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HEREDITARY PROPERTIES OF MINIMAL ISOMETRIC DILATIONS AND MINIMAL COISOMETRIC EXTENSIONS*

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The notation and terminology employed herein agree with those in [1], [3], and [9]. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Throughout this paper, we write $\mathbf{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2. For T in $\mathcal{L}(\mathcal{H})$ we let $\mathrm{Lat}(T)$ denote the lattice of subspaces invariant for T. If $\mathcal{M} \in \mathrm{Lat}(T)$ we write $T|\mathcal{M}$ for the restriction of T to $\mathcal{M}$. A subspace $\mathcal{K}$ is *semi-invariant* for T if there exist $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$. If $\mathcal{K}$ is semi-invariant for T, we write

$$T_{\mathcal{K}} = P_{\mathcal{K}} T|\mathcal{K} \tag{1}$$

for the *compression* of T to $\mathcal{K}$, where $P_{\mathcal{K}}$ is the orthogonal projection whose range is $\mathcal{K}$. Note that by (1) we have

$$T \cong \begin{pmatrix} * & * & * \\ 0 & \tilde{T} & * \\ 0 & 0 & * \end{pmatrix} \tag{2}$$

relative to the decomposition $\mathcal{N} \oplus \mathcal{K} \oplus \mathcal{M}^{\perp}$, where $\tilde{T} = T_{\mathcal{K}}$. We say that an operator B is an *extension* of T if there exists $\mathcal{M}$ in $\mathrm{Lat}(B)$ such that $T = B|\mathcal{M}$; B is a *dilation* of T if there is a semi-invariant subspace $\mathcal{K}$ for B such that $T = B_{\mathcal{K}}$.

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Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$. Then it follows from [9, Theorem I.4.2] that there exist a Hilbert space $\mathcal{K}$ and an isometry B_T in $\mathcal{L}(\mathcal{K})$ such that $\mathcal{K} \supset \mathcal{H}$ and

$$(3) \qquad B_T \cong \begin{pmatrix} * & * \\ 0 & T \end{pmatrix}$$

relative to the decomposition $(\mathcal{K}\ominus\mathcal{H})\oplus\mathcal{H}$. Furthermore, we may suppose B_T to be minimal, which means that for subspaces $\mathcal{M}$ of $\mathcal{K}$,

$$\{(\mathcal{H} \subset \mathcal{M} \subset \mathcal{K}) \wedge (B_T\mathcal{M} \subset \mathcal{M}) \wedge (B_T|\mathcal{M} \text{ is an isometry})\} \Rightarrow \mathcal{M} = \mathcal{K}.$$

By (3) and the above statements, it is easy to show that B_T^* is a minimal coisometric extension of T. Of course, this minimality means that for subspaces $\mathcal{M}'$ of $\mathcal{K}$,

$$\{(\mathcal{H} \subset \mathcal{M}' \subset \mathcal{K})\wedge(B_T^*\mathcal{M}' \subset \mathcal{M}')\wedge(B_T^*|\mathcal{M}' \text{ is a coisometry})\}\Rightarrow \mathcal{M}' = \mathcal{K}.$$

A contraction operator $T \in \mathcal{L}(\mathcal{H})$ is *absolutely continuous* if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0) (cf. [2]). We write $\mathbf{D}$ for the open unit disc in the complex space $\mathbf{C}$ and $\mathbf{T}$ for the boundary of $\mathbf{D}$. Let $\mathcal{C}_1(\mathcal{H})$ be the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm. Then the dual algebra $\mathcal{A}$ can be identified with the dual space of $Q_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing

$$(4) \qquad \langle T, [L]_{\mathcal{A}}\rangle = \operatorname{tr}(TL), \quad T \in \mathcal{A}, \quad [L]_{\mathcal{A}} \in Q_{\mathcal{A}}.$$

We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. The space L^p, $1 \le p \le \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\mathbf{T}$. The space H^p, $1 \le p \le \infty$, is the usual Hardy space on $\mathbf{T}$. It is well-known (cf. [6]) that the space H^∞ is the dual space of L^1/H_0^1, where

$$(5) \quad H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \quad \text{for} \quad n = 0, 1, 2, \dots\}$$

and the duality is given by the pairing

$$(6) \qquad \langle f, [g] \rangle = \int fg\, dm, \quad f \in H^\infty,\ [g] \in L^1/H_0^1.$$

Recall that a ***dual algebra*** is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ coincides with the weak*–topology on $\mathcal{L}(\mathcal{H})$ (cf. [5]). For $T \in \mathcal{L}(\mathcal{H})$ we denote by $\mathcal{A}_T$ the dual algebra generated by T.

