Comm. Korean Math. Soc. 8 (1993), No. 2, pp. 233–237

ON THE REDUCED INDEX OF RELATIVELY OPEN OPERATORS

CHUN IN CHOI AND HONG YOUL LEE

It is well known that if T is a semi-Fredholm operator between Banach spaces and if T has an index then $T^\dagger$, the adjoint of T, has an index with $\text{index}(T) = -\text{index}(T^\dagger)$. In this note we extend this result for incomplete spaces.

Throughout this note suppose X and Y are normed spaces, write $\mathcal{L}(X,Y)$ for the bounded linear operators from X to Y and $X^\dagger$ for the dual space of X. If $T \in \mathcal{L}(X,Y)$ write $T^\dagger \in \mathcal{L}(Y^\dagger, X^\dagger)$ for the adjoint of T. It is also known that if $T \in \mathcal{L}(X,Y)$ then

$$\dim (T^\dagger)^{-1}(0) = \dim Y/\operatorname{cl} T(X) \tag{0.1}$$

(cf.[1] Theorem IV.2.3 (i)). If $\dim T^{-1}(0)$ and $\dim Y/\operatorname{cl} T(X)$ are not both infinite, we shall say that T *has a reduced index*. The *reduced index* of T is defined by

$$\bar{i}(T) = \dim T^{-1}(0) - \dim Y/\operatorname{cl} T(X),$$

with the understanding that for any natural number n,

$$\infty - n = \infty, \quad n - \infty = -\infty, \quad \text{and} \quad -(-\infty) = \infty.$$

We recall [2] that $T \in \mathcal{L}(X,Y)$ is said to be *bounded below* if there is $k > 0$ for which

$$\|x\| \le k\|Tx\| \quad \text{for each} \quad x \in X,$$

is said to be *open* if there is $k > 0$ for which

$$y \in \{Tx : \|x\| \le k\|y\|\} \quad \text{for each} \quad y \in Y,$$

Received March 25, 1992.

and is said to be *relatively open* if its truncation $T^\wedge : X \longrightarrow T(X)$ is open. Evidently,

$$(0.2) \qquad T \text{ relatively open and one-one} \Longleftrightarrow T \text{ bounded below.}$$

We also recall that ([4] Theorem 1; [2] (5.5.3.2))

$$(0.3) \qquad T \text{ relatively open} \Longrightarrow T^\dagger \text{ relatively open}$$

and

$$(0.4) \qquad T \text{ bounded below} \Longleftrightarrow T^\dagger \text{ open.}$$

Relative openness can be tested with the *reduced minimum modulus* (cf. [1], [3])

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, T^{-1}(0)) \geq 1\} \quad \text{if } 0 \neq T \in \mathcal{L}(X,Y);$$

if $T = 0$ we may take $\gamma(T) = \infty$. Evidently,

$$(0.5) \qquad T \text{ relatively open} \Longleftrightarrow \gamma(T) > 0.$$

If X and Y are both complete then T is relatively open if and only if T has a closed range ((0.5) and [1] Theorem IV.1.6).

We are ready to meet:

THEOREM 1. *If $T \in \mathcal{L}(X,Y)$ is relatively open and has a reduced index then $T^\dagger$ has a reduced index and*

$$(1.1) \qquad \bar{i}(T) = -\bar{i}(T^\dagger).$$

Proof. Suppose $T \in \mathcal{L}(X,Y)$ is relatively open. If Z is a subspace of X, write $Z^\perp$ for the annihilator of Z in $X^\dagger$. We now claim that

$$(1.2) \qquad T^{-1}(0)^\perp \cong T^\dagger(Y^\dagger).$$

Indeed, if we define the mapping $T_1 : X/T^{-1}(0) \longrightarrow Y$ by setting

$$T_1(x + T^{-1}(0)) = Tx \in Y \quad \text{for each } x \in X,$$

then, by (0.2), T_1 is bounded below, so that, by (0.4), $T_1^\dagger$ is open and hence onto; we thus have

$$T^{-1}(0)^\perp \cong (X/T^{-1}(0))^\dagger = T_1^\dagger(Y^\dagger) \cong T^\dagger(Y^\dagger),$$

where the first isomorphism follows from [1] Theorem I.6.4 (ii) and the last isomorphism follows from the observation that, for each $g \in Y^\dagger$,

$$(T_1^\dagger g)(x + T^{-1}(0)) = gT_1(x + T^{-1}(0)) = gT(x) = (T^\dagger g)(x).$$

