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# $\Delta$–PRIME RINGS AND DIFFERENTIAL OPERATOR RINGS

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## 0. Introduction

In 1984, D. R. Malm([8]) developed necessary and sufficient conditions for certain differential operator rings $R[\theta_1, \ldots, \theta_k; \delta_1, \ldots, \delta_k]$ in $k$ indeterminates and $k$ commuting derivations on a ring $R$ to be simple in his Ph. D. thesis. In 1990, K. A. Brown, K. R. Goodearl and T. H. Lenagan ([2]) studied the prime ideal structure of rings of this type.

The object of this paper is to find some equivalent conditions for certain rings of this type to be prime and then to obtain some results concerning $\Delta$-prime rings by applying these equivalent conditions.

## 1. Preliminaries

Throughout this paper $R$ will denote a ring with 1. In particular, throughout section 2 $R$ will denote a right noetherian ring with 1. Let $\Delta = \{\delta_1, \ldots, \delta_k\}$ be a finite set of commuting derivations from $R$ to itself. Let

$$T = R[\theta_1, \ldots, \theta_k; \delta_1, \ldots, \delta_k] = R[\Theta; \Delta]$$

be the formal linear differential operator ring (Thus the elements of $T$ may be written uniquely as left $R$-linear combinations of the ordered monomials in $\theta_1, \ldots, \theta_k$. Multiplication in $T$ is defined by extending the multiplication from $R$ according to the rules

$$\begin{aligned} \theta_i\theta_j &= \theta_j\theta_i, \\ \theta_i r &= r\theta_i + \delta_i(r) \end{aligned}$$

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for all $i$, $j$ and all $r \in R$). The following multi-index notations will be used: for $I = (i(1), \ldots, i(k))$ with each $i(j)$ a nonnegative integer we set

$$\begin{aligned} \theta^I &= \theta_1^{i(1)} \cdots \theta_k^{i(k)}, \\ \delta^I &= \delta_1^{i(1)} \cdots \delta_k^{i(k)}, \\ \delta^{-I} &= \delta_1^{-i(1)} \cdots \delta_k^{-i(k)}. \end{aligned}$$

In this notation, $T$ is a free left $R$-module with basis $\{\theta^I\}$ ( We omit the subscripts $R$, $\Delta$ as long as $T$ is the only differential operator ring under discussion).

Throughout this paper $N_0$ will be the set of all nonnegative integers and $M_n$ be the set $\{0, 1, \ldots, n\}$. We now fix the following Dixmier ordering ([2; p.757]) on our multi-indices in $(N_0)^k$. Given any multi-indices $I$ and $J$ in $(N_0)^k$, we set $I < J$ in the Dixmier ordering if and only if either $|I| = i(1) + \cdots + i(k) < |J| = j(1) + \cdots + j(k)$ or else $|I| = |J|$ and $I$ is lexicographically less than $J$ (that is, $I \neq J$ and if $m$ is the smallest index for which $i(m) \neq j(m)$, then $i(m) < j(m)$). If $\theta^I$ is any monomial in $T$ and $I \in (N_0)^k$, the order of $\theta^I$ is said to be $I$. For a nonzero $t \in T$, the coefficient of the monomial of maximal order in $t$ with respect to the Dixmier ordering on the multi-indices is called the leading coefficient of $t$. We will usually denote an element $t \in T$ by either

$$t = \sum a_I \theta^I$$

or

$$t = \sum_{j=0}^{n} a_{I_j} \theta^{I_j},$$

where $a_I \in R$, all but finitely many $a_I$ are zero, $a_{I_j} \in R$, $I$, $I_j \in (N_0)^k$ and $0 = I_0 < I_1 < \cdots < I_n$. If $a_J$ is the leading coefficient of $t$, the order of $t$ is said to be $J$. We extend the derivations $\delta_1, \ldots, \delta_k$ on $R$ to derivations on $T$ by setting

$$\delta_i(t) = \theta_i t - t\theta_i$$

for $t \in T$ . Note that this implies $\delta_i(\theta_j) = 0$ for all $i, j$ and if $t = \sum a_I \theta^I$, then $\delta_i(t) = \sum \delta_i(a_I)\theta^I$. The following Leibniz's formulas are introduced in [8; p.4] and [1; p.557] respectively:

$$\delta_j^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta_j^i(a)\delta_j^{n-i}(b),$$
$$\delta^I(ab) = \sum_{M+N=I} ((M,N))\delta^M(a)\delta^N(b)$$

where $j = 1, \dots, k$, $a, b \in R$, $I, M, N \in (N_0)^k$,

$$((M,N)) = (M+N)!/M!N!,\ M! = \prod_{i=1}^{k} m(i)!.$$

We now examine some basic results concerning $\Delta$-ideals and $\Delta$-prime ideals of $R$.

DEFINITION 1.1. Let $\Delta = \{\delta_1, \dots, \delta_k\}$ be a finite set of commuting derivations on $R$. An ideal $A$ of $R$ is said to be a $\Delta$-*ideal* ([8]) if $\delta_i(A) \subset A$ for all $i = 1, \dots, k$.

THEOREM 1.2. *Let $A$ be an ideal of $R$. Then the following statements are true:*

(a) $E_n = \cap_{I \in (M_n)^k} \delta^{-I}(A)$ *for each* $n \in N_0$ *is an ideal of* $R$.
(b) $E = \cap_{I \in (N_0)^k} \delta^{-I}(A)$ *is the largest* $\Delta$-*ideal of* $R$ *contained in* $A$.
(c) $F = R(\cup_{I \in (N_0)^k} \delta^I(A))R$ *is the smallest* $\Delta$-*ideal containing* $A$.

The proof is straightforward.

If $U$ is an ideal of $T$ we denote by $\tau(U)$ the set:

$$\{a \in R \mid a\theta^J + \sum a_I\theta^I \in U, \text{ where } I, J \in (N_0)^k, J \neq 0,\ I < J \text{ and } a_I \in R\}.$$

Clearly $U \cap R \subset \tau(U)$. If a multi-index $L \in (N_0)^k$ is the minimal order of nonzero elements in $U$ then we denote by $\tau_0(U)$ the set:

$$\{a \in R \mid a\theta^L + \sum a_I\theta^I \in U, \text{ where } I \in (N_0)^k,\ I < L \text{ and } a_I \in R\}.$$

LEMMA 1.3. *If $U$ is an ideal of $T$, then $\tau(U)$ and $\tau_0(U)$ are $\Delta$-ideals of $R$.*

*Proof.* Let $a \in \tau(U)$. Then there exists $t$ in $U$ such that

$$t = a\theta^J + \sum a_I\theta^I,$$

where $J \neq 0, I < J$ and $a_I \in R$. Thus for each $i$,

$$\begin{aligned}\theta_i t - t\theta_i &= (\theta_i a - a\theta_i)\theta^J + \text{terms of lower order}\\ &= \delta_i(a)\theta^J + \text{terms of lower order}\end{aligned}$$

is in $U$ and so $\delta_i(a) \in \tau(U)$. Therefore $\tau(U)$ is a $\Delta$-ideal of $R$. Similarly $\tau_0(U)$ is a $\Delta$-ideal of $R$.

DEFINITION 1.4. A proper $\delta_i$-ideal $A$ of $R$ said to be *$\delta_i$-prime* ([6]) if for all $\delta_i$-ideals $B, C$ of $R$ such that $BC \subset A$ either $B \subset A$ or $C \subset A$. A proper $\Delta$-ideal $A$ of $R$ is said to be *$\Delta$-prime* ([2]) if for all $\Delta$-ideals $B, C$ of $R$ such that $BC \subset A$ either $B \subset A$ or $C \subset A$ (It is enough to check this for $\Delta$-ideals $B, C$ that contain $A$). $R$ is called a *$\Delta$-prime ring* if 0 is a $\Delta$-prime ideal of $R$. Certainly, a $\delta_i$-prime ideal of $R$ is $\Delta$-prime.

