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EMBEDDING BORDERED RIEMANN SURFACES IN RIEMANNIAN MANIFOLDS

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0. Introduction

C^∞–embedded surfaces are called *classical surfaces* if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way : the local coordinates are those which preserve angles and orientation.

In his lectures, F. Klein emphasized that classical surfaces should be viewed as Riemann surfaces, that is, as domains of analytic functions and integrals. In 1882, Klein posed the question of whether every Riemann surface is conformally equivalent to a classical surface (see Klein [7, p. 635]).

The first non-trivial result was obtained by Teichmüller ([19]). He deformed an embedded surface by moving each point in the normal direction and studied the dependence of the conformal structures of the perturbed surface on deformation parameters.

Around 1960, A. Garsia ([6]) proved that every compact Riemann surface can be conformally immersed in Euclidean 3-space $\mathbf{R}^3$. He stated that he had found a realization of every compact Riemann surface as a classical surface although Klein required that classical surfaces be embedded. Garsia's proof uses Teichmüller's idea, results, and constructions inspired by Nash's embedding theorem and Brouwer's fixed point theorem.

In 1970, Rüedy extended Garsia's result to open Riemann surfaces $\mathbf{S}$ by applying Garsia's techniques to compact exhaustions of $\mathbf{S}$ ([16]) and later he proved that every compact Riemann surface can be conformally embedded in $\mathbf{R}^3$ ([17], [18]).

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In 1989, author apply Teichmüller theory to prove that we can find a conformally equivalent model surface in an orientable Riemannian manifold $\mathbf{M}$ of dim $\mathbf{M} \geq 3$ for every compact Riemann surface ([8]).

We here prove the *extension of the Embedding theorem for compact Riemann surfaces (Ko [8]) to finite topological type Riemann surfaces in orientable Riemannian manifolds.*

1. The Main Results

Let $\mathbf{M}$ be an orientable Riemannian manifold of dim $\mathbf{M} \geq 3$ and let $\mathbf{S}$ be a closed $\mathcal{C}^\infty$–embedded Riemann surface in $\mathbf{M}$. We briefly examine one method of constructing deformations of $\mathbf{S}$ in $\mathbf{M}$. For computations of some facts, see Ko [8].

Let $\Gamma : \mathbf{S} \hookrightarrow \mathbf{NS} \setminus \Gamma_0$ be a nowhere vanishing smooth section (with unit length) of the normal bundle $\mathbf{NS}$ of $\mathbf{S}$ in $\mathbf{M}$. Let $h : \mathbf{S} \to (-\epsilon, \epsilon)$ be a $\mathcal{C}^\infty$–function on $\mathbf{S}$ and call $\{h(x)\Gamma(x)\}$ a normal vector field on $\mathbf{S}$.

Let $\mathbf{M}_1$ be the subset of $\mathbf{NS}$ consisting of all pairs

$$(x, r) := (x, r\Gamma(x)) \quad \text{for all} \quad x \in \mathbf{S}, \quad \text{where} \quad |r| < 2\epsilon.$$

Then $\mathbf{M}_1$ contains the pair $(x, h(x)\Gamma(x))$. Also let $\mathbf{M}_2$ be the set of all points

$$\{y \in \mathbf{M} : y = \exp r\Gamma(x), r \in (-2\epsilon, 2\epsilon),\ (x, r) \in \mathbf{M}_1\},$$

then $\mathbf{M}_2$ is a Riemannian submanifold of $\mathbf{M}$ for ϵ sufficiently small. Again, for sufficiently small ϵ, the map $\beta : \mathbf{M}_1 \to \mathbf{M}_2$, defined by the exponential map $\beta(x, r) = \exp r\Gamma(x)$, is a diffeomorphism.

By Nash's Embedding Theorem, there is a $\mathcal{C}^\infty$–isometric embedding $j : \mathbf{M} \hookrightarrow \mathbf{R}^m$ for some sufficiently large m. This allows us to consider $\mathbf{S}$ and $\mathbf{M}$ as subsets of $\mathbf{R}^m$.

