Comm. Korean Math. Soc. 8 (1993), No. 2, pp. 191–204

GENERIC THETA–SERIES OF HALF INTEGRAL WEIGHT

MYUNG-HWAN KIM AND YOUNGSO KO

0. Introduction and Notations

In this article, we prove that the generic theta-series of half integral weight is simultaneous eigen-functions with respect to a certain Hecke ring. An analogous result was given by A. N. Andrianov [A1] for integral weight theta-series in 1979.

For $g \in M_{2n}(\mathbf{R})$, let A_g, B_g, C_g, and D_g denote the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively.

Let $G_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R})\,;\, J_n[g] = rJ_n,\ r > 0\}$ where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, $J_n[g] = {}^tgJ_ng$, and $r = r(g)$ is a real number determined by g. Let $\Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z})\,;\, J_n[M] = J_n\}$ and $\mathcal{H}_n = \{Z = {}^tZ \in M_n(\mathbf{C})\,;\, \mathrm{Im}(Z) \text{ is positive definite}\}$. For $g \in G_n$ and $Z \in \mathcal{H}_n$, we set

$$g\langle Z\rangle = (A_gZ + B_g)(C_gZ + D_g)^{-1} \in \mathcal{H}_n.$$

For $Z \in M_n(\mathbf{C})$, let $e(Z) = \exp(2\pi i\sigma(Z))$ where $\sigma(Z)$ is the trace of Z.

For other standard terminologies and basic facts, we refer the readers [A2], [M], [O].

Received April 29, 1992.

This was partially supported by KOSEF Research Grant (Grant No. 901-0101-016-1).

1. Lifted Hecke Rings

Let n, q be positive integers and p be a prime with $gcd(p,q) = 1$. Let $L^n = L^n_p = \{g \in M_{2n}(\mathbf{Z}[p^{-1}])\ ;\ J_n[g] = p^\delta J_n,\ \delta \in \mathbf{Z}\}$ where $\delta = \delta(g)$ is an integer determined by g. Let $\Gamma^n_0(q) = \{M \in \Gamma^n\,;\, C_M \equiv 0 \pmod q\}$ and $L^n_0(q) = L^n_{0,p}(q) = \{g \in L^n\,;\, C_g \equiv 0 \pmod q\}$. Let $\Gamma^n_0 = \{M \in \Gamma^n\,;\, C_M = 0\}$ and $L^n_0 = L^n_{0,p} = \{g \in L^n\,;\, C_g = 0\}$. Then $(\Gamma^n_0(q), L^n_0(q))$ and (Γ^n_0, L^n_0) are Hecke pairs. We denote their corresponding Hecke rings by $\mathcal{L}^n_0(q) = \mathcal{L}^n_{0,p}(q)$ and $\mathcal{L}^n_0 = \mathcal{L}^n_{0,p}$, respectively. We let $E^n = E^n_p = \{g \in L^n\,;\, \delta(g) \in 2\mathbf{Z}\}$, $E^n_0(q) = E^n_{0,p}(q) = E^n \cap L^n_0(q)$, and $E^n_0 = E^n_{0,p} = E^n \cap L^n_0$. Then $(\Gamma^n_0(q), E^n_0(q))$ and (Γ^n_0, E^n_0) are also Hecke pairs, whose corresponding Hecke rings are denoted by $\mathcal{E}^n_0(q) = \mathcal{E}^n_{0,p}(q)$ and $\mathcal{E}^n_0 = \mathcal{E}^n_{0,p}$, respectively. They are the even subrings of $\mathcal{L}^n_0(q)$, and $\mathcal{L}^n_0$, respectively.

Let $\hat{G}_n = \{(g, \alpha(Z))\,;\, g \in G_n, \alpha(Z)^2 = t(\det g)^{-1/2}\det(C_g Z + D_g)$ for some $t \in \mathbf{C}, |t| = 1, Z \in \mathcal{H}_n\}$. $\hat{G}_n$ is a multiplicative group under the multiplication defined by $(g, \alpha(Z))(h, \beta(Z)) = (gh, \alpha(h\langle Z\rangle)\beta(Z))$ and is called the universal covering group of G_n.

Let $\gamma : \hat{G}_n \to G$ be the projection $\gamma(g, \alpha(z)) = g$. We define an action of $\hat{G}_n$ on $\mathcal{H}_n$ by $\zeta\langle Z\rangle = \gamma(\zeta)\langle Z\rangle$ for $\zeta \in \hat{G}_n$, $Z \in \mathcal{H}_n$.

