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ON THE UNIPOTENT RADICALS OF PARABOLIC SUBGROUPS IN CHEVALLEY GROUPS

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1. Introduction

Let (E, Φ, Δ) be a root system of rank ℓ where E is a Euclidean space, Φ is the set of roots and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is a set of simple roots. Let $I = \{1, \ldots, \ell\}$ and J be a subset of I. Consider any type of Chevalley group G, over an arbitrary field F, defined by a complex semisimple Lie algebra with the root system Φ (see [7]). Then G is generated by the elements $x_\alpha(t)$ where α runs over Φ and t runs over F. Let P_J be the parabolic subgroup of G corresponding to J (see [1, §2.1]). Then we have a Levi decomposition $P_J = L_J U_J$ with a Levi subgroup L_J of P_J and the unipotent radical U_J of P_J. The unipotent radical U_J is a subgroup of P_J generated by $\{x_\alpha(t) | \alpha \in \Phi^+ - \Phi_J,\ t \in F\}$ where Φ^+ is the set of all positive roots and Φ_J is the subsystem of Φ spanned by $\{\alpha_j | j \in J\}$ ([1, Proposition 2.6.4]).

The structure of U_J and the automorphism group of U_J has been studied by Gibbs([2]) when $J = \emptyset$, Khor([6]) when Φ is of type A_ℓ and Im([4] and [5]) when Φ is of type B_ℓ, C_ℓ or D_ℓ. In these cases Khor and Im showed that $\Phi^+ - \Phi_J$ can be partitioned into blocks of roots and the set of these blocks can be identified with some (not necessarily reduced) root system of rank $\ell - |J|$. However the works of Khor and Im strongly depend on the ordinary matrix representation of G so that they had to restrict themselves to special types of G, for example $SL_{\ell+1}(F)$ or $Sp_{2\ell}(F)$.

In this paper, we study the structure of U_J when Φ is a simply-laced root system, that is, of type A, D and E. We consider W_J-orbits of $\Phi^+ - \Phi_J$, where W_J is the subgroup of the Weyl group W generated by $\{s_j | j \in J\}$ and s_j is the simple reflection corresponding to the simple root

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α_j. We show that the set of W_J-orbits of $\Phi^+ - \Phi_J$ can be identified with a subset of the lattice $(\mathbb{Z}^+)^{\ell-|J|}$ containing the standard unit vectors. We also show that the addition of W_J-orbits (see Definition 3.2) can be characterized by the addition of corresponding lattice points. Therefore we have an obvious notion of the height function (or the *level* function) in this orbit space. The notion of level function plays an important role when we study the automorphism group of U_J (see [2], [6] and [4]).

2. The W_J-orbits of $\Phi^+ - \Phi_J$

As in the Section 1, let (E, Φ, Δ) be a simply-laced root system (see [3]) where E is a Euclidean space equipped with a positive definite scalar product (,) and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is a set of simple roots. We denote by Φ^+ the set of positive roots in Φ. Let W be the Weyl group of the root system Φ generated by the simple reflections s_i, $i = 1, \ldots, \ell$, corresponding to the simple roots α_i. For $x, y \in E$ we denote $\langle x, y\rangle = 2(x,y)/(y,y)$.

Let $I = \{1, \ldots, \ell\}$ and J be a subset of I. Let Φ_J be the subsystem of Φ spanned by $\{\alpha_j | j \in J\}$ and W_J be the Weyl group of Φ_J. Then W_J is the subgroup of W generated by $\{s_j | j \in J\}$. Since W_J acts on Φ_J and s_j, $j \in J$ permutes $\Phi^+ - \{\alpha_j\}$ ([3, 10.2.B]), W_J acts on $\Phi^+ - \Phi_J$.

Since the case when $J = I$ or $J = \emptyset$ is trivial, we may assume $J \neq I$ and $J \neq \emptyset$. Set $I - J = \{i_1, \ldots, i_n\}$. For $\alpha \in \Phi^+ - \Phi_J$, we denote by $[\alpha]$ the W_J-orbit containing α.

