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INTEGRALITY OVER WEAKLY SEPARABLE GRADED RINGS*

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In [3], Paré and Schelter proved a Cayley-Hamilton theorem for matrices over noncommutative rings, that is, a matrix ring is integral over the coefficient ring. D. S. Passman [4] showed that a finite group graded ring is integral over the basis ring.

In contrast to commutative rings, integrality of noncommutative rings is not transitive [1]. But D. Quinn [7] proved the following theorem ;

THEOREM. *Let G be a finite group of order n and let R be a G-graded ring with $|G|^{-1} \in R$. Then $M_G(R)$ is fully integral over R_1, where $M_G(R)$ is the $n \times n$ matrices over R with the rows and columns indexed by the elements of G.*

The main object of this paper is to give a generalization of the theorem.

For our purpose, we introduce a few concepts as follows; Let R be a ring with the identity 1 and let S be a subring of R. If $r_1, r_2, \dots, r_m \in R$, then an *$S$-monomial in* $r_1, r_2, \dots, r_m$ is a product each of whose factors is either one of the r_i or the element from S, with at least one element of S occurring. By the *degree* of any such monomial we mean the total number of factors from the r_i occurring. We say that R is *fully integral of degree m over S* if for any $r_1, r_2, \dots, r_m \in R$, $r_1 r_2 \cdots r_m = \psi(r_1, r_2, \dots, r_m)$, where $\psi(r_1, r_2, \dots, r_m)$ is a sum of S-monomials in the r_i's of degree less than m. In particular, by setting $r_1 = r_2 = \cdots = r_m = r \in R$, it follows that fully integral implies Schelter integral of bounded degree m. Let G be a finite group with the identity 1 and let $R = \oplus \sum_{g \in G} R_g$ be a G-graded ring. R is called *weakly separable over* R_1 if there is a family $\{a^{(g)} \in Z(R_1) \mid g \in G\}$ such that

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1. $\sum_{g\in G} a^{(g)} = 1$.

2. For all $h \in G$ and $\lambda_h \in R_h$, $a^{(g)}\lambda_h = \lambda_h a^{(h^{-1}g)}$.

This definition is the equivalent condition of a separable functor for a group graded ring. If $|G|^{-1} \in R$, then R is weakly separable over R_1. But the converse is not true [2]. For undefined terminologies, we refer [2] and [5].

First we need a key lemma.

LEMMA 1 [6, PROPOSITION 1.2]. *Let $S \subseteq R \subseteq T$ be rings with R Schelter integral (resp., fully integral of degree m) over S. Suppose that $e \in T$ is an idempotent such that $eRe \subseteq R$. Then eRe is Schelter integral (resp., fully integral of degree m) over eSe.*

In order to prove our main theorem, we need the following theorem. We use similar methods to those in [7].

THEOREM 2. *Let G be a finite group of order n and let R be a G-graded ring such that R is weakly separable over R_1. Suppose that H is a finite group of automorphisms of R that permute the homogeneous components and fix the elements of the centre of R_1. If $G \cong H$, then R is fully integral over R_1^H of degree $m(|G|^2)$, where $R_1^H = \{r \in R_1 \mid r^h = r$ for all $h \in H\}$ and r^h means the image of r under h.*

Proof. Let $\widetilde{R}\sharp G$ be the smash product of R and G. Then $\widetilde{R}\sharp G = \sum_{g,h\in G} R_{gh^{-1}}\, e_{g,h} \subseteq M_G(R)$, where $e_{g,h}$ is the matrix unit with 1_R in the (g,h)-position and zeros elsewhere.

Let $T = M_H(\widetilde{R}\sharp G)$, then T is a subring of $M_{G\times H}(R)$. If $x, y \in H$, let $E_{x,y} \in M_H(\widetilde{R}\sharp G) \subseteq M_{G\times H}(R)$ be the matrix unit with $1 = 1_{\widetilde{R}\sharp G}$ in the (x,y)-position and zeros elsewhere. Let $A = \{\sum_{h\in H} r^h E_{h,h} | r \in R_1\}$, then $A \subseteq T \subseteq M_{G\times H}(R)$ and A is a subring of diagonal matrices. From [7, Theorem 1], we obtain that T is fully integral over A of degree $m(|G|^2)$. Since R is weakly separable over R_1 and $G \cong H$, there is a family $\{a^{(x_i)} \in Z(R_1) | x_i \in H\}$ such that $\sum_{i=1}^n a^{(x_i)} = 1$. Let

$$e = \begin{bmatrix} X_1 & X_1 & \cdots & X_1 \\ X_2 & X_2 & \cdots & X_2 \\ \vdots & \vdots & & \vdots \\ X_n & X_n & & X_n \end{bmatrix},$$

where $X_i = a^{(x_i)}1_{\widetilde{R}\sharp G}$, then $e \in T$ is an idempotent. By lemma 1, eTe is fully integral over eAe. We define a ring homomorphism $\phi : R_1^H \to eAe$ by $\phi(r) = e(\sum_{h\in H} r^h E_{h,h})e$. Clearly, if $\phi(r) = 0$, then $X_i r = 0$ for $i = 1, 2, \ldots, n$. Thus $0 = \sum_{i=1}^n X_i r = (\sum_{i=1}^n X_i)r = r$. Let $t = e(\sum_{h\in H} r^h E_{h,h})e \in eAe$. Put $x = \sum_{h\in H} a^{(h)} r^h$, we have $x \in R_1^H$ and $\phi(x) = t$. Hence $R_1^H \cong eAe$ as rings. Similarly, if we define $\psi : \widetilde{R}\sharp G \to eTe$ by $\psi(r) = ere$, we have that ψ is a one to one ring homomorphism. Hence $\widetilde{R}\sharp G$ is fully integral over R_1^H. Now $R \subseteq \widetilde{R}\sharp G$, it follows that R is fully integral over R_1^H.

THEOREM 3. *Let G be a finite group and let R be a G-graded ring such that R is weakly separable over R_1. Then $M_G(R)$ is fully integral over R_1.*

Proof. $M_G(R)$ is a $G \times G$-graded ring, where $G \times 1$ grades R and the graded $(1, gh^{-1}) \in G \times G$ is assigned the matrix unit $e_{g,h}$. Clearly, $M_G(R)$ is weakly separable over $(M_G(R))_1$ and $(M_G(R))_1$ is the set of diagonal matrices with entries from R_1. Let $\overline{G} = \{\bar{g} = \sum_{x\in G} e_{x,xg} | g \in G\}$ be the group of permutation matrices. Then $G \cong \overline{G}$, $\overline{G}$ acts by conjugation, as graded automorphisms of $M_G(R)$ and fixes the elements of the centre of R_1. Moreover $(M_G(R))^{\overline{G}} \cong R_1$. By theorem 2, $M_G(R)$ is fully integral over R_1. This completes the proof.

REMARK. We don't know whether theorem 2 holds under the condition $|G| = |H|$ instead of $G \cong H$.

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