For a moment, we assume $4|q$. Let

$$\theta^n(Z) = \sum_{M \in M_{1,n}(\mathbf{Z})} e({}^tMMZ) = \sum_{N \in M_{n,1}(\mathbf{Z})} e(Z[N]),\ Z \in \mathcal{H}_n. \tag{1.1}$$

$\theta^n(Z)$ is called the standard theta-function. For $M \in \Gamma^n_0(q)$, we define

$$j(M,Z) = \frac{\theta^n(M\langle Z\rangle)}{\theta^n(Z)},\ Z \in \mathcal{H}_n. \tag{1.2}$$

The map $j : \Gamma^n_0(q) \to \hat{G}_n$ defined by $j(M) = (M, j(M,Z))$ is a well defined injective homomorphism [S] such that $\gamma \circ j = 1$ on $\Gamma^n_0(q)$. According to Zhuravlev's argument [Zh1] we may conclude that $(\hat{\Gamma}^n_0(q), \hat{L}^n_0(q))$, $(\hat{\Gamma}^n_0, \hat{L}^n_0)$ are Hecke pairs, where $\hat{\Gamma}^n_0(q) = j(\Gamma^n_0(q)), \hat{\Gamma}^n_0 = j(\Gamma^n_0), \hat{L}^n_0(q) = \gamma^{-1}(L^n_0(q))$, and $\hat{L}^n_0 = \gamma^{-1}(L^n_0)$. We denote their corresponding Hecke rings by $\hat{\mathcal{L}}^n_0(q) = \hat{\mathcal{L}}^n_{0,p}(q), \hat{\mathcal{L}}^n_0 = \hat{\mathcal{L}}^n_{0,p}$, respectively. Similary, $(\hat{\Gamma}^n_0(q), \hat{E}^n_0(q)), (\hat{\Gamma}^n_0, \hat{E}^n_0)$ are Hecke pairs where $\hat{E}^n_0(q) = \gamma^{-1}(E^n_0(q)), \hat{E}^n_0 =$

$\gamma^{-1}(E_0^n)$, and we denote their corresponding Hecke rings by $\hat{\mathcal{E}}_0^n(q) = \hat{\mathcal{E}}_{0,p}^n(q), \hat{\mathcal{E}}_0^n = \hat{\mathcal{E}}_{0,p}^n$, which are the even subrings of $\hat{\mathcal{L}}_0^n(q), \hat{\mathcal{L}}_0^n$, respectively.

It is well known [A1] that there exists an injective homomorphism $\hat{\beta}^n : \hat{\mathcal{L}}_0^n(q) \to \hat{\mathcal{L}}_0^n$ defined by

$$(1.3) \qquad \hat{\beta}^n\left(\sum a_i(\hat{\Gamma}_0^n(q)\zeta_i)\right) = \sum a_i(\hat{\Gamma}_0^n\zeta_i)$$

for any $X = \sum a_i(\hat{\Gamma}_0^n(q)\zeta_i) \in \hat{\mathcal{L}}_0^n(q)$ where ζ_i are chosen to be in $\hat{L}_0^n$.

We also have a well defined surjective ring homomorphism $\pi_k^n : \hat{\mathcal{L}}_0^n \to \mathcal{L}_0^n$ satisfying

$$(1.4) \qquad \pi_k^n(\hat{\Gamma}_0^n\zeta\hat{\Gamma}_0^n) = \tau(\zeta)^{-2k}(\Gamma_0^n g \Gamma_0^n)$$

where k is a positive half integer, i.e., $k = m/2$ for some odd integer $m \geq 1$, $\zeta = (g, \alpha(Z)) \in \hat{L}_0^n$, and $\tau(\zeta) = \dfrac{\alpha(Z)}{|\alpha(Z)|}$.

Let $g_s^n = \mathrm{diag}(I_{n-s}, pI_s, p^2 I_{n-s}, pI_s) \in E_0^n$, $s = 0, 1, \ldots, n$. Let $\hat{T}_s^n = (\hat{\Gamma}_0^n(q)\hat{g}_s^n\hat{\Gamma}_0^n(q)) \in \hat{\mathcal{E}}_0^n(q)$, where $\hat{g}_s^n = (g_s^n, p^{(n-s)/2}) \in \hat{E}_0^n$, and let $\hat{\mathcal{L}}_0^n(T) = \hat{\mathcal{L}}_{0,p}^n(T)$ be the subring $\mathbf{C}[\hat{T}_0^n, \ldots, \hat{T}_{n-1}^n, (\hat{T}_n^n)^{\pm 1}]$ of $\hat{\mathcal{E}}_0^n(q)$. We define

$$\mathbf{L}_0^n(T) = \mathbf{L}_{0,p}^n(T) = (\pi_k^n \circ \hat{\beta}^n)(\hat{\mathcal{L}}_0^n(T)) \subset \mathcal{E}_0^n.$$

Let S_n be the permutation group on $\{x_1, x_2, \ldots, x_n\}$. Let W_n be the group of automorphisms of $\mathbf{C}_n[\underline{x}] = \mathbf{C}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, generated by S_n and σ_i, $i = 0, \ldots, n$, where σ_i are automorphisms of $\mathbf{C}_n[\underline{x}]$ defined by

$$\sigma_0 : x_0 \mapsto -x_0\,;\; x_j \mapsto x_j,\ {}^\forall j \neq 0$$
$$\sigma_i : x_0 \mapsto x_0 x_i\,;\; x_i \mapsto x_i^{-1}\,;\; x_j \mapsto x_j,\ {}^\forall j \neq 0, i,\ \text{for}\, i = 1, \ldots, n.$$

Let $W_n[\underline{x}]$ be the subring of $\mathbf{C}_n[\underline{x}]$ consisting of all W_n-invariant elements. Then we have an isomorphism

$$(1.5) \qquad \psi_n : \mathbf{L}_0^n(T) \to W_n[\underline{x}].$$

For the precise definition of this map, see [A1], [Zh2]. Note that this implies $\mathbf{L}_0^n(T)$ is a commutative ring.