Assume that $\widetilde{\mathbf{S}}$ is the holomorphic universal covering of $\mathbf{S}$. Let $X : \widetilde{\mathbf{S}} \to \mathbf{M} \subset \mathbf{R}^m$ be a local parametrization of $\mathbf{S}$ in the orientable Riemannian manifold $\mathbf{M} \subset \mathbf{R}^m$.

For $X(z) \in \mathbf{S}$, let

(1.1)

$$\begin{aligned} \alpha_{X(z)} : (-2, 2) &\longrightarrow \mathbf{M} \subset \mathbf{R}^m \\ t &\longmapsto \beta(X(z), th(X(z))) := \exp th(X(z))\Gamma(X(z)). \end{aligned}$$

Then $\alpha_{X(z)}(t)$ is a $\mathcal{C}^\infty$–curve for which $\alpha_{X(z)}(0) = X(z)$ and

$$\alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = \exp h(X(z))\Gamma(X(z)). \tag{1.2}$$

Let $\tilde{\Gamma}(X(z)) \in \mathrm{T}_{X(z)}\mathbf{R}^m$ be a unit tangent vector in $\mathbf{R}^m$ to the curve $\alpha_{X(z)}(t)$ at the point $X(z) \in \mathbf{S}$. We then have

$$\alpha_{X(z)}(1) = X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z))), \quad |h(X(z))| < \epsilon.$$

This is precisely the statement that $h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z)))$ is the tangent vector in $\mathbf{R}^m$ to the curve $\alpha_{X(z)}(t)$ at the point $\alpha_{X(z)}(0) = X(z)$.

Now, for any given sufficiently small $\epsilon > 0$, we define a normal deformation of $\mathbf{S}$.

DEFINITION 1.1. In the above notation, a surface $\mathbf{S}_h \hookrightarrow \mathbf{M}$ is called an *ϵ–normal deformation* of $\mathbf{S} \hookrightarrow \mathbf{M}$ if, for a given small $\epsilon > 0$, h is a $\mathcal{C}^\infty$– real-valued function on $\mathbf{S}$ such that $\|h\|_\infty < \epsilon$ and $\mathbf{S}_h$ is defined by the map :

$$\begin{aligned} \mathcal{S}_h : \quad \mathbf{S} &\longrightarrow \mathbf{M} \subset \mathbf{R}^m \\ X(z) &\longmapsto \alpha_{X(z)}(1) = \beta(X(z), h(X(z))) \\ &= X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2) \in \mathbf{M}. \end{aligned} \tag{1.3}$$

The number $\|h\|_\infty$ is called the *size of the deformation* h.

We prove the Embedding Theorem for compact Riemann surfaces in Ko [8].

THEOREM 1.1. *Assume that* $\mathbf{S}$ *is a compact Riemann surface* $\mathcal{C}^\infty$–*embedded in the orientable Riemannian manifold* $\mathbf{M}$ *of dim* $\mathbf{M} \geq 3$. *Let* $\mathbf{S}_0$ *be any Riemann surface structure on* $\mathbf{S}$. *If there exists a nowhere vanishing smooth section of the normal bundle* $\mathbf{NS}$ *of* $\mathbf{S}$ *in* $\mathbf{M}$, *then*

(1) **[Existence of normal deformations of S]** *There exists an* $\epsilon = \epsilon(\mathbf{S})$ *so that there is an embedded* ϵ–*normal deformation* $\mathbf{S}_h$ *of* $\mathbf{S}$, *of the form given in (1.3).*

(2) **[Existence of Conformal Models]** *There exists an* ϵ*-normal deformation* $\mathbf{S}_h$ *of* $\mathbf{S}$ *which is conformally equivalent to the given Riemann surface* $\mathbf{S}_0$.