The following theorem gives a good relationship between Hardy space H^∞ and a dual algebra generated by an absolutely continuous contraction.

THEOREM 1 [2, THEOREM 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism Φ_T : $H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*

(a) $\Phi_T(1) = 1,\ \Phi_T(\xi) = T$,

(b) $\|\Phi_T(f)\| \le \|f\|_\infty,\ f \in H^\infty$,

(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*

(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*

(e) *there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L^1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*

(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L^1/H_0^1.*

Recall that $T \in C_{\cdot 0}$ if $\|T^{*n}x\| \to 0$ for any $x \in \mathcal{H}$. We say $T \in C_{0\cdot}$ if $T^* \in C_{\cdot 0}$. And we denote that $C_{00} = C_{\cdot 0} \cap C_{0\cdot}$. And recall (cf. [1]) that a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of class C_0 if there exists $u \in H^\infty$, $u \not\equiv 0$, such that the functional calculus $u(T) = 0$ in Theorem 2.1. It follows from [1, Corollary II.4.2] that $C_0 \subset C_{00}$.

Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let $B_T \in \mathcal{L}(\mathcal{K})$ be a minimal isometric dilation of T. Then it follows from the Wold decomposition theorem (cf. [9, Theorem I.1.1]) that

$$(7) \qquad B_T = U_T \oplus R_T,$$

where $U_T \in \mathcal{L}(\mathcal{U}_T)$ is a (forward) unilateral shift operator of some multiplicity and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is a unitary operator. Note that by (3)

$$(8) \qquad B_T^* \cong \begin{pmatrix} * & 0 \\ * & T^* \end{pmatrix}$$

relative to the decomposition $(\mathcal{K} \ominus \mathcal{H}) \oplus \mathcal{H}$. Moreover, by (7) and (8) it is obvious that

$$(9) \qquad B_T^* = U_T^* \oplus R_T^*$$

is a minimal coisometric extension of T^*.

Suppose $T \in \mathcal{L}(\mathcal{H})$ has a non-zero semi-invariant subspace $\mathcal{M}$ (i.e., $\mathcal{M} \neq (0)$). Then by (2) a minimal isometric dilation $B_T \in \mathcal{L}(\mathcal{K})$ is an isometric dilation of $T_{\mathcal{M}}$. Hence $T_{\mathcal{M}}$ has a minimal isometric dilation $B_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ such that $\mathcal{M} \subset \widetilde{\mathcal{K}} \subset \mathcal{K}$ with $\widetilde{\mathcal{K}}$ in $\mathrm{Lat}(B_T)$ and $B_{T_{\mathcal{M}}} = B_T|\widetilde{\mathcal{K}}$.

Now we are ready to define hereditary properties of minimal isometric dilations and minimal coisometric extensions.

DEFINITION 2. Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$.

(a) T has property (H_1) if, for any non-zero semi-invariant subspace $\mathcal{M}$ for T, the minimal isometric dilation $B_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B_T|\widetilde{\mathcal{K}}$ of the minimal isometric dilation B_T of T with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B_T)$ satisfies $\mathcal{U}_{T_{\mathcal{M}}} \subset \mathcal{U}_T$.

(a*) T has property (H_1^*) if, for any non-zero invariant subspace $\mathcal{M}$ for T, the minimal coisometric extension $B'_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B'_T|\widetilde{\mathcal{K}}$ of the minimal coisometric extension B'_T of T with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B'_T)$ satisfies $\mathcal{U}_{T_{\mathcal{M}}} \subset \mathcal{U}_T$.