Let $n_1\alpha_1 + n_2\alpha_2 + \cdots + n_\ell\alpha_\ell$ be the maximal positive root in Φ ([3, 10.4.A]).

DEFINITION 2.1. For $\alpha \in \Phi^+ - \Phi_J$ and $\mathbf{A} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we say that α is of *type* $\mathbf{A}$ if

$$\alpha = a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_n\alpha_{i_n} + (\text{terms in } \alpha_j,\ j \in J).$$

We set Λ to be the set consisting of elements $(a_1, \ldots, a_n)$ in $\mathbb{Z}^n$, where $0 \leq a_k \leq n_{i_k}$ for $k = 1, \ldots, n$, such that there exists a root in $\Phi^+ - \Phi_J$ of type $(a_1, \ldots, a_n)$.

The main result of the present section is that the set of W_J-orbits of $\Phi^+ - \Phi_J$ is parameterized by Λ. In fact, we have the following.

THEOREM 2.2. *Let α and β be in $\Phi^+ - \Phi_J$. Assume α is of type* **A** *and β is of type* **B** *respectively. Then $[\alpha] = [\beta]$ if and only if* $\mathbf{A} = \mathbf{B}$.

Proof. Suppose $\mathbf{A} \neq \mathbf{B}$. Then β cannot be a W_J-conjugate of α, for the action of s_j possibly change the coefficients of α_j for $j \in J$. Conversely, assume $\mathbf{A} = \mathbf{B}$. We have to show that α is a W_J-conjugate of β. This is a result of the following three lemmas.

LEMMA 2.3. *Let $\alpha - \beta = \alpha_j$ for some $j \in J$. Then $s_j(\alpha) = \beta$.*

Proof. Since we are assuming that Φ is simply-laced, $s_j(\alpha)$ is equal to α, $\alpha + \alpha_j$ or $\alpha - \alpha_j$ and $s_j(\beta)$ is equal to β, $\beta + \alpha_j$ or $\beta - \alpha_j$. We find that $s_j(\alpha) - s_j(\beta) = -\alpha_j$ is true only when $s_j(\alpha) = \alpha - \alpha_j$ and $s_j(\beta) = \beta + \alpha_j$. This proves the lemma.

LEMMA 2.4. *Let $\alpha - \beta = \alpha_{j_1} + \cdots + \alpha_{j_t}$ for some $j_1, \ldots, j_t$ in J which are not necessarily distinct. Then $s_{j_{\sigma(1)}} s_{j_{\sigma(2)}} \cdots s_{j_{\sigma(t)}}(\alpha) = \beta$ for some permutation σ of $\{1, \ldots, t\}$.*

Proof. We prove the lemma by the induction on t. There is nothing to prove if $t = 1$ by the previous lemma and let us assume $t \geq 2$. Then it is enough to show that $\beta + \alpha_{j_u}$ is a root for some u. Because if $\beta + \alpha_{j_u}$ is a root then $\alpha - (\beta + \alpha_{j_u})$ is a sum of $t-1$ simple roots and by the induction hypothesis $\beta + \alpha_{j_u}$ is a W_J-conjugate of α. Again by the previous lemma, we have $s_{j_u}(\beta + \alpha_{j_u}) = \beta$ and thus our assertion can be shown. Now, on the contrary, suppose that $(\beta, \alpha_{j_u}) \geq 0$ for all $u = 1, \ldots, t$. Then $(\beta, \alpha - \beta) \geq 0$ and hence $\langle \alpha, \beta \rangle \geq 2$, which is a contradiction. Therefore $(\beta, \alpha_{j_u}) < 0$ for some u and $\beta + \alpha_{j_u}$ is a root.