2. Siegel Modular Forms of Half Integral Weight

Let n, q be a positive integers with $4|q$. Let χ be a Dirichlet character modulo q. Let p be a prime with $gcd(p,q)=1$. Let k be a positive half integer. For a complex valued function F on $\mathcal{H}_n$ and $\zeta=(g,\alpha(Z))\in\hat{G}_n$, we set

$$(2.1)\qquad (F|_k\zeta)(Z)=r(g)^{nk/2-<n>}\alpha(Z)^{-2k}F(g\langle Z\rangle),\ Z\in\mathcal{H}_n.$$

where $<n>=n(n+1)/2$.

A function $F:\mathcal{H}_n\to\mathbf{C}$ is called a Siegel modular form of degree n, weight k, level q, with character χ if the following conditions hold : (i) F is holomorphic on $\mathcal{H}_n$, (ii) $F|_k\hat{M}=\chi(\det D_M)F$ for every $\hat{M}=(M,j(M,Z))\in\hat{\Gamma}_0^n(q)$, and (iii) $F|_k(M,\alpha(z))$ is bounded as $\operatorname{Im} z\to\infty$, $z\in\mathcal{H}_1$, for every $(M,\alpha(z))\in\hat{G}_1$ with $M\in SL_2(\mathbf{Z})$ when $n=1$. We denote the set of all such Siegel modular forms by $\mathcal{M}_k^n(q,\chi)$. This is a finite dimensional vector space over $\mathbf{C}$ [Si2].

A function $F:\mathcal{H}_n\to\mathbf{C}$ is called an even or odd modular form of degree n if F satisfies (i), (ii)$'$ $(\det D_M)^sF(M\langle Z\rangle)=F(Z)$, $Z\in\mathcal{H}_n$ for every $M\in\Gamma_0^n$, where $s=0$ for even and $s=1$ for odd modular forms, and (iii)$'$ $F(z)$ is bounded as $\operatorname{Im} z\to\infty$, $z\in\mathcal{H}_1$ when $n=1$. We denote the set of all even modular forms by $\mathcal{M}_0^n$ and odd modular forms by $\mathcal{M}_1^n$. They are also vector spaces over $\mathbf{C}$.

Let $F\in\mathcal{M}_k^n(q,\chi)$ and $\chi(-1)=(-1)^s$ for $s=0$ or 1. For $M\in\Gamma_0^n$, we have $\hat{M}=(M,j(M,Z))=(M,1)$ and $\det D_M=\pm1$. So, F satisfies (ii)$'$ (iii)$'$ and hence

$$\mathcal{M}_k^n(q,\chi)\subset\mathcal{M}_s^n \text{ if } \chi(-1)=(-1)^s.$$

For $F\in\mathcal{M}_k^n(q,\chi)$ and $\hat{X}=\sum a_i(\hat{\Gamma}_0^n(q)\zeta_i)\in\hat{\mathcal{E}}_0^n(q)$, we set

$$(2.2)\qquad F|_{k,\chi}\hat{X}=\sum a_i\chi(\det A_i)F|_k\zeta_i,$$

where $A_i=A_{\gamma(\zeta_i)}$.

As for $F\in\mathcal{M}_s^n$ and $X=\sum a_i(\Gamma_0^n g_i)\in\mathcal{L}_0^n$, we set

$$(2.3)\qquad F|_{k,\chi}X=\sum a_i\chi(\det A_i)F|_k\tilde{g}_i$$

where

$$\tilde{g}_i = (g_i, (\det g_i)^{-1/4} |\det D_i|^{1/2}) \in \hat{L}_0^n, \tag{2.4}$$

$A_i = A_{g_i}$, and $\chi(-1) = (-1)^s$.

$\hat{X}$ and X in (2.2) and (2.3) are well defined operators acting on $\mathcal{M}_k^n(q,\chi)$ and $\mathcal{M}_s^n$, respectively, which are called Hecke operators.

Let $\chi(-1) = (-1)^s$, with $s = 0$ or 1, $F \in \mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_s^n$, and $\hat{X} = \sum a_i(\hat{\Gamma}_0^n(q)\zeta_i) \in \hat{\mathcal{E}}_0^n(q)$, where $\zeta_i = (g_i, \alpha_i(Z)) \in \hat{E}_0^n$ with $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}$. Since $j(M,Z) = 1$ for any $M \in \Gamma_0^n$, from (1.3) and (1.4) follows that

$$(\pi_k^n \circ \hat{\beta}^n)(\hat{X}) = \sum a_i (t_i \varepsilon_i)^{-2k} (\Gamma_0^n g_i) \in \mathcal{E}_0^n$$

where $\varepsilon_i = 1$ or $\sqrt{-1}$ according to $\det D_i > 0$ or $\det D_i < 0$, respectively. So from (2.1)–(2.4) follows

$$\begin{aligned} F\big|_{k,\chi}(\pi_k^n \circ \hat{\beta}^n)(\hat{X}) &= \sum a_i (t_i\varepsilon_i)^{-2k} \chi(\det p^{\delta_i} D_i^*) F\big|_k \tilde{g}_i \\ &= \sum a_i (t_i\varepsilon_i)^{-2k} \chi(\det p^{\delta_i} D_i^*)(p^{\delta_i})^{nk/2-<n>} \cdot \\ &\qquad (p^{-n\delta_i/4} |\det D_i|^{1/2})^{-2k} F(g_i\langle Z\rangle) \\ &= \sum a_i \chi(\det p^{\delta_i} D_i^*)(p^{\delta_i})^{nk-<n>} (t_i (\det D_i)^{1/2})^{-2k} F(g_i\langle Z\rangle) \end{aligned}$$

so that

$$F\big|_{k,\chi}\hat{X} = F\big|_{k,\chi}(\pi_k^n \circ \hat{\beta}^n)(\hat{X}). \tag{2.5}$$