(b) T has property (H_2) if, for any non-zero semi-invariant subspace $\mathcal{M}$ for T, the minimal isometric dilation $B_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B_T|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B'_T)$ satisfies $\mathcal{R}_{T_{\mathcal{M}}} \subset \mathcal{R}_T$.

(b*) T has property (H_2^*) if, for any non-zero invariant subspace $\mathcal{M}$ for T, the minimal coisometric extension $B'_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B'_T|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B'_T)$ satisfies $\mathcal{R}_{T_{\mathcal{M}}} \subset \mathcal{R}_T$.

REMARK 3. According to the notation of Definition 2(a*) and (b*) it is not difficult to show that if $\mathcal{M}$ is a semi-invariant subspace for T and the minimal coisometric extension of $T_{\mathcal{M}}$ is obtained as a restriction $B'_T|\widetilde{\mathcal{K}}$ for some $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B'_T)$, then $\mathcal{M} \in \mathrm{Lat}(T)$.

Note from Definition 2 that (a) and (b) are related with (a*) and (b*) as dual properties, respectively. But it is interesting to see that by some examples and Theorem 7 there are some gaps between (a) and (a*).

LEMMA 4. *If $T \in C_{\cdot 0}$, then T has property (H_1).*

Proof. Let $\mathcal{M}$ be a non-zero semi-invariant subspace for T. Then it is not difficult to show that $T_{\mathcal{M}} \in C_{\cdot 0}$. Hence by [1, Corollary I.2.11], B_T is a unilateral shift operator of some multiplicity and $B_{T_{\mathcal{M}}}$ is a unilateral shift operator of some multiplicity. Therefore T has property (H_1).

EXAMPLE 5. If $U \in \mathcal{L}(\mathcal{H})$ is a unilateral shift operator of multiplicity one, then by Lemma 4 U has property (H_1). Furthermore, the fact that U has property (H_1^*) will be proved in Theorem 7. But its adjoint operator U^* doesn't have property (H_1). (Indeed, there is a nontrivial invariant subspace $\mathcal{M}$ for U^* (i.e., $(0) \neq \mathcal{M} \neq \mathcal{H}$). If we denote $\tilde{U} = U^*|\mathcal{M}$, then $\tilde{U} \in C_0 \subset C_{\cdot 0}$ (cf. [1] or [8, Theorem 1]). Hence by [1, Corollary I.2.11] $B_{\tilde{U}}$ is a unilateral shift operator of multiplicity one. But B_{U^*} is a bilateral shift operator of multiplicity one. Therefore U^* doesn't have property (H_1).)

By the above example, in general, the fact that an operator T has property (H_1^*) doesn't always mean that T^* has property (H_1). The following proposition should be compared with Example 5.

PROPOSITION 6. *If $U \in \mathcal{L}(\mathcal{H})$ is a unilateral shift of multiplicity one and $\mathcal{K}$ is a nontrivial semi-invariant subspace for U^*, then $U^*_{\mathcal{K}}$ has property (H_1).*

Proof. Let $\mathcal{K}$ be a nontrivial semi-invariant subspace for U^*. Then there exist $\mathcal{M}$, $\mathcal{N} \in \mathrm{Lat}(U^*)$ with $\mathcal{M} \supset \mathcal{N}$ such that

$$U \cong \begin{pmatrix} * & 0 & 0 \\ * & \tilde{U}^* & 0 \\ * & * & * \end{pmatrix} \tag{10}$$

relative to the decomposition $\mathcal{N} \oplus \mathcal{K} \oplus \mathcal{M}^{\perp}$, where $\tilde{U} = U_{\mathcal{K}}^{*}$. Since $\mathcal{K} \neq (0) \neq \mathcal{H}$, by [4, Proposition I.7.13] we have

$$U \cong \begin{pmatrix} \tilde{U}^* & 0 \\ * & * \end{pmatrix} \tag{11}$$

relative to the decomposition $\mathcal{K} \oplus \mathcal{M}^{\perp}$. Hence $\tilde{U}^* \in C_0 \subset C_{00}$ and $\tilde{U}^* \in C_{\cdot 0}$. By Lemma 4, we have this proposition.