LEMMA 2.5. *Let $\alpha - \beta = \alpha_{j_1} + \cdots + \alpha_{j_t} - \alpha_{j_{t+1}} - \cdots - \alpha_{j_{t+s}}$ for some $j_1, \ldots, j_{t+s}$ in J which are not necessarily distinct but $j_u \neq j_{t+v}$ for $u = 1, \ldots, t$ and $v = 1, \ldots, s$. Then $[\alpha] = [\beta]$.*

Proof. We use the induction on $t + s$. If $t + s = 1$, our assertion is already proved in Lemma 2.3. If $t + s \geq 2$, we may assume $t \geq 1$ and $s \geq 1$ by Lemma 2.4. Using the similar argument as in the proof of Lemma 2.4, we can show that it is enough to prove that $(\alpha, \alpha_{j_u}) > 0$ for some $u = 1, \ldots, t$ or $(\alpha, \alpha_{j_{t+v}}) < 0$ for some $v = 1, \ldots, s$. Suppose, on the contrary, that $(\alpha, \alpha_{j_u}) \leq 0$ for all $u = 1, \ldots, t$ and $(\alpha, \alpha_{j_{t+v}}) \geq 0$ for all $v = 1, \ldots, s$. Then $(\alpha - \beta, \alpha) \leq 0$ and thus $0 < (\alpha, \alpha) \leq (\alpha, \beta)$. This implies that $\alpha - \beta$ is a root which is impossible.

This completes the proof of Theorem 2.2.

Theorem 2.2 implies that the set of W_J-orbits of $\Phi^+ - \Phi_J$ is in one-to-one correspondence with Λ.

3. The addition between W_J-orbits

We begin with the following interesting observation.

PROPOSITION 3.1. *Let α be a root of type* **A** *in $\Phi^+ - \Phi_J$. Assume that α has the maximal height among roots of type* **A**. *Then for any root β of type* **A**, *$\alpha - \beta$ is a sum of simple roots α_j for $j \in J$. Therefore such α is uniquely determined.*

Proof. The uniqueness follows easily from the first statement. Suppose $(\alpha_j, \alpha) < 0$ for some $j \in J$. Then $\alpha + \alpha_j$ is a root but this is impossible for α has the maximal height. Thus we have $(\alpha_j, \alpha) \geq 0$ for all $j \in J$. Now let $\beta = w(\alpha)$ for some $w \in W_J$. Write $w = s_{j_1} \cdots s_{j_m}$ where $j_1, \ldots, j_m$ are in J. We show that $\alpha - \beta$ is a sum of simple roots α_j for $j \in J$ by the induction on m. Since $s_j(\alpha) = \alpha$ or $\alpha - \alpha_j$ for all $j \in J$, we can easily verify our assertion when $m = 1$ or 2. So let us assume $m \geq 3$. If $s_{j_m}(\alpha) = \alpha$, then the induction hypothesis completes the proof. If $s_{j_m}(\alpha) = \alpha - \alpha_{j_m}$, then again by the induction hypothesis, we may assume that $s_{j_1} \cdots s_{j_{m-1}}(\alpha_{j_m})$ is a negative root. In this case, by [3, 10.2.C], we have

$$s_{j_1} \cdots s_{j_{m-1}} = s_{j_1} \cdots s_{j_{t-1}} s_{j_{t+1}} \cdots s_{j_m}$$

for some $t = 1, \ldots, m-1$. Thus we have

$$w(\alpha) = s_{j_1} \cdots s_{j_{t-1}} s_{j_{t+1}} \cdots s_{j_{m-1}}(\alpha)$$

and the induction hypothesis applies.

By the above proposition, the root of maximal height of given type is well-defined. By the same argument, we can also show that there is a unique root of minimal height of given type.

Now, we define the addition between the W_J-orbits in $\Phi^+ - \Phi_J$.

DEFINITION 3.2. For α, β and γ in $\Phi^+ - \Phi_J$, we say $[\alpha] + [\beta] = [\gamma]$ if there exist $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ such that $\alpha' + \beta' \in [\gamma]$.

We note that the addition of two orbits is not always defined.

We consider Λ as a subset of the additive group $\mathbb{Z}^n$. The following result shows that the addition of W_J-orbits can be characterized by the addition in Λ.