Therefore (1.2) gives

$$\begin{aligned} \dim X^\dagger/\operatorname{cl} T^\dagger(Y^\dagger) &= \dim X^\dagger/T^\dagger(Y^\dagger) = \dim X^\dagger/T^{-1}(0)^\perp \\ &= \dim(T^{-1}(0))^\dagger = \dim T^{-1}(0), \end{aligned} \tag{1.3}$$

where the first equality follows from the fact that if T is relatively open then by (0.3) $T^\dagger$ is relatively open, and hence has a closed range because $X^\dagger$ and $Y^\dagger$ are complete, and the third equality follows from [1] Theorem I.6.4 (i). We therefore have, by (0.1) and (1.3),

$$\begin{aligned} \bar{\imath}(T) &= \dim T^{-1}(0) - \dim Y/\operatorname{cl} T(X) \\ &= \dim X^\dagger/\operatorname{cl} T^\dagger(Y^\dagger) - \dim (T^\dagger)^{-1}(0) = -\bar{\imath}(T^\dagger). \end{aligned}$$

If X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ then the assumption on T in Theorem 1 is equivalent to the condition that T is *semi-Fredholm*, in the sense that $T(X)$ is closed and T has an index; therefore, in this case, Theorem 1 reduces to [1] Theorem V.2.3 (ii).

We recall that, between normed spaces, the bounded below operators form an open set:

LEMMA 2. *Let S and T be in $\mathcal{L}(X,Y)$. If T is bounded below and $\|S\| < \gamma(T)$ then $T-S$ is bounded below and*

$$\dim Y/\mathrm{cl}\, T(X) = \dim Y/\mathrm{cl}\,(T-S)(X). \tag{2.1}$$

Proof. See [1] Corollary V.1.3 and [3] Theorem 3.

We conclude with:

THEOREM 3. *If S and T are in $\mathcal{L}(X,Y)$ and if $\dim T^{-1}(0) = \dim(T-S)^{-1}(0) < \infty$ there is implication*

$$\|S\| < \gamma(T) \Longrightarrow \bar{i}(T-S) = \bar{i}(T).$$

Proof. Suppose $\|S\| < \gamma(T)$; thus T is relatively open. If $\dim T^{-1}(0) = \dim(T-S)^{-1}(0) < \infty$, there is a closed subspace W of X such that

$$X = W \oplus T^{-1}(0).$$

If $\widetilde{T}$ is the restriction of T to W then, by (0.2), $\widetilde{T}$ is bounded below and $\widetilde{T}(W) = T(X)$. If $\widetilde{S}$ is the restriction of S to W then, by assumption, $\|\widetilde{S}\| \le \|S\| < \gamma(T) \le \gamma(\widetilde{T})$; thus, by Lemma 2, $\widetilde{T} - \widetilde{S}$ is also bounded below. We thus have, by (0.2) and (2.1),

$$\dim(\widetilde{T}-\widetilde{S})^{-1}(0) = \dim \widetilde{T}^{-1}(0) = 0 \quad \text{and}$$
$$\dim Y/\mathrm{cl}(\widetilde{T}-\widetilde{S})(W) = \dim Y/\mathrm{cl}\, \widetilde{T}(W),$$

which gives, by Lemma V.1.5 (iii) in [1],

$$\bar{i}(T) = \bar{i}(\widetilde{T}) + \dim T^{-1}(0) = \bar{i}(\widetilde{T}-\widetilde{S}) + \dim(T-S)^{-1}(0) = \bar{i}(T-S).$$

What is not so obvious, and looks like an interesting problem is to speculate under what conditions on T the "classical"punctured neighborhood theorem or its extended form ([3], [5], [6]) holds. We invite the reader to consider the problem.

Acknowledgement. The authors are grateful to Prof. Woo Young Lee for the suggestions concerning this paper.

References

1. S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York, 1966.
2. R. E. Harte, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988.
3. R. E. Harte and W. Y. Lee, *The punctured neighbourhood Theorem for incomplete spaces*, J. Operator Theory (to appear).
4. W. Y. Lee, *Relatively open mappings*, Proc. Amer. Math. Soc. **108** (1990), 93–94.
5. ———, *Boundaries of the spectra in $\mathcal{L}(X)$*, Proc. Amer. Math. Soc. **116** (1992), 185–189.
6. ———, *A generalization of the punctured neighborhood theorem*, Proc. Amer. Math. Soc. (to appear).

Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea