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SIMPLE COMODULE OF THE BIALGEBRA $\mathbf{C}[x, y]$

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Recently, the coordinate ring of quantum affine n-space is defined by I. Yu. Manin, N. Yu. Reshetikhin and M. Takeuchi, independently. The spectrum and the primitive spectrum of this algebra are known and all its prime ideals are completely prime. In general, it admits no bialgebra structure. However, a peculiar feature for the coordinate ring of quantum affine 2-space $\mathbf{C}[x, y]$ is that there is a bialgebra structure. It is the following:

$$yx = qxy, \Delta x = x \otimes x, \Delta y = y \otimes 1 + x \otimes y, \varepsilon x = 1, \varepsilon y = 0,$$

where q is a nonzero complex number. This bialgebra structure appears in Example 2.1.2 of [1].

In this note, we show that: (1) every simple comodule of a bialgebra R is isomorphic to a sub-comodule of R and (2) we deduce that every simple comodule of $\mathbf{C}[x, y]$ is one dimensional from the result of (1).

All considered algebras are defined over the complex numbers.

LEMMA 1. *Let R be a bialgebra and let $\tau : V \longrightarrow V \otimes R$ and $\sigma : W \longrightarrow W \otimes R$ be two comodule structure maps. Then*

$$\phi = (1 \otimes 1 \otimes \mu)s_{(23)}(\tau \otimes \sigma) : V \otimes W \longrightarrow V \otimes W \otimes R$$

is a comodule structure map, where μ is the multiplication map on R and $s_{(23)}(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$.

LEMMA 2. *Let $V_i, i \in I$, be comodules of a bialgebra R. Then every canonical epimorphism $\pi_j : \oplus_{i \in I} V_i \longrightarrow V_j$ is a comodule homomorphism.*

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PROPOSITION 3. *Every simple comodule of a bialgebra R is isomorphic to a sub-comodule of R.*

Proof. Since every comodule is locally finite, simple comodule is finite dimensional. Let $\tau : V \longrightarrow V \otimes R$ be a simple R-comodule structure map. First, let us prove that $\theta = (1 \otimes \mu)(\tau \otimes 1) : V \otimes R \longrightarrow V_{(0)} \otimes R$ is a comodule homomorphism, where $V_{(0)} = V$ is the R-comodule with comodule structure $v \longrightarrow v \otimes 1$ and μ is the multiplication map on R. If we put $\phi : V \otimes R \longrightarrow V \otimes R \otimes R$ the comodule structure map as defined in lemma 1, then

$$\begin{aligned}(\theta \otimes 1)\phi(v \otimes a) &= (\theta \otimes 1)(1 \otimes 1 \otimes \mu)s_{(23)}(\tau \otimes \Delta)(v \otimes a)\\ &= \sum v_{(0)} \otimes b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)}\\ &= (1 \otimes \Delta)\theta(v \otimes a)\end{aligned}$$

where $(\tau \otimes 1)\tau(v) = \sum v_{(0)} \otimes b_{(1)} \otimes b_{(2)}$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. Thus θ is a comodule homomorphism. Clearly, the map $j : V \longrightarrow V \otimes R$ given by $v \longrightarrow v \otimes 1$ is a comodule homomorphism. Therefore, the map from V into R

$$V \xrightarrow{j} V \otimes R \xrightarrow{\theta} V_{(0)} \otimes R \cong \oplus_n R \xrightarrow{\pi_i} R$$

is an injective comodule homomorphism for some π_i because V is simple, where $n = \dim_C V$. Hence V is isomorphic to a sub-comodule of R.

Let I be the (two-sided) ideal of $\mathbf{C}[x, y]$ generated by y. Since $\Delta(y) = y \otimes 1 + x \otimes y \in I \otimes \mathbf{C}[x, y] + \mathbf{C}[x, y] \otimes I$, I is a bi-ideal and thus $\mathbf{C}[x, y]/I$ is a bialgebra. Let $\pi : \mathbf{C}[x, y] \longrightarrow \mathbf{C}[x, y]/I$ be the natural epimorphism.

LEMMA 4. *Let $\tau : V \longrightarrow V \otimes \mathbf{C}[x, y]$ be a comodule structure map. Then*

$$(1 \otimes \pi)\tau : V \longrightarrow V \otimes (\mathbf{C}[x, y]/I)$$

is a $\mathbf{C}[x, y]/I$-comodule structure map.

THEOREM 5. *All simple comodules of $\mathbf{C}[x, y]$ are one dimensional. In fact, all simple sub-comodules of $\mathbf{C}[x, y]$ are $\{\mathbf{C}x^i | i = 0, 1, 2, \dots\}$.*

Proof. By proposition 3, it is enough to show the last statement. Let V be a simple sub-comodule of $\mathbf{C}[x, y]$. Then V is a $\mathbf{C}[x, y]/I$-comodule by lemma 4. If $f \in V$, f can be written as $f = f_0 + xf_1 + \cdots + x^t f_t$, where $f_i \in \mathbf{C}[y]$. Since we have

$$(1 \otimes \pi)\Delta(f) = f_0 \otimes \overline{1} + xf_1 \otimes \overline{x} + \cdots + x^t f_t \otimes \overline{x^t},$$

V can be decomposed as following:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n,$$

where $V_i = \{f \in V | f = x^i g \text{ for some } g \in \mathbf{C}[y]\}$. Choose $0 \neq f \in V_n$ such that the degree of f with respect to y is maximal in V_n. Set $f = x^n(a_k y^k + \cdots + a_1 y + a_0)$. Since

$$\begin{aligned}\Delta(f) &= x^n \otimes x^n[a_k(y \otimes 1 + x \otimes y)^k + \cdots + a_1(y \otimes 1 + x \otimes y) + a_0] \\ &= a_k(x^{n+k} \otimes x^n y^k) + * \in V \otimes \mathbf{C}[x, y],\end{aligned}$$

we have $k = 0$ and so $x^n \in V$. But $\mathbf{C}x^n$ is a sub-comodule of V and thus we have the conclusion.

References

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