THEOREM 3.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be (not necessarily distinct) elements of* Λ *such that* $\mathbf{A} + \mathbf{B}$ *is also an element of* Λ. *Let* α *be the root of type* $\mathbf{A}$ *with minimal height and let* β *be the root of type* $\mathbf{B}$ *with maximal height. Then* $\alpha + \beta$ *is a root.*

The following corollary is clear from the above theorem.

COROLLARY 3.4. *Let* α, β *and* γ *be roots of type* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *respectively. Then* $[\alpha] + [\beta] = [\gamma]$ *if and only if* $\mathbf{A} + \mathbf{B} = \mathbf{C}$.

Proof of Theorem 3.3. We set $\alpha = \gamma_1$ and $\beta = \gamma_{-1}$. Suppose that $\gamma_1 + \gamma_{-1} \notin \Phi$ but

$$\gamma = \gamma_1 + \gamma_{-1} + \sum_{j \in J_1} x_j \alpha_j - \sum_{j \in J_{-1}} x_j \alpha_j,$$

is a root, where J_1 and J_{-1} are disjoint subsets of J and x_j are positive integers. We choose γ such that the positive integer

$$\sum_{j \in J_1 \cup J_{-1}} x_j$$

is minimal. We will show that this leads to a contradiction.

If $\langle \gamma, \alpha_j \rangle > 0$ for some $j \in J_1$, then $\gamma - \alpha_j \in \Phi$ but this is impossible because of the minimality of γ. Thus we may assume $\langle \gamma, \alpha_j \rangle \leq 0$ for all $j \in J_1$ and similarly $\langle \gamma, \alpha_j \rangle \geq 0$ for all $j \in J_{-1}$. We also note that $\langle \gamma_1, \alpha_j \rangle \leq 0$ and $\langle -\gamma_{-1}, \alpha_j \rangle \leq 0$ for all $j \in J$ by Proposition 3.1. For $j \in J_1 \cup J_{-1}$, we can write

$$\langle \gamma, \alpha_j \rangle = \langle \gamma_1, \alpha_j \rangle + \langle \gamma_{-1}, \alpha_j \rangle + y_j,$$

where y_j is a $\mathbb{Z}$-linear combination of $\{x_j | j \in J_1 \cup J_{-1}\}$ which depends on the location of α_j in the Dynkin diagram of $\{\alpha_j | j \in J_1 \cup J_{-1}\}$. By the preceding remarks, we find that

$$-2 \leq \varepsilon y_j \leq 1 \quad \text{for } j \in J_\varepsilon \text{ and } \varepsilon \in \{-1, 1\}.$$

We also note that if $j \in J_\varepsilon$ then $\varepsilon y_j = 1$ implies $\langle \varepsilon\gamma_\varepsilon, \alpha_j \rangle = -1$ for $\varepsilon \in \{-1, 1\}$. We choose $j_1 \in J_1 \cup J_{-1}$ such that there is at most one α_j $(j_1 \neq j \in J_1 \cup J_{-1})$ which is not perpendicular to α_{j_1}. This is possible, since the Dynkin diagram does not contain a cycle.

Since y_j, $j \in J_1 \cup J_{-1}$, is determined by the connected component of α_j in the Dynkin diagram of $\{\alpha_j | j \in J_1 \cup J_{-1}\}$, we let C be the connected component of α_{j_1} in the Dynkin diagram of $\{\alpha_j | j \in J_1 \cup J_{-1}\}$. We consider the following four possibilities for C. In the following description of C, the number k on a vertex represents the simple root α_{j_k} $(j_k \in J_1 \cup J_{-1})$. We shall assume that the numbering of vertices in the Dynkin diagram is
(a) that of [3, p.58] for types A_m, E_6, E_7 and E_8,
(b) that of [3, p.58] in the reverse order for type D_m.