The following Theorem 1.5 says the relationship between the ideals of $R$ and those of $T$.

THEOREM 1.5. (a) *If $A$ is a $\Delta$-ideal of $R$, then $AT$ is an ideal of $T$.*
(b) *If $P$ is a $\Delta$-prime ideal of $R$, then $PT$ is a prime ideal of $T$.*
(c) *If $U$ is an ideal of $T$, then $U \cap R$ is a $\Delta$-ideal of $R$.*
(d) *If $Q$ is a prime ideal of $T$, then $Q \cap R$ is a $\Delta$-prime ideal of $R$.*

*Proof.* (a) and (c) follows from the definition of multiplication in $T$.
(b) Let $U, V$ be ideals of $T$ such that $U \supset PT, V \supset PT$ and $UV \subset PT$. Then $\tau(U)\tau(V) \subset \tau(UV) \subset \tau(PT) = P$. Since $\tau(U)$ and $\tau(V)$ are $\Delta$-ideals of $R$ by Lemma 1.3 we have either $\tau(U) \subset P$ or $\tau(V) \subset P$. Suppose that $\tau(U) \subset P$. Let $t \in U$ and write $t = \sum_{j=0}^{n} a_{I_j}\theta^{I_j}$, where $a_{I_j} \in R$ and $0 = I_0 < I_1 < \cdots < I_n$. Then $a_{I_n} \in \tau(U) \subset P$ and so $a_{I_n}\theta^{I_n} \in PT \subset U$. Thus

$$a_{I_{n-1}}\theta^{I_{n-1}} + \cdots + a_{I_1}\theta^{I_1} + a_0$$

is in $U$ and $a_{I_{n-1}} \in \tau(U) \subset P$. If we continue this processing it follows that $a_{I_j} \in P$ for all $j = 0, 1, \ldots, n$ and hence $U \subset PT$. Similarly if $\tau(V) \subset P$ then $V \subset PT$.

(d) $Q \cap R$ is a $\Delta$-ideal of $R$ by (c). Let $A, B$ be $\Delta$-ideals of $R$ such that $AB \subset Q \cap R$. Then $AT$ and $BT$ are ideals of $T$ by (a). Also

$$(AT)(BT) \subset (AB)T \subset (Q \cap R)T \subset QT \subset Q$$

and so we have either $AT \subset Q$ or $BT \subset Q$. Therefore either $A \subset Q \cap R$ or $B \subset Q \cap R$.

For a $\Delta$-ideal $A$ of $R$ the derivations $\delta'_1, \ldots, \delta'_k$ on the ring $R/A$ are defined as follows:

$$\delta'_i(r + A) = \delta_i(r) + A$$

for all $i$ and all $r \in R$. The derivations $\delta'_1, \ldots, \delta'_k$ induced on $R/A$ by $\delta_1, \ldots, \delta_k$ again will be denoted by $\delta_1, \ldots, \delta_k$ respectively.

LEMMA 1.6. *If $A$ is a $\Delta$-ideal of $R$ then*

$$T/AT \cong R/A[\Theta; \Delta].$$

*Proof.* Let a mapping $F : T \to R/A[\Theta; \Delta]$ be defined by

$$F(\sum_{j=0}^{n}(a_{I_j}\theta^{I_j})) = \sum_{j=0}^{n}(a_{I_j} + A)\theta^{I_j}$$

for all polynomials $\sum_{j=0}^{n} a_{I_j}\theta^{I_j} \in T$, where $a_{I_j} \in R$, $I_j \in (N_0)^k$ and $0 = I_0 < I_1 < \cdots < I_n$. Then it is easy to show that $F$ is an epimorphism and the kernel of $F$ is $AT$. Thus $T/AT \cong R/A[\Theta; \Delta]$.

LEMMA 1.7. *If $U$ is an ideal of $T$ then*

$$T/(U \cap R)T \cong (R/U \cap R)[\Theta; \Delta].$$

The Proof follows from Theorem 1.5 and Lemma 1.6.

## 2. $\Delta$-prime rings

Throughout this section $R$ will denote a right noetherian ring with 1.

LEMMA 2.1. *$T = R[\theta_1, \ldots, \theta_k; \delta_1, \ldots, \delta_k]$ is a right noetherian ring.*

*Proof.* This follows from [6; Lemma 1.1] and the fact that

$$R[\theta_1, \ldots, \theta_k; \delta_1, \ldots, \delta_k] \cong S[\theta_2, \ldots, \theta_k; \delta_2, \ldots, \delta_k],$$

where $S = R[\theta_1; \delta_1]$.

DEFINITION 2.2. $R$ is said to *have no $Z$-torsion* ([8]) if for all $r \in R$ and all positive integer $n$, $nr = 0$ if and only if $r = 0$. $R$ is called a *Ritt ring* ([4]) if $R$ contains the field of rational numbers as a subring of $R$. It is clear that a Ritt ring has no $Z$-torsion.

We denote the nilpotent radical of $R$ by $N(R)$ and the Jacobson radical of $T$ by $J(T)$.

LEMMA 2.3. *If $R$ has no $Z$-torsion then $N(R)$ is a $\Delta$-ideal of $R$.*

*Proof.* By [6; Lemma 2.1] $\delta_i(N(R)) \subset N(R)$ for all $i$ and hence $N(R)$ is a $\Delta$-ideal of $R$.

We have some equivalent conditions for $T$ to be prime. These are applied to investigate the results concerning $\Delta$-prime rings.

THEOREM 2.4. *The following are equivalent:*
(a) *$T = R[\Theta; \Delta]$ is prime.*
(b) *$R$ is $\Delta$-prime.*
(c) *$N(R)$ is a prime ideal of $R$ and*

$$\cap_{I \in (M_n)^k} \delta^{-I}(N(R)) = 0$$

*for some nonnegative integer $n$.*
(d) *$R$ contains a prime ideal $P$ such that $\cap_{I \in (N_0)^k} \delta^{-I}(P) = 0$.*

*Proof.* (a) $\Leftrightarrow$ (b). This follows from Theorem 3.5.
(b) $\Rightarrow$ (c). Let $R$ be $\Delta$-prime. If $R$ is prime then (c) certainly holds. Suppose that $R$ is not prime. Then, since $R$ is right noetherian, we can choose a nonzero ideal $P$ of $R$ maximal with respect to the property that

$$r[P] = \{a \in R \mid Pa = 0\} \neq 0.$$

To show that $P$ is a prime ideal, assume that $P$ is not a prime ideal of $R$. Then there exist ideals $A, B$ of $R$ such that $AB \subset P$, $A \not\subset P$ and $B \not\subset P$. Since $P$ is maximal with respect to the above property, $r[A+P] = 0$ and $r[B+P] = 0$ so that for all nonzero elements $a \in R$, $(A+P)a \neq 0$ and $(B+P)a \neq 0$. But, since $(A+P)(B+P) \subset P$ and $r[P] \neq 0$, for some nonzero element $b \in R$, $(A+P)(B+P)b = 0$, which is a contradiction and so $P$ is a prime ideal of $R$. Therefore $N(R) \subset P$. We claim that $P$ is nilpotent. For each $l \in N_0$, let

$$P_l = \cap_{I \in (M_l)^k} \delta^{-I}(P),$$

where $M_l = \{0, 1, \ldots, l\}$. Then we have the ascending chain of ideals :

$$0 \neq r[P] = r[P_0] \subset r[P_1] \subset \cdots \subset r[P_l] \subset \cdots.$$

