|g_2(y)\|\ | < 1 + (c-1)\varepsilon'.$$

We have

$$\liminf_\alpha \|Tf^i_\alpha + y_i\| \geq 1 + M,\ i = 1,2.$$

By Proposition A,

$$\liminf_\alpha \|f^i_\alpha + T^{-1}g_i\| \geq 1 + M - \varepsilon', \quad i = 1,2.$$

Since $(f^i_\alpha)$ is a peaking sequence,

$$\|e + T^{-1}g_i(x_0)\| \geq 1 + M - \varepsilon', \quad i = 1,2$$

and so

$$\|T^{-1}g_i(x_0)\| \geq M - \varepsilon', \quad i = 1,2.$$

By the definition of $G$ and Proposition A, we get

$$\max(\|T^{-1}g_1 + T^{-1}g_2\|, \|T^{-1}g_1 - T^{-1}g_2\|) \geq c(M - \varepsilon'),$$

$$\max(\|g_1+g_2\|, \|g_1-g_2\|) \geq c(M-\varepsilon') - \varepsilon' > 1+(c-1)\varepsilon',$$

which contradicts the assumption

$$\sup_{y\in Y} |\ \|g_1(y)\| + \|g_2(y)\|\ | < 1+(c-1)\varepsilon'.$$

In the remaining part of the proof we assume that

$$\frac{1}{c} + 2\varepsilon' < M = 1-\varepsilon-\varepsilon'$$

and we define $\phi : ChA \to Y$ by $\{\phi(x)\} = S_x$.
We denote $\partial G_1 = \{a \in G|\ \|a\| = 1\}$.

STEP 3. *Fix* $x_0 \in ChA$ *and* $f_0 \in A$ *with* $\|f_0\| = 1, f_0(x_0) = e$. *For any* $a \in \partial G_1$ *we define* $\chi(a) = T(af_0)(\phi(x_0))$. *Then*
(*i*) *For every* $a \in \partial G_1$ $\|\chi(a)\| \geq M$,
(*ii*) $\{\frac{\chi(a)}{\|\chi(a)\|} : a \in \partial G_1\} = \partial G_1$.

*Proof of Step 3.* (i) Choose a net $(f_\alpha)_{\alpha\in\Lambda}$ in $P_A(x_0)$. As in Step 1, we have $\|af_\alpha + af_0\| = 2$ for every $\alpha \in \Lambda$. Hence

$$\|T(af_\alpha) + T(af_0)\| \geq 2-\varepsilon', \quad (\alpha \in \Lambda)$$

and therefore

$$\sup\{\|T(af_0)(y)\| : y \in Y, \quad \|T(af_\alpha)(y)\| \geq 1-\varepsilon-\varepsilon'\} \geq 1-\varepsilon-\varepsilon'.$$

It follows that there is a net $(y_\alpha)$ such that

$$\|T(af_\alpha)(y_\alpha)\| \ \geq 1-\varepsilon-\varepsilon', \qquad \|T(af_0)(y_\alpha)\| \geq 1-\varepsilon-\varepsilon'.$$

By the definition of $S_{x_0}$, $\sup\{\|T(af_0)(y)\| : y \in S_{x_0}\} \geq 1-\varepsilon'-\varepsilon$. Since $S_{x_0} = \{\phi(x_0)\}$, we get $\|\chi(a)\| \geq M$.
(ii) If $b \in \partial G_1$, let $a = \frac{b\cdot(Tf_0)(\phi(x_0))^{-1}}{\|b\cdot(Tf_0)(\phi(x_0))^{-1}\|} \in \partial G_1$. Then

$$\frac{\chi(a)}{\|\chi(a)\|} = \frac{a\cdot Tf_0(\phi(x_0))}{\|a\cdot Tf_0(\phi(x_0))\|} = b.$$

So it holds.