3. Zharkovskaya Operator

Let n, q, χ, p and k be as above. Let $F \in \mathcal{M}_s^n$. We define $\Phi : \mathcal{M}_s^n \to \mathcal{M}_s^{n-1}$ by

$$(\Phi F)(Z') = \lim_{\lambda \to +\infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}\right), \quad Z' \in \mathcal{H}_{n-1} \text{ and } \lambda > 0. \tag{3.1}$$

Φ is well defined and is called the Siegel operator ($\mathcal{M}_s^0 = \mathbf{C}$, $\mathcal{H}_0 = \{0\}$). It is well known [Si1] that

$$\text{(3.2)} \qquad \Phi F \in \mathcal{M}_k^{n-1}(q,\chi) \text{ if } F \in \mathcal{M}_k^n(q,\chi).$$

Let $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$ where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$. By multiplying $\begin{pmatrix} U_i^* & 0 \\ 0 & U_i \end{pmatrix} \in \Gamma_0^n$ for a suitable $U_i \in GL_n(\mathbf{Z})$ from the left of g_i, we may assume that all the D_i are of the form $D_i = \begin{pmatrix} D_i' & * \\ 0 & p^{d_i} \end{pmatrix}$, $d_i \in \mathbf{Z}$, where $D_i' \in V^{n-1}$ is upper triangular. We set

$$\text{(3.3)} \qquad \Psi(X,u) = \sum a_i u^{-\delta_i}(up^{-n})^{d_i}(\Gamma_0^{n-1} g_i') \in \mathcal{L}_0^{n-1}[u^{\pm 1}]$$

where $g_i' = \begin{pmatrix} p^{\delta_i}(D_i')^* & B_i' \\ 0 & D_i' \end{pmatrix} \in L_0^{n-1}$ and $\mathcal{L}_0^{n-1}[u^{\pm 1}]$ is the polynomial ring in u, u^{-1} over $\mathcal{L}_0^{n-1}$. Here B_i' and D_i' denote the blocks of size $(n-1)\times(n-1)$ in the upper left corner of B_i and D_i, respectively. If $n=1$, we set $\Psi(X,u) = \sum a_i u^{-\delta_i}(up^{-1})^{d_i}$. Note that δ_i, d_i are uniquely determined by the left coset $(\Gamma_0^n g_i)$ for each i. $\Psi(-,u)$ is a well defined ring homomorphism : $\mathcal{L}_0^n \to \mathcal{L}_0^{n-1}[u^{\pm 1}]$ (see [Z]).

We define a ring homomorphism $\eta(-,u) : \mathbf{C}_n[\underline{x}] \to \mathbf{C}_{n-1}[\underline{x}, u^{\pm 1}]$ by

$$\begin{cases} x_0 \mapsto x_0 u^{-1}\,;\ x_n \mapsto u\,;\ x_i \mapsto x_i,\ i \neq 0,n & \text{when } n > 1, \\ x_0 \mapsto u^{-1}\,;\ x_1 \mapsto u & \text{when } n = 1\ (\mathbf{C}_0[\underline{x}] = \mathbf{C}). \end{cases}$$

Then the following diagram commutes :

$$\text{(3.4)} \qquad \begin{array}{ccc} \mathcal{L}_0^n & \xrightarrow{\psi_n} & \mathbf{C}_n[\underline{x}] \\ {\scriptstyle \Psi(-,u)}\downarrow & & \downarrow{\scriptstyle \eta(-,u)} \\ \mathcal{L}_0^{n-1}[u^{\pm 1}] & \xrightarrow{\psi_{n-1}\times 1_u} & \mathbf{C}_{n-1}[\underline{x}][u^{\pm 1}] \end{array}$$

where $\psi_{n-1} \times 1_u$ is the ring homomorphism that coincides with ψ_{n-1} on $\mathcal{L}_0^{n-1}$ and fixes u.

We state the following theorem concerning a commuting relation between Hecke operators and the Siegel operator acting on Siegel modular forms of half integral weight.

THEOREM 3.1. *Let $F \in \mathcal{M}_k^n(q,\chi)$ and $\hat{X} \in \hat{\mathcal{E}}_0^n(q)$, where k is a half integer. Then*

$$\Phi(F|_{k,\chi}\hat{X}) = (\Phi F)|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1})$$

where $Y = (\pi_k^n \circ \hat{\beta}^n)(\hat{X}) \in \mathcal{E}_0^n$. (If $n = 1$, then the action on the right hand side is nothing but a multiplication of complex numbers.)

Proof. See [KKO].