For the simplicity, we set $\alpha_{j_1} = a_1$, $\alpha_{j_k} = a_k$, $x_{j_k} = X_k$ and $y_{j_k} = Y_k$. We say $a_k \in J_\varepsilon$ if $j_k \in J_\varepsilon$. We define $\varepsilon_k = 1$ if $a_k \in J_1$ and $\varepsilon_k = -1$ if $a_k \in J_{-1}$.

CASE I. $C = A_m$, $m \geq 1$.

If $m = 1$, then $\varepsilon_1 Y_1 = 2X_1 \geq 2$ which is absurd. Thus we assume $m \geq 2$. First, we assume that $\varepsilon_k = \varepsilon$ for all $k = 1, \ldots, m$. Then we have the following system of linear inequalities:

$$\begin{array}{ccccccc} -2 & \leq & \varepsilon Y_1 & = & 2X_1 - X_2 & \leq & 1 \\ -2 & \leq & \varepsilon Y_s & = & -X_{s-1} + 2X_s - X_{s+1} & \leq & 1 \quad (s = 2, \ldots, m-1) \\ -2 & \leq & \varepsilon Y_m & = & -X_{m-1} + 2X_m & \leq & 1 \end{array}$$

Summing up all the middle terms in the above system of inequalities, we get

$$\sum_{k=1}^{m} \varepsilon Y_k = X_1 + X_m \geq 2.$$

Hence εY_k are equal to 1 for at least two values of k, say $\varepsilon Y_s = 1 = \varepsilon Y_t$. Then, as we have seen above, $\langle \varepsilon\gamma_\varepsilon, a_s\rangle = -1 = \langle \varepsilon\gamma_\varepsilon, a_t\rangle$. Considering the Dynkin diagram of $\{\varepsilon\gamma_\varepsilon, a_1, \ldots, a_m\}$, we conclude that this is absurd. (See the proof of the classification theorem of Dynkin diagrams, for example, [3, pp. 58–63].) Next, we assume that ε_k are not all same. We observe that if $\varepsilon_1 \neq \varepsilon_2$ then $\varepsilon_1 Y_1 = 2X_1 + X_2 \geq 3$ which is absurd. Hence we may assume $\varepsilon_1 = \varepsilon_2$ and similarly we may assume $\varepsilon_{m-1} = \varepsilon_m$. We also observe that if $\varepsilon_s \neq \varepsilon_{s-1} = \varepsilon_{s+1}$ for some s then $\varepsilon_s Y_s = X_{s-1} + 2X_s + X_{s+1} \geq 4$ which is impossible. Thus we may assume that ε_k remain unchanged for more than two consecutive terms. In particular we must have $m \geq 4$. Now, if

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_r \neq \varepsilon_{r+1} = \varepsilon_{r+2} = \cdots,$$

then we have the following system of linear inequalities:

$$\begin{array}{rclclcll}
-2 & \leq & \varepsilon_1 Y_1 & = & 2X_1 - X_2 & \leq & 1 & \\
-2 & \leq & \varepsilon_s Y_s & = & -X_{s-1} + 2X_s - X_{s+1} & \leq & 1 & (2 \leq s \leq r-1) \\
-2 & \leq & \varepsilon_r Y_r & = & -X_{r-1} + 2X_r + X_{r+1} & \leq & 1 & \\
-2 & \leq & \varepsilon_{r+1} Y_{r+1} & = & X_r + 2X_{r+1} - X_{r+2} & \leq & 1 & \\
 & & & \vdots & & & & \\
-2 & \leq & \varepsilon_m Y_m & = & -X_{m-1} + 2X_m & \leq & 1 &
\end{array}$$

Summing up all the middle terms, we get

$$\sum_{k=1}^{m} \varepsilon_k Y_k \geq X_1 + 2X_r + 2X_{r+1} + X_m \geq 6.$$

Therefore $\varepsilon_k Y_k$ must be equal to 1 for at least six values of k, and this gives a contradiction if we consider the Dynkin diagram of $\{\gamma_1, a_1, \ldots, a_m\}$ or $\{-\gamma_{-1}, a_1, \ldots, a_m\}$.

CASE II. $C = D_m$, $m \geq 4$.