We will recall the idea of the proof as we needed in the proof of the main results (Theorem 1.2) in this paper. The **main result** we want to prove in this paper is

THEOREM 1.2. *Theorem 1.1 is valid if* $\mathbf{S}$ *is a hyperbolic Riemann surface,* C^∞*−em-bedded in the orientable Riemannian manifold* $\mathbf{M}$ *of dim* $\mathbf{M} \geq 3$*, of genus* g *with* m *punctures and* n *boundary points and if there exists a nowhere vanishing smooth section of the normal bundle* $N\overline{\mathbf{S}}$ *of* $\overline{\mathbf{S}}$ *in* $\mathbf{M}$*, where* $\overline{\mathbf{S}}$ *is a compactification of* $\mathbf{S}$*.*

It can be shown that if dim $\mathbf{M} \neq 4$, then there always exists a nowhere vanishing section of the normal bundle $\mathbf{NS}$ of $\mathbf{S}$ in $\mathbf{M}$ if $\mathbf{S}$ is compact. When dim $\mathbf{M} = 4$, the nowhere vanishing section of the normal bundle $\mathbf{NS}$ exists if there are no obstructions. In this case the obstruction lies in the Euler class $e(\mathbf{NS})$ of the normal bundle $\mathbf{NS}$. That is, if $e(\mathbf{NS}) = 0$, then there is always such a section. For the proof see Ko [8].

2. Embedding Bordered Riemann Surfaces in Riemannian Manifolds

As usual, $\mathbf{H}$ is a upper half-plane and $\mathbf{L}$ denotes a lower half-plane.

2.1. Teichmüller Space of Bordered Riemann Surfaces

We say that $\mathbf{S}$ is a Riemann surface of *finite conformal type* (g, m) if $\mathbf{S}$ is a surface of genus g with m punctures. A Riemann surface $\mathbf{S}$ is said to be of *finite topological type* (g, m, n) if $\mathbf{S}$ is biholomorphic to a compact genus g surface $\overline{\mathbf{S}}$ from which m points and $n(> 0)$ hyperbolic disks have been removed with $2g - 2 + m + n > 0$.

The surfaces of finite topological type have the finitely generated fundamental group. Two surfaces with finitely generated fundamental group are exactly quasiconformally equivalent if they are of the same type.

In the following special cases, surfaces of the same type are always conformally equivalent;

a. $(0, i, 0)$, $i = 0, 1, 2, 3$: i−components punctured sphere.
b. $(0, i, 1)$, $i = 0, 1$: i−components punctured disk.

c. $(0,0,2)$.

Therefore, for the above cases, the Theorem 1.2 is already proved. Henceforth we omit the above cases. That is assume that $2g-2+m+n>0$.

To define a Teichmüller space for the Riemann surfaces of the type (g,m,n), let $\mathbf{S}_1$ be a topological oriented surface obtained from a closed surface $\overline{\mathbf{S}_1}$ of genus g by removing $m+n$ distinct points which we divide into two sets consisting of m and $n>0$ points repectively. Form a pair $(\mathbf{S}_1, f_1)$ consisting of $\mathbf{S}_1$ and an orientation-preserving quasiconformal mapping $f_1 : \mathbf{S} \to \mathbf{S}_1$. It is required that f_1 takes the punctures of $\mathbf{S}$ into the first set of punctures of $\mathbf{S}_1$. The pair $(\mathbf{S}_1, f_1)$ is called a *marked Riemann surface of the type* (g,m,n). $(\mathbf{S}_1, f_1)$ and $(\mathbf{S}_2, f_2)$ are *conformally equivalent* if and only if $f_2 \circ f_1^{-1}$ is homotopic to the conformal map of $\mathbf{S}_1 \to \mathbf{S}_2$. Then for the Riemann surface $\mathbf{S}$ of the type (g,m,n), $2g-2+m+n>0$, we get

DEFINITION 2.1. The *Teichmüller space* $\mathcal{T}^{\#}(\mathbf{S})$, which is called *the (reduced) Teichmüller space*, is the set of all conformal equivalence classes of marked surfaces $(\mathbf{S}_1, f_1)$ of the type (g,m,n), where $f_1 : \mathbf{S} \to \mathbf{S}_1$. We denote the equivalence class $[(\mathbf{S}_1, f_1)]$ as $[f_1]$.