STEP 4. *For every* $f_0 \in A$ *with* $\|f_0\| \leq 1+\varepsilon'$ *and* $x \in ChA$ *we have*

$$\|Tf_0(\phi(x))\| \leq \|f_0(x)\| + \varepsilon + 3\varepsilon'.$$

*Proof of Step 4.* We have

$$\begin{aligned}&1 + \|f_0(x)\| + \varepsilon' \\ &\qquad \geq \inf\{\sup\{\|f_0 + af\| : a \in \partial G_1\} : f \in A, \|f\| = 1, f(x) = e\}\end{aligned}$$

and by Step 3,

$$\begin{aligned}&M + \|Tf_0(\phi(x))\| \\ &\quad \leq \|T(af)(\phi(x))\| + \|Tf_0(\phi(x))\| \\ &\quad \leq \inf\{\sup\{\|Tf_0 + T(af)\| : a \in \partial G_1\} : f \in A, \|f\| = 1, f(x) = e\} \\ &\quad \leq 1 + \|f_0(x)\| + 2\varepsilon'.\end{aligned}$$

Hence by the assumption that $M = 1 - \varepsilon - \varepsilon'$,

$$\|Tf_0(\phi(x))\| \leq \|f_0(x)\| + \varepsilon + 3\varepsilon'.$$

STEP 5. *Fix* $x_0 \in ChA$ *and* $y \in B$ *with* $g(\phi(x_0)) = e$ *and* $\|y\| = 1$. *For any* $a \in \partial G_1$ *we define* $K(a) = T^{-1}(ag)(x_0)$. *Then we have*

(i) *For any* $a \in \partial G_1, \|K(a)\| \geq 1 - 3\varepsilon' - \varepsilon$,

(ii) $\{\frac{K(a)}{\|K(a)\|} : a \in \partial G_1\} = \partial G_1$.

*Proof of Step 5.* It is an immediate consequence of Step 4.

STEP 6. *For any* $f_0 \in A$ *with* $\|f_0\| \leq 1 + 2\varepsilon$ *and* $x_0 \in ChA$ *we have*

$$\|Tf_0(\phi(x_0))\| \geq \|f_0(x_0)\| - 5\varepsilon'.$$

*Proof of Step 6.* Since $B$ is extremely regular, there is a $g \in B$ such that $\|g\| = 1$, $g(\phi(x_0)) = e$ and

$$\|Tf_0(y)\| + \|g(y)\| \leq 1 + \|Tf_0(\phi(x_0))\| + 4\varepsilon, \quad y \in Y.$$

By Step 5, there is a $a \in \partial G_1$ such that

$$\|f_0 + T^{-1}(ag)\| \geq \|f_0(x_0)\| + 1 - 3\varepsilon' - \varepsilon.$$

Hence by Proposition A,

$$\|Tf_0 + ag\| \geq \|f_0(x_0)\| + (1 - 3\varepsilon' - \varepsilon) - \varepsilon'.$$

Therefore

$$\|Tf_0(\phi(x_0))\| \geq \|Tf_0 + ag\| - 1 + 4\varepsilon \geq \|f_0(x_0)\| - 5\varepsilon'.$$

Following steps are immediate consequences of proofs of Theorem 1 in [11].

STEP 7. *$\phi$ can be extended to a continuous function from $X$ into $Y$. We denote the extended function by the same symbol.*

STEP 8. *If $X$ is non-compact, then neither is $Y$ and $\phi$ can be extended to a continuous map from $X^* = X \cup \{\infty\}$ (one point compactification of $X$) into $Y^* = Y \cup \{\infty\}$.*

STEP 9. *$\phi$ maps $X$ onto $Y$.*

STEP 10. *$\phi$ is injective.*

Now by Steps 7-10, $\phi$ is a homeomorphism from $X$ onto $Y$. And from Step 4 and Step 6 with $c(\varepsilon) = 5\varepsilon' = 500\varepsilon^{\frac{1}{10}}$,

$$|\ \|(Tf)\phi(x)\| - \|f(x)\|\ | \leq c(\varepsilon)\|f\| \quad (f \in A,\ x \in X).$$

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