For the integral weight Siegel modular forms, the anology was given by Andrianov [A1]. The following result is also given by Andrianov.

THEOREM 3.2. *$\Psi(-,u) : \mathbf{L}^n(T) \to \mathbf{L}^{n-1}(T)$ is a surjective ring homomorphism for any $u \in \mathbf{C}$, $u \neq 0$.*

Proof. See [A1].

4. Theta-Series of Half Integral Weight

Let $Q \in \mathcal{N}_m^+$, where $\mathcal{N}_m^+$ is the set of all positive definite (eigenvalues > 0) semi-integral (diagonal entries and twice of non-diagonal entries are integers) $m \times m$ symmetric matrices. The level q of Q is defined to be the smallest positive integer such that $q(2Q)^{-1}$ is integral with even diagonal entries. It is well known [Og] that q is divisible by 4 when m is odd. We define the theta-series of degree n associated to Q by

$$(4.1)\quad \theta^n(Z,Q) = \sum_{X \in M_{m,n}(\mathbf{Z})} e(Q[X]Z) = \sum_{N \in \mathcal{N}_n} r(N,Q)e(NZ), \quad Z \in \mathcal{H}_n$$

where $r(N,Q) = |\{X \in M_{m,n}(\mathbf{Z});\, Q[X] = N\}| < \infty$.
We have the following theorem :

THEOREM 4.1. *Let $Q \in \mathcal{N}_m^+$, m odd. Then*

$$\theta^n(Z,Q) \in \mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_0^n$$

where $k = m/2$ is a half integer, q is the level of Q, and $\chi = \chi_Q$ is the Dirichlet character modulo q defined by

$$\chi_Q(d) = \left(\frac{2\det 2Q}{|d|}\right)_{\mathrm{Jac}}.$$

Proof. See [K].

Let m, n be positive integers. Let Θ_m^n be the vector space over $\mathbf{C}$ spanned by $\theta^n(Z,Q), Q \in \mathcal{N}_m^+$, and let $\Theta_m^n(q,d)$ be its subspace spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$ with $d = \det 2Q$ and $q =$ the level of Q for given positive integers d and q. If m is odd, then from Theorem 4.1 follows

$$\Theta_m^n \subset \mathcal{M}_0^n \quad \text{and} \quad \Theta_m^n(q,d) \subset \mathcal{M}_k^n(q,\chi)$$

where $\chi(\det D_M) = \left(\frac{2d}{|\det D_M|}\right)_{\text{Jac}}$ for any $M \in \Gamma_0^n(q)$.

Let $Q \in \mathcal{N}_m^+$. We denote the class and the genus of Q by (Q) and $[Q]$, respectively. Obviously $(Q) \subset [Q]$. It is well known that $[Q]$ contains a finite number of classes (see, for instance, [O]). Note that $\theta^n(Z,Q_1) = \theta^n(Z,Q)$ for any $Q_1 \in (Q)$. Also note that $\det 2Q$ and the level of Q are invariants of $[Q]$ and hence

$$\Theta_m^n[Q] \subset \Theta_m^n(q,d) \subset \Theta_m^n$$

if $q =$ the level of Q and $d = \det 2Q$, where $\Theta_m^n[Q]$ is the subspace of Θ_m^n spanned by $\theta^n(Z,Q_i)$, $Q_i \in [Q]$.

It is well known [Si1] that

$$\Phi(\theta^n(Z,Q)) = \theta^{n-1}(Z',Q)$$

where Φ is the Siegel operator (3.1) and $Z = \begin{pmatrix} Z' & * \\ * & * \end{pmatrix} \in \mathcal{H}_n$, $Z' \in \mathcal{H}_{n-1}$. In particular, $\Phi : \Theta_m^n[Q] \to \Theta_m^{n-1}[Q]$, $\Phi : \Theta_m^n(q,d) \to \Theta_m^{n-1}(q,d)$ are epimorphisms for all $n \geq 1$ and isomorphisms [F] if $n > m$.

We now introduce theta operators. Let $m, n \geq 1$ and let p be a prime with $gcd(p,q) = 1$. Let $\alpha : L_0^m \to \mathbf{C}^\times$ be a character such that $\alpha(\Gamma_0^m) = 1$. For $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ O & D_0 \end{pmatrix} \in L_0^m$ and $\theta^n(Z,Q) \in \Theta_m^n$ with $Q \in \mathcal{N}_m^+$, we set

$$(4.2) \qquad \theta^n(Z,Q) \circ_\alpha X = \alpha(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda/\Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D)\theta^n(Z, p^\delta Q[D^*])$$

where $\Lambda = \Lambda^m = SL_m(\mathbf{Z})$ and

$$(4.3) \qquad l_\chi(Q,D) = \sum_{B \in B_\chi(D)/\mathrm{mod}\, D} e(QBD^{-1}).$$

Here $B_\chi(D) = \{B \in M_m(\mathbf{Z}[p^{-1}])\,;\, \begin{pmatrix} p^\delta D^* & B \\ O & D \end{pmatrix} \in \Gamma_0^m g_0 \Gamma_0^m\}$ and B_1, $B_2 \in B_\chi(D)$ are said to be congruent modulo D on the right if $(B_1 - B_2)D^{-1} \in M_m(\mathbf{Z})$. This congruence is obviously an equivalent relation and the summation in (4.3) is over equivalent classes in $B_\chi(D)$ modulo D on the right. We extend (4.2) by linearity to the whole space Θ_m^n and the whole ring $\mathcal{L}_0^m$.