REMARK. $\mathcal{T}^{\#}(\mathbf{S}) = \mathcal{T}(\mathbf{S})$ if the boundary of $\mathbf{S} = \emptyset$.

Given any Riemann surface $\mathbf{S}$ of type (g,m,n), we define the Teichmüller distance between $[f_1], [f_2] \in \mathcal{T}^{\#}(\mathbf{S})$ by

$$d([f_1],[f_2]) = \inf_h \left\{ \frac{1}{2} \log(\sup_z K_h(z)) \mid h \simeq f_2 \circ f_1^{-1} \right\}, \tag{2.1}$$

where $K_h(z)$, the dilatation of h at z, is defined by

$$K_h(z) = \frac{|h_z(z)| + |h_{\bar{z}}(z)|}{|h_z(z)| - |h_{\bar{z}}(z)|} \tag{2.2}$$

and $\simeq$ denotes the free homotopy.

Since the dilatation of a $K-$quasiconformal mapping is invariant under conformal transformations, this distance is well defined.

PROPOSITION 2.1. *The Teichmüller space $\mathcal{T}^{\#}(\mathbf{S})$ of the Riemann surface of the type (g,m,n), $2g-2+m+n>0$, is a complete metric space under the Teichmüller metric.*

Teichmüller theorem for $\mathbf{S}$ of type (g,m,n), $2g-2+m+n>0$, can be deduced from the theorem for finite conformal type (see Abikoff [1]) applied to $\mathbf{S}^d$ which is a Riemann surface of type $(2g+n-1,2m)$ since we are doubling over the boundary curves. Indeed, any quasiconformal homeomorphism $f:\mathbf{S}\to\mathbf{S}_1$ extends to a quasiconformal homeomorphism $f^d:\mathbf{S}^d\to\mathbf{S}_1^d$ of course, by reflection.

Let G be a nonelementary torsion-free (as usual, normalized) Fuchsian group (of the second kind but finitely generated) uniformizing $\mathbf{S}=\mathbf{H}/G$; then $\mathbf{S}^d=\Omega/G$, where $\Omega=\mathbf{H}\cup\mathbf{L}\cup\{\text{discontinuity region on }\mathbf{R}\}$ is the full region of discontinuity for G. The complex dilatation of $f^d\in L_1^\infty$ is symmetric with respect to conjugation since f^d (and its lift to Ω) has this symmetry.

We wish to find Teichmüller mapping $f_T:\mathbf{S}\to\mathbf{S}_1$ in the homotopy class of f with minimal K_f. The double $f_T^d:\mathbf{S}^d\to\mathbf{S}_1^d$ will in fact then be a Teichmüller mapping between finite conformal type surfaces.

The requirements of the symmetry on *the complex dilatation of* f_T^d implies that its dilatation *is some Teichmüller-Beltrami form on* $\mathbf{S}^d$ *formed from a "symmetric" integrable quadratic differential* $\omega=\phi dz^2\in Q(\mathbf{S}^d)$. But we know that $Q(\mathbf{S}^d)=Q(\Omega)$ corresponds to meromorphic quadratic differentials on $\overline{\mathbf{S}^d}$ (which is not a two sheeted covering of $\mathbf{S}^d$ but a compact Riemann surface of genus $(2g+n-1)$) with at worst simple poles at the $2m$ distinguished points. The *symmetric elements correspond to those* ϕ *which, when lifted to* Ω, *are real on the discontinuity portions of the real axis.* Thus we look at the *real* subspace of symmetric integrable quadratic differentials:

$$Q^{sym}(\Omega)\equiv Q^{sym}(\mathbf{S}^d)=\{\omega\in Q(\Omega):\omega \text{ is real on } \Omega\cap\mathbf{R}\},$$

and then $[f]\in\mathcal{T}(\mathbf{S})$ corresponds uniquely to a Teichmüller-Beltrami form $\mu_f(\lambda,\phi)$ with $\omega=\phi dz^2\in Q^{sym}(\mathbf{S}^d)$ and $0<\lambda<1(\lambda=0$ only for the base point of $\mathcal{T}^{\#}(\mathbf{S}))$. This becomes Teichmüller theorem for $\mathbf{S}$ of finite topological type and as before we get Teichmüller homeomorphism

$$H_T:Q_1^{sym}(\mathbf{S}^d)\to\mathcal{T}^{\#}(\mathbf{S}),$$

where $Q_1^{sym}(\mathbf{S}^d)$ is the open unit ball in a Banach space $Q^{sym}(\mathbf{S}^d)$. To complete the arguments we only need to establish that $Q^{sym}(\mathbf{S}^d)$ is $(6g-6+2m+3n)$-dimensional (over $\mathbf{R}$). In fact, we have