THEOREM 7. *Every contraction operator in $\mathcal{L}(\mathcal{H})$ has*
(a) *property (H_1^*),*
(b) *property (H_2) and*
(c) *property (H_2^*).*

Proof. (a) Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let $\mathcal{M}$ be a non-zero invariant subspace for T. Let $B'_T \in \mathcal{L}(\mathcal{K})$ and let $B'_{\tilde{T}} \in \mathcal{L}(\tilde{\mathcal{K}})$ be minimal coisometric extensions of T and $\tilde{T}$, respectively, such that $B'_T|\tilde{\mathcal{K}} = B'_{\tilde{T}}$, where $\tilde{\mathcal{K}} \in \mathrm{Lat}(B'_T)$. Then we have

$$\begin{aligned} B'_T &= U_T^* \oplus R_T^* \in \mathcal{L}(\mathcal{U}_T \oplus \mathcal{R}_T) \\ &\cong \begin{pmatrix} \tilde{T} & * \\ 0 & * \end{pmatrix} \end{aligned} \tag{12}$$

relative to the decomposition $\mathcal{M} \oplus (\mathcal{K} \ominus \mathcal{M})$, and

$$\begin{aligned} B'_{\tilde{T}} &= U_{\tilde{T}}^* \oplus R_{\tilde{T}}^* \in \mathcal{L}(U_{\tilde{T}} \oplus \mathcal{R}_{\tilde{T}}) \\ &\cong \begin{pmatrix} \tilde{T} & * \\ 0 & * \end{pmatrix} \end{aligned} \tag{13}$$

relative to the decomposition $\mathcal{M} \oplus (\tilde{\mathcal{K}} \ominus \mathcal{M})$. Now we shall claim that $\mathcal{U}_{\tilde{T}} \subset \mathcal{U}_T$. To do so, let $x = s \oplus r \in \mathcal{U}_T \oplus \mathcal{R}_T$. Since

$$B'_{\tilde{T}} = B'_T|\tilde{\mathcal{K}}, \tag{14}$$

we have that

$$\begin{aligned} \|U_{\tilde{T}}^{*n} x\|^2 &= \|B_{\tilde{T}}^{\prime n} x\|^2 = \|B_T^{\prime n} x\|^2 \\ &= \|(U_T^{*n} \oplus R_T^{*n})(s \oplus r)\|^2 \\ &= \|U_T^{*n} s\|^2 + \|R_T^{*n} r\|^2 \\ &= \|U_T^{*n} s\|^2 + \|r\|^2. \end{aligned} \tag{15}$$

Letting $n \to \infty$ on the equation (15), we have that $\|r\| = 0$. So $x \in \mathcal{U}_T$. This proves that $\mathcal{U}_{\tilde{T}} \subset \mathcal{U}_T$.

(c) Using notation in the proof of (a), we shall show that $\mathcal{R}_{\tilde{T}} \subset \mathcal{R}_T$. Let $x \in \mathcal{R}_{\tilde{T}}$ and let $x = s \oplus r \in \mathcal{U}_T \oplus \mathcal{R}_T$. Then we have

$$\begin{aligned} \|s\|^2 + \|r\|^2 &= \|x\|^2 = \|R_{\tilde{T}}^{*n} x\|^2 = \|B_{\tilde{T}}'^{\,n} x\|^2 = \|B_T'^{\,n} x\|^2 \\ &= \|U_T^{*n} s\|^2 + \|R_T^{*n} r\|^2 = \|U_T^{*n} s\|^2 + \|r\|^2 \end{aligned} \tag{16}$$

for any $n \in \mathbf{N}$. Since $\|U_T^{*n} s\| \to 0$, $s = 0$. This proves that $\mathcal{R}_{\tilde{T}} \subset \mathcal{R}_T$.