Since $R$ is right noetherian there exists an integer $n$ such that $r[P_n] = r[P_{n+1}]$. To show that $r[P_n]$ is a $\Delta$-ideal of $R$, let $a \in r[P_n] = r[P_{n+1}]$ and $b \in P_{n+1}$. Then $ba = 0$ and so $0 = \delta_i(ba) = \delta_i(b)a + b\delta_i(a)$ for all $i$. But $\delta^I(\delta_i(b)) \in P$ for all $i$ and all $I \in (M_n)^k$ since $b \in P_{n+1}$, so that $\delta_i(b) \in P_n$ for all $i$. Thus $\delta_i(b)a = 0$ and $b\delta_i(a) = 0$ for all $i$, so that $\delta_i(a) \in r[P_{n+1}] = r[P_n]$ for all $i$. Therefore $r[P_n]$ is a $\Delta$-ideal of $R$. Also $l[r[P_n]] = \{s \in R \mid s(r[P_n]) = 0\}$ is a $\Delta$-ideal of $R$ in a similar way. Because $R$ is $\Delta$-prime and $(l[r[P_n]])(r[P_n]) = 0$ we have either $l[r[P_n]] = 0$ or $r[P_n] = 0$. But $0 \neq r[P] \subset r[P_n]$ and $P_n \subset l[r[P_n]]$, so that $P_n = 0$. Using the rule :

$$\delta_i(a_1 \ldots a_{kn+1}) = \sum_{j=1}^{kn+1} a_1 \ldots a_{j-1}\delta_i(a_j)a_{j+1} \ldots a_{kn+1}$$

for all $i$ and all $a_1, \ldots, a_{kn+1} \in P$, it follows that $\delta^I(P^{kn+1}) \subset P$ for all $I \in (M_n)^k$. Thus $P^{kn+1} \subset P_n = 0$ and $P$ is nilpotent, so that $P \subset N(R)$ since $N(R)$ is the largest nilpotent ideal of $R$. Therefore $N(R) = P$ is prime and

$$\cap_{I \in (M_n)^k} \delta^{-I}(N(R)) = P_n = 0.$$

(c) $\Rightarrow$ (b). This is clear.

(d) $\Rightarrow$ (b). Suppose that there exists a prime ideal $P$ of $R$ such that

$$\cap_{I\in(N_0)^k}\delta^{-I}(P)=0.$$

Let $A, B$ be $\Delta$-ideals of $R$ such that $AB=0$. Then either $A \subset P$ or $B \subset P$. If $A \subset P$ then $\delta^I(A) \subset P$ for all $I \in (N_0)^k$ since $A$ is a $\Delta$-ideal of $R$. Thus $A \subset \cap_{I\in(N_0)^k}\delta^{-I}(P)=0$ and $A=0$. Similarly if $B \subset P$ then $B=0$. Therefore $R$ is $\Delta$-prime.

COROLLARY 2.5. *A $\Delta$-prime ring $R$ with no $Z$-torsion is prime.*

*Proof.* Let $R$ be a $\Delta$-prime ring with no $Z$-torsion. Then $N(R)$ is a $\Delta$-ideal of $R$ by Lemma 2.3. Because $R$ is a right noetherian ring $(N(R))^n=0$ for some positive integer $n$. It is easy to check that $N(R)=0$. By Theorem 2.4 $R$ is prime.

In the following theorem we show that prime ideals, $\delta_i$-prime ideals and $\Delta$-prime ideals under certain conditions are the same.

THEOREM 2.6. *Let $R$ be a Ritt ring and $A$ be a $\Delta$-ideal of $R$. Then the following are equivalent:*
(a) *$A$ is a prime ideal of $R$.*
(b) *For each $i$ $A$ is a $\delta_i$-prime ideal of $R$.*
(c) *$A$ is a $\Delta$-prime ideal of $R$.*

*Proof.* (a) $\Leftrightarrow$ (b). This is clear by [7; Proposition 1].
(a) $\Rightarrow$ (c). This is straightforward.
(c) $\Rightarrow$ (a). Let $A$ be a $\Delta$-prime ideal of $R$. Then $R/A$ is a $\Delta$-prime ring. By Theorem 2.4 the nilpotent radical $N(R/A)$ of $R/A$ is a prime ideal of $R/A$ and $\cap_{I\in(M_n)^k}\delta^{-I}(N(R/A))=0$ for some nonnegative integer $n$, that is, $\cap_{I\in(N_0)^k}\delta^{-I}(N(R/A))=0$. Because $R/A$ is also a Ritt ring, $N(R/A)$ is a $\Delta$-ideal by Lemma 2.3. From Theorem 1.2 (b) we have $N(R/A)=0$. Thus it follows that $A$ is a prime ideal of $R$.