First, suppose that $\varepsilon_k = \varepsilon$ for all $k = 1, \ldots, m$. Then we have the following system of linear inequalities:

$$\begin{array}{rclclcll}
-2 & \leq & \varepsilon Y_1 & = & 2X_1 - X_3 & \leq & 1 & \\
-2 & \leq & \varepsilon Y_2 & = & 2X_2 - X_3 & \leq & 1 & \\
-2 & \leq & \varepsilon Y_3 & = & -X_1 - X_2 + 2X_3 - X_4 & \leq & 1 & \\
-2 & \leq & \varepsilon Y_s & = & -X_{s-1} + 2X_s - X_{s+1} & \leq & 1 & (4 \leq s \leq m-1) \\
-2 & \leq & \varepsilon Y_m & = & -X_{m-1} + 2X_m & \leq & 1 &
\end{array}$$

Note that the coefficient matrix of the above system is nothing but the Cartan matrix of the type D_m. Since the inverse of the Cartan matrix is known, we find that $\varepsilon Y_t = 1$ for some $t = 2, \ldots, m-1$ if $m \geq 5$, and that $\varepsilon Y_t = 1$ for at least two values of t if $m = 4$. But the Dynkin diagram of $\{\varepsilon\gamma_e, a_1, \ldots, a_m\}$ gives a contradiction. Second, suppose that ε_k are not all same. In this case, we have already seen in Case I that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$. If $\varepsilon_3 \neq \varepsilon_4$, then by the same method as in the Case I we get

$$\sum_{k=1}^{m} \varepsilon_k Y_k \geq X_1 + X_2 + X_3 + X_m \geq 4$$

and this is impossible. Hence we may assume that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$. Then, again using the method of Case I, we get

$$\sum_{k=1}^{m} \varepsilon_k Y_k \geq X_1 + X_2 - X_3 + 2X_r + 2X_{r+1} + X_m$$

for some $r \geq 4$. Suppose that

$$\sum_{k=1}^{m} \varepsilon_k Y_k \leq 2.$$

Then, since $X_3 \leq X_1 + X_2 + 2$, we have

$$5 \leq 2X_r + 2X_{r+1} + X_m \leq 4$$

and this is absurd. Thus we must have

$$\sum_{k=1}^{m} \varepsilon_k Y_k \geq 3,$$

but this implies $\varepsilon_k Y_k = 1$ for at least three values of k, which is also absurd.

CASE III. $C = E_6$.

If $\varepsilon_1 = \varepsilon_3 \neq \varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$ or $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6 \neq \varepsilon_2 = \varepsilon_4$ then we have

$$\sum_{k=1}^{6} \varepsilon_k Y_k \geq X_1 + X_2 + X_4 + X_6 \geq 4$$

and this is impossible. Thus, by the symmetry, we may assume that $\varepsilon_k = \varepsilon$ for all $k = 1, \ldots, 6$. In this case, the coefficient matrix of the system of inequalities is nothing but the Cartan matrix of the type E_6. Since we know the inverse of the Cartan matrix, we can easily verify that there are too many edges in the Dynkin diagram of $\{\varepsilon\gamma_\varepsilon, a_1, \ldots, a_6\}$.

CASE IV. $\mathcal{C} = E_7$.

This case occurs only when $\Phi = E_8$ and $J = \{1, 2, \ldots, 7\}$. Then there are two W_J-orbits, namely, $[\alpha_8]$ and $[\alpha_{\max}]$ where $\alpha_{\max}$ is the maximal positive root. Note that $\gamma_1 = \alpha_8$ is the root of minimal height in $[\alpha_8]$ and $\gamma_{-1} = \alpha_{\max} - \alpha_8$ is the root of maximal height in $[\alpha_8]$. Therefore we have $\gamma_1 + \gamma_{-1} = \alpha_{\max} \in \Phi$. This completes the proof of Theorem 3.3.

4. The level function on J-roots

Next, we define a projection of the W_J-orbit space onto $\mathbf{R}^n$ (or $\mathbf{Z}^n$) in a natural way.