We set

$$\mathcal{L}_{00}^m = \left\{\sum a_i(\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^m\,;\, \delta_i m - 2b_i = 0, b_i = \log_p |\det D_i|\right\}$$

where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^m$ and let $\mathcal{E}_{00}^m = \mathcal{E}_0^m \cap \mathcal{L}_{00}^m$.

We have the following theorem :

THEOREM 4.2.

(1) *The action (4.2) is a well defined action of $\mathcal{L}_0^m$ on Θ_m^n.*
(2) *$\Theta_m^n(q,d)$ is invariant under the theta operators of $\mathcal{L}_{00}^m$ if p is relatively prime to q, the level of Q.*
(3) *$\Theta_m^n[Q]$ is invariant under the theta operators of $\mathcal{E}_{00}^m$ if p is relatively prime to $2q$.*

Proof. See [A1] for m even. Here we assume that m is odd. Let

$$(4.4) \qquad \varepsilon(Z,Q) = \sum_{U \in \Omega} e(Q[U]Z), \ Z \in \mathcal{H}_m,$$

where $\Omega = \Omega^m = GL_m(\mathbf{Z})$. $\varepsilon(Z,Q)$ is called the epsilon-series of Q. For every $M = \begin{pmatrix} D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m$ with $D \in \Omega$, we have

$$(4.5) \qquad \varepsilon(M\langle Z\rangle, Q) = \sum_{U \in \Omega} e(Q[UD^*]Z)e(Q[U]BD^{-1}) = \varepsilon(Z,Q)$$

Note that $e(Q[U]BD^{-1}) = 1$ because $Q[U] \in \mathcal{N}_m^+$ and BD^{-1} is integral symmetric [M]. From (4.5) and the definition of even modular forms follows that $\varepsilon(Z,Q) \in \mathcal{M}_0^m$. Let

$$\mathcal{A}_m = \left\{ \sum a_i \varepsilon(Z,Q_i);\ Q_i \in \mathcal{N}_m^+ \right\} \subset \mathcal{M}_0^m.$$

Let $k = m/2$ and χ be a character satisfying $\chi(-1) = 1$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$. Then

$$X = \sum_{\substack{D \in \Omega \backslash \Omega D_0 \Omega \\ B \in B_\chi(D)/\mathrm{mod}\, D}} (\Gamma_0^m g)$$

where $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}$. From (2.1) and (2.4) one can deduce

$$\varepsilon(Z,Q)\big|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda / \Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_\chi(Q,D)\varepsilon(Z, p^\delta Q[D^*]). \tag{4.6}$$

where $\alpha_{k,\chi} : L_0^m \to \mathbf{C}^\times$ is a character defined by

$$\alpha_{k,\chi}(g) = \chi(p^{\delta m - b}) p^{\delta(mk - <m>) - bk} \tag{4.7}$$

We now define a linear map

$$\vartheta_m^n : \mathcal{A}_m \to \Theta_m^n \text{ by } \vartheta_m^n(\varepsilon(Z,Q)) = \theta^n(Z,Q),\ Q \in \mathcal{N}_m^+.$$

Obviously ϑ_m^n is a well-defined epimorphism. From (4.2) and (4.6) follows

$$\vartheta_m^n(\varepsilon(Z,Q)\,|_{k,\chi} X) = \theta^n(Z,Q) \circ_\alpha X,\ X \in \mathcal{L}_0^m, \tag{4.8}$$

where $\alpha = \alpha_{k,\chi}$ is the character (4.7). Oviously $\theta^n(Z,Q) \circ_\alpha X_1 \circ_\alpha X_2 = \theta^n(Z,Q) \circ_\alpha X_1 X_2$. From the surjectivity of ϑ_m^n, (4.6) and the above follows (1).

For (2) and (3), exactly the same arguments for the case m even [A1] apply here.

5. Main Theorem

Let $Q \in \mathcal{N}_m^+$ with m odd. We set $\Psi = \Psi_Q : \hat{\mathcal{L}}_0^n(T) \to \hat{\mathcal{L}}_0^{n-1}(T)$ by requiring the following diagram commutes :

$$\begin{array}{ccc} \hat{\mathcal{L}}_0^n(T) & \xrightarrow[\pi_k^n \circ \hat{\beta}^n]{\sim} & \mathbf{L}_0^n(T) \\ {\scriptstyle \Psi=\Psi_Q}\downarrow & & \downarrow{\scriptstyle \Psi(-,p^{n-k}\chi_Q^{-1}(p))} \\ \hat{\mathcal{L}}_0^{n-1}(T) & \xrightarrow[\pi_k^{n-1} \circ \hat{\beta}^{n-1}]{\sim} & \mathbf{L}_0^{n-1}(T) \end{array} \tag{5.1}$$

where $k = m/2$ and χ_Q is the character in Theorem 4.1. Since the right vertical arrow is surjective by Theorem 3.2, Ψ is also surjective. We let Ψ^r be the r-th iteration of Ψ for $r > 0$ and $\Psi^0 =$ the identity map. For $\hat{X} \in \hat{\mathcal{L}}_0^{n-r}(T)$, $0 \le r \le n$, let $\Psi^{-r}(\hat{X})$ denote any element in $\hat{\mathcal{L}}_0^n(T)$ whose image under Ψ^r is $\hat{X}$.