(b) Let $\mathcal{M}$ be a non-zero semi-invariant subspace for T. Then there exist $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(T)$ with $\mathcal{M}_1 \supset \mathcal{M}_2$ such that $\mathcal{M} = \mathcal{M}_1 \ominus \mathcal{M}_2$. Furthermore, we have

$$T \cong \begin{pmatrix} * & * & * \\ 0 & \tilde{T} & * \\ 0 & 0 & * \end{pmatrix} \tag{17}$$

relative to the decomposition $\mathcal{M}_2 \oplus \mathcal{M} \oplus \mathcal{M}_1^{\perp}$, where $\tilde{T} = T_{\mathcal{M}}$. Let $B_T \in \mathcal{L}(\mathcal{K})$ and $B_{\tilde{T}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ be minimal isometric dilations of T and $\tilde{T}$, respectively, such that $B_T|\mathcal{K} = B_{\tilde{T}}$ and $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B_T)$. By (7), we have that

$$\begin{aligned} B_T &= U_T \oplus R_T \in \mathcal{L}(\mathcal{U}_T \oplus \mathcal{R}_T) \\ &\cong \begin{pmatrix} * & * & * \\ 0 & \tilde{T} & * \\ 0 & 0 & * \end{pmatrix} \end{aligned} \tag{18}$$

relative to the decomposition $((\widetilde{\mathcal{K}} \ominus \mathcal{H}) \oplus \mathcal{M}_2) \oplus \mathcal{M} \oplus \mathcal{M}_1^{\perp}$, and

$$\begin{aligned} B_{\tilde{T}} &= U_{\tilde{T}} \oplus R_{\tilde{T}} \in \mathcal{L}(\mathcal{U}_{\tilde{T}} \oplus \mathcal{R}_{\tilde{T}}) \\ &\cong \begin{pmatrix} * & * \\ 0 & \tilde{T} \end{pmatrix} \end{aligned} \tag{19}$$

relative to the decomposition $(\widetilde{\mathcal{K}} \ominus \mathcal{M}) \oplus \mathcal{M}$. Now we shall claim that $\mathcal{R}_{\tilde{T}} \subset \mathcal{R}_T$. Let $x \in \mathcal{R}_{\tilde{T}}$ and let $x = s \oplus r \in \mathcal{U}_T \oplus \mathcal{R}_T$. Since $B_{\tilde{T}} = B_T|\widetilde{\mathcal{K}}$, we have

$$B_T^{*n} = \begin{pmatrix} B_{\tilde{T}}^{*n} & 0 \\ A_n & * \end{pmatrix} \tag{20}$$

relative to the decomposition $\widetilde{\mathcal{K}} \oplus (\mathcal{K} \ominus \widetilde{\mathcal{K}})$, where A_n is some bounded operator from $\widetilde{\mathcal{K}}$ to $\mathcal{K} \ominus \widetilde{\mathcal{K}}$, for any $n \in \mathbf{N}$. Furthermore, by (19) we have that

$$\begin{aligned} \|x\|^2 &\le \|x\|^2 + \|A_n x\|^2 = \|R_{\widetilde{T}}^{*n} x\|^2 + \|A_n x\|^2 \\ &= \|B_{\widetilde{T}}^{*n} x \oplus A_n x\|^2 = \|B_T^{*n} x\|^2 \le \|x\|^2. \end{aligned} \tag{21}$$

Hence $A_n x = 0$ for any $n \in \mathbf{N}$. This proves that

(22)

$$\begin{aligned} \|s\|^2 + \|r\|^2 &= \|x\|^2 = \|R_{\widetilde{T}}^{*n} x\|^2 = \|B_{\widetilde{T}}^{*n} x\|^2 \\ &= \|B_{\widetilde{T}}^{*n} x\|^2 + \|A_n x\|^2 = \|B_{\widetilde{T}}^{*n} x \oplus A_n x\|^2 \\ &= \|B_T^{*n} x\|^2 = \|U_T^{*n} s\|^2 + \|R_T^{*n} r\|^2 = \|U_T^{*n} s\|^2 + \|r\|^2. \end{aligned}$$

Letting $n \to \infty$ on the right side of (22), we have that $s = 0$. So $x \in \mathcal{R}_T$. Hence the proof is complete.

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