DEFINITION 4.1. Let V be the real vector space whose basis is consisting of W_J-orbits of $\Phi^+ - \Phi_J$ and R be the subspace of V spanned by all possible relations $[\alpha] + [\beta] = [\gamma]$ among the orbits. We set $\mathcal{E} = V/R$. By Theorem 2.2 and Corollary 3.4, we may identify V with the real vector space with a basis Λ and R with the subspace spanned by all possible relations $\mathbf{A} + \mathbf{B} = \mathbf{C}$ in Λ. We call the image of $[\alpha]$ (or $\mathbf{A}$) in $\mathcal{E}$ a *J-root* and denote it by $\langle\alpha\rangle$ or $\langle\mathbf{A}\rangle$.

Our situation is now similar to that of constructing a base of a root system.

DEFINITION 4.2. A J-root is called an *indecomposable* J-root if it cannot be written as a sum of two (not necessarily distinct) J-roots.

Note that the unit vectors $\mathbf{E}_i = (0, \ldots, 1, \ldots, 0)$ are contained in Λ for all $i = 1, \ldots, n$.

PROPOSITION 4.3. *$\langle\alpha_i\rangle$, $i \in I - J$, in $\mathcal{E}$ are the only indecomposable J-roots and every J-root is a sum of indecomposable J-roots. Moreover, for any J-root $\langle\alpha\rangle$, we can write*

$$\langle\alpha\rangle = \langle\alpha_{k_1}\rangle + \cdots + \langle\alpha_{k_t}\rangle, \qquad k_1, \ldots, k_t \in I - J$$

such that

$$\langle\alpha_{k_1}\rangle + \cdots + \langle\alpha_{k_s}\rangle$$

is a J-root for all $s = 1, \ldots, t$.

Proof. It is enough to prove the second statement. Let α be of type $A = (a_1, \ldots, a_n)$. We show that $\langle\alpha\rangle$ can be written as in the proposition by induction on $a = a_1 + \cdots + a_n$. If $a = 1$, we are done. If $a > 1$, we write $\alpha = \alpha_{k_1} + \cdots + \alpha_{k_t}$ such that $\alpha_{k_1} + \cdots + \alpha_{k_s} \in \Phi$ for all $s = 1, \ldots, t$ ([3, 10.2.A]). Since $a > 1$, the number of indices k_s in $I - J$ is greater than 1. Choose the maximal u such that $k_u \in I - J$. Then $[\alpha] = [\alpha_{k_1} + \cdots + \alpha_{k_u}] = [\alpha_{t_1} + \cdots + \alpha_{k_{u-1}}] + [\alpha_{k_u}]$ and hence $\langle\alpha\rangle = \langle\alpha_{k_1} + \cdots + \alpha_{k_{u-1}}\rangle + \langle\alpha_{k_u}\rangle$. Now the induction hypothesis completes the proof.

PROPOSITION 4.4. *$\mathcal{E}$ is an n-dimensional vector space over $\mathbb{R}$ with a basis $\{\langle\alpha_i\rangle | i \in I - J\}$.*

Proof. First, we define a map φ from $\mathbb{R}^n$ onto $\mathcal{E}$ by $\varphi(\varepsilon_i) = \langle\alpha_i\rangle$, where ε_i are the standard unit vectors, for $i = 1, \ldots, n$. Next, we define a map ψ from V onto $\mathbb{R}^n$ by $\psi([\alpha]) = (a_1, \ldots, a_n)$ where α is of type $(a_1, \ldots, a_n)$. Then ψ induces a mapping from $\mathcal{E}$ onto $\mathbb{R}^n$ by Corollary 3.4. Now it is clear that ψ is the inverse mapping of φ.

Hence we have a natural notion of the *level* function on J-roots. That is we define the level of $\langle\alpha\rangle$ by the sum of coefficients with respect to the basis $\{\langle\alpha_i\rangle | i \in I - J\}$ of $\mathcal{E}$.

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