Let $X = (\Gamma_0^m g \Gamma_0^m) \in \mathcal{L}_0^m$ for $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. We define the signature $s(X)$ of X by $s(X) = 2b - m\delta$ where $b = \log_p |\det D|$. A linear combination of double cosets with the same signature $s \in \mathbf{Z}$ in $\mathcal{L}_0^m$ is said to be s-homogeneous of signature s. For general $X = \sum_i a_i(\Gamma_0^m g_i) \in \mathcal{L}_0^m$ with $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}$ and $b_i = \log_p |\det D_i|$, we denote the s-homogeneous part of signature s in X by $X_{(s)}$, i.e.,

$$X_{(s)} = \sum_{i, 2b_i - m\delta_i = s} a_i(\Gamma_0^m g_i).$$

Let $\hat{X} \in \hat{\mathcal{L}}_0^m(T)$ and $Y = (\pi_k^m \circ \hat{\beta}^m)(\hat{X}) \in \mathbf{L}_0^m(T)$. We define a homomorphism $\xi^m = \xi_Q^m : \hat{\mathcal{L}}_0^m(T) \to \mathcal{E}_{00}^m$ by

$$\xi^m(\hat{X}) = \sum_{s \ge 0} (\chi_Q(p) p^{m/2})^s Y_{(-2s)} X_m^{+s} \tag{5.2}$$

where

$$X_m^{+s} = p^{-sm} \sum_{\substack{D \in \Lambda^m \backslash M_m(\mathbf{Z})/\Lambda^m \\ \det D = p^s}} \left(\Gamma_0^m \begin{pmatrix} D^* & 0 \\ 0 & D \end{pmatrix} \Gamma_0^m\right) \in \mathcal{E}_0^m.$$

From Theorems 3.1, 4.2, and the above, one can deduce the following by the same argument for even m as in [A1] : Let $m, n \geq 1$ be integers, m odd, $m \geq n$. Let $Q \in \mathcal{N}_m^+$ with level q, $4 \mid q$. Let p be a prime with $gcd(p, q) = 1$. Then for $\hat{X} \in \hat{\mathcal{L}}_0^n(T)$, we have

$$\theta^n(Z, Q)\big|_{k,\chi}\hat{X} = \theta^n(Z, Q) \circ_\alpha \xi^m(\Psi^{n-m}(\hat{X})) \tag{5.3}$$

where $k = m/2$, $\chi = \chi_Q$, and $\alpha = \alpha_{k,\chi}$ (see Theorem 4.1 and (4.7)).

Theorem 4.2 and (5.3) say that $\theta^n(Z, Q)$, $Q \in \mathcal{N}_m^+$, applied by a Hecke operator $\hat{X} \in \hat{\mathcal{L}}_0^n(T)$, can be written as a linear combination of $\theta^n(Z, Q_i)$, $Q_i \in [Q]$.

Let $Q \in \mathcal{N}_m^+$. Let $Q_1, \cdots, Q_h$ be the full set of representatives of the classes in the genus $[Q]$ of Q. We define the generic theta-series of degree n associated to $[Q]$ by

$$\theta^n(Z, [Q]) = \left(\sum_{i=1}^{h} \frac{\theta^n(Z, Q_i)}{e_i}\right)\left(\sum_{i=1}^{h} \frac{1}{e_i}\right)^{-1}, \ Z \in \mathcal{H}_n \tag{5.4}$$

where e_i is the order of the orthogonal group $O(Q_i)$.

THEOREM 5.1. *Let $m \geq n \geq 1$ be integers with m odd. Let $Q \in \mathcal{N}_m^+$. Let q and $\chi = \chi_Q$ be the level and the character of Q, respectively. Let p be a prime relatively prime to q. Then for any $\hat{X} \in \hat{\mathcal{L}}_0^n(T)$,*

$$\theta^n(Z, [Q])\big|_{k,\chi}\hat{X} = \lambda(\hat{X}, \chi)\theta^n(Z, [Q]) \tag{5.5}$$

where $k = m/2$ and the eigenvalue $\lambda(\hat{X}, \chi)$ is determined by :

$$\lambda(\hat{X}, \chi) = f(p^{nk-<n>}\chi(p)^n, p^{1-k}\chi(p)^{-1}, \cdots, p^{n-k}\chi(p)^{-1}). \tag{5.6}$$

where $f(x_0, x_1, \cdots, x_n) = (\psi_n \circ \pi_k^n \circ \hat{\beta}^n)(\hat{X}) \in W_n[\underline{x}]$.