If $A$ is an ideal of $R$ we denote by $C_R(A)$ the set:

$$\{c \in R \mid c+A=[c] \text{ is regular in } R/A\}.$$

COROLLARY 2.7. *If $R$ is $\Delta$-prime then $\mathcal{C}_R(0) = \mathcal{C}_R(N(R))$.*

*Proof.* We need only show that $\mathcal{C}_R(N(R)) \subset \mathcal{C}_R(0)$ since the reverse inclusion is known in [3;p.240]. Let $c \in \mathcal{C}_R(N(R))$ and suppose that $cr = 0$ for some $r \in R$. We claim that $\delta^I(r) \in N(R)$ for all $I \in (N_0)^k$ by induction on $|I| = i(1) + \cdots + i(k)$ and hence we have $r = 0$ by Theorem 2.4 (c). By hypothesis $[c][r] = [cr] = [0]$ and $[r] = [0] = N(R)$. Thus $\delta^I(r) = r \in N(R)$ for $|I| = 0$. Assume that $\delta^I(r) \in N(R)$ for all $|I| \leq m$. Let $|I| = m + 1$. Then

$$\begin{aligned} 0 &= \delta^I(cr) \\ &= \sum_{M+N=I} ((M,N))\delta^M(c)\delta^N(r) \\ &= \sum_{M+N=I,|N|\leq m} ((M,N))\delta^M(c)\delta^N(r) + c\delta^I(r) \end{aligned}$$

is in $N(R)$. Thus $c\delta^I(r) \in N(R)$ and $[c][\delta^I(r)] = [c\delta^I(r)] = [0]$. Since $c \in \mathcal{C}_R(N(R))$, $\delta^I(r) \in N(R)$. Therefore $\delta^I(r) \in N(R)$ for all $I \in (N_0)^k$ and so we have $r = 0$ by Theorem 2.4 (c). Similarly we can show that $c$ is regular on the right. Hence $\mathcal{C}_R(0) = \mathcal{C}_R(N(R))$.

COROLLARY 2.8. *Let $R$ be $\Delta$-prime. If $A$ is a nonzero $\Delta$-ideal of $R$ then $A \cap \mathcal{C}_R(0) \neq \phi$.*

*Proof.* Let $A$ be a nonzero $\Delta$-ideal of $R$. Then $\cap_{I\in(N_0)^k}\delta^{-I}(A) = A$ by Theorem 1.2 (b). Thus it follows from Theorem 2.4 (b) that $A \not\subset N(R)$. Therefore the ideal $A+N(R)/N(R)$ of $R/N(R)$ is essential since $R/N(R)$ is prime by Theorem 2.4 (b), so that $A+N(R)/N(R)$ contains a regular element by [5; Lemma 7.2.5], which implies that $A \cap \mathcal{C}_R(N(R)) \neq \phi$. By Corollary 2.7 $A \cap \mathcal{C}_R(0) \neq \phi$.

THEOREM 2.9. *If $R$ is $\Delta$-prime then $T$ is semiprimitive.*

*Proof.* Let $J = J(T)$ and let $0 \neq a \in \tau(J)$. Then $a$ is the leading coefficient of some nonconstant element $t_1$ in $J$. Write $t_1 = a\theta^L + \sum a_I\theta^I$, where $0 \neq L > I$ and $a_I \in R$. Then there exists an element $t_2$ in $T$ such that $(1 + t_1)t_2 = 1$. Comparing leading coefficients of both sides we see that $a$ must be a zero divisor in $R$. Thus $\tau(J)$ consists of 0 and zero divisors. Assume that $J \neq 0$. Then $\tau(J)$ is a nonzero $\Delta$-ideal by Lemma

1.3 and so $\tau(J) \cap \mathcal{C}_R(0) \neq \phi$ by Corollary 2.8, which is a contradiction and hence $J = 0$.

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