Proof. According to (5.3), it suffices to show that $\theta^n(Z, [Q])$ is an eigenform of any theta operator $X \in \mathcal{E}_{00}^m$. Then by (4.8), this is equivalent to show that $\varepsilon(Z, [Q])$ is an eigenform of any Hecke operator $X \in \mathcal{E}_{00}^m$, where

$$\varepsilon(Z, [Q]) = \left(\sum_{i=1}^{h} \frac{\varepsilon(Z, Q_i)}{e_i}\right)\left(\sum_{i=1}^{h} \frac{1}{e_i}\right)^{-1}.$$

From definition of $\varepsilon(Z,Q)$ in (4.4) follows

$$\varepsilon(Z,[Q]) = \mu^{-1} \sum_{N\in[Q]} e(NZ)$$

where $\mu = \sum_{i=1}^{h} \frac{1}{e_i}$, the mass of $[Q]$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{E}_{00}^m$, $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in E_0^m$. Then

$$X = \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_\chi(D)/\mathrm{mod}\, D}} \left(\Gamma_0^m \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}\right)$$

and hence from (2.3) and (2.4) follows

$$\varepsilon(Z,[Q])\big|_{k,\chi} X = \sum_{D,B} \chi(\det p^\delta D^*)\varepsilon(Z,[Q])\big|_k \tilde{g}$$

where $\tilde{g} = (g, p^{-\delta m/4}|\det D|^{1/2})$, $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}$, and the summation is over $D \in \Omega\backslash\Omega D_0 \Omega$, $B \in B_\chi(D)/\mathrm{mod}\, D$. So from (2.1), (4.4), (4.5), (4.6), and (4.7) follows

$$\begin{aligned}
&\varepsilon(Z,[Q])\big|_{k,\chi} X \\
&= \sum_{D,B} \chi(\det p^\delta D^*)(p^\delta)^{mk/2-<m>}(p^{-\delta m/4}|\det D|^{1/2})^{-2k}\varepsilon(g\langle Z\rangle,[Q]) \\
&= \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{\substack{Q_0\in[Q] \\ D,B}} e(Q_0(p^\delta Z[D^{-1}] + BD^{-1})) \\
&= \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{\substack{Q_0,D \\ p^\delta Q_0[D^*]\in\mathcal{N}_m^+}} l_\chi(Q_0,D) e(p^\delta Q_0[D^*]Z).
\end{aligned}$$

According to Theorem 4.2, $p^\delta Q_0[D^*] \in [Q]$. So we have

$$\varepsilon(Z,[Q])\big|_{k,\chi} X = \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{Q_1\in[Q]} \left(\sum_D l_\chi(p^\delta Q_1[{}^tD],D)\right) e(Q_1 Z).$$

But it is easy to check that $\sum_D l_\chi(p^\delta Q_1[{}^tD], D)$ is independent on $Q_1 \in [Q]$. This proves that $\theta^n(Z,[Q])$ is an eigenform of any Hecke operator $\hat{X} \in \hat{\mathcal{L}}_0^n(T)$. To prove (5.6), we apply Φ^n to (5.5) so that

$$\Phi^n(\theta^n(Z,[Q]))\big|_{k,\chi}\Psi^n(\hat{X}) = \lambda(\hat{X},\chi)\Phi^n(\theta^n(Z,[Q])).$$

But $\Phi^n(\theta^n(Z,[Q])) = 1$ since Q is positive definite. Therefore, we have $\lambda(\hat{X},\chi) = \Psi^n(\hat{X})$ and (5.6) follows immediately from the diagram (5.1).

References

[A1] A. N. Andrianov, *The Multiplicative Arithmetic of Siegel Modular Forms*, Russian Math. Surveys **34** (1979), 75–148.

[A2] ———, *Quadratic Forms and Hecke Operators*, Grundlehren Math. Wiss., vol. 286, Springer-Verlag, 1987.

[F] E. Freitag, *Siegelsche Modulfunktionen*, Springer-Verlag, 1983.

[K] M.-H. Kim, *The Behavior of Theta-Series under Slash Operators*, Comm. Korean Math. Soc. **2** (1987), 9–16.

[KKO] M.-H. Kim, J. K. Koo, Y. Y. Oh, *Hecke Operators and the Siegel Operator*, J. Korean Math. Soc. **26** (1989), 323–334.

[M] H. Maass, *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Math., vol. 216, Springer-Verlag, 1971.

[Og] A. Ogg, *Modular Forms and Dirichlet Series*, Benjamin, 1969.

[O] O. T. O'Meara, *Introduction to Quadratic Forms*, Grundlehren Math. Wiss., vol. 117, Springer-Verlag, 1963.

[S] G. Shimura, *On Certain Reciprocity Laws for Theta Functions and Modular Forms*, Acta Math. **141** (1978), 35–71.

[Si1] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen, I, II, III*, Ann. of Math. **36** (1935), 527–606, ibid., **37** (1936), 230–263, ibid., **38** (1937), 212–291.

[Si2] ———, *Lectures on Analytic Theory of Quadratic Forms*, revised ed., IAS and Princeton Univ. Press, 1949.

[Z] N. A. Zharkovskaya, *The Siegel Operator and Hecke Operators*, Functional Anal. Appl. **8** (1974), 113–120.

[Zh1] V. G. Zhuravlev, *Hecke Rings for a Covering of the Symplectic Groups*, Math. USSR-Sb. **49** (1984), 379–399.

[Zh2] ———, *Expansions of Theta-Transformations of Siegel Modular Forms of Half Integral Weight and their Analytic Properties*, Math. USSR-Sb. **51** (1985), 169–190..

Department of Mathematics
Seoul National University,
Seoul 151-742, Korea