LEMMA 2.1. *$Q(\Omega)$ is a (real) direct sum of the real subspaces $Q^{sym}(\Omega)$ and $iQ^{sym}(\Omega)$. Here $iQ^{sym}(\Omega)$ of course denotes those elements of $Q(\Omega)$ that take pure imaginary values on $\Omega \cap \mathbf{R}$. Note that multiplication by i is a (real-linear) isomorphism of $Q^{sym}(\Omega)$ on $iQ^{sym}(\Omega)$.*

Proof. Since G is a group of real Möbius transformations, one sees that if $\phi \in Q(\Omega)$, then $\overline{\phi(\bar{z})} = \psi(z)$ is also in $Q(\Omega)$. Then $\phi(z) = \frac{1}{2}(\phi(z) + \overline{\phi(\bar{z})}) + \frac{1}{2}(\phi(z) - \overline{\phi(\bar{z})})$ gives the required direct sum decomposition.

When $\mathbf{S}^d$ is of finite conformal type $(2g+n-1, 2m)$, then $Q(\Omega) = Q(\mathbf{S}^d)$ has finite complex dimension $3(2g+n-1)-3+2m = 6g-6+2m+3n$. By the lemma above the symmetric elements $Q^{sym}(\mathbf{S}^d)$ form a real subspace of half the real dimension of $Q(\Omega)$—we are therefore done. Thus far we have shown

THEOREM 2.1. *Suppose $\mathbf{S}$ is a Riemann surface of finite topological type (g, m, n), where $2g-2+m+n > 0$, $n \geq 1$. Then $\mathcal{T}^{\#}(\mathbf{S})$ embeds as the open unit ball $Q_1^{sym}(\mathbf{S}^d)$ in a normed linear vector space $Q^{sym}(\mathbf{S}^d)$ of real dimensions $6g-6+2m+3n$. Furthermore, this $\mathcal{T}^{\#}(\mathbf{S})$ contains that of $\widehat{\mathbf{S}^d}$, the two sheeted covering of the double $\mathbf{S}^d$ of $\mathbf{S}$.*

We can therefore define $\mathcal{T}^{\#}(\mathbf{S})$ of a surface $\mathbf{S}$ of type (g, m, n), where $2g-2+m+n > 0$, $n \geq 1$, identify them with $Q_1^{sym}(\mathbf{S})$. Furthermore, this $\mathcal{T}^{\#}(\mathbf{S})$ contains that of $\widehat{\mathbf{S}^d}$, the two sheeted covering of $\mathbf{S}^d$ of $\mathbf{S}$.

2.2. The Continuity Lemma

Good estimates of the distance between two points in $\mathcal{T}^{\#}(\mathbf{S})$ will be crucial in our arguments. The following Lemma, due to Garsia ([6]), serves this purpose. In order to formulate it, we have to fix, in the holomorphic universal covering space $\widetilde{\mathbf{S}}$ of $\mathbf{S}$, a fundamental domain P for the covering group G. Assume that $\omega \in \mathcal{T}^{\#}(\mathbf{S})$ is a local coordinate for a neighborhood of $[\mathrm{id}_{\mathbf{S}}]$ in $\mathcal{T}(\mathbf{S})$ provided $\|\omega\| \leq 2\epsilon < 1$.

If $f_\omega : \mathbf{S} \to \mathbf{S}_\omega$ is a quasiconformal map and $[f_\omega] \in \mathcal{T}^\#(\mathbf{S})$, then we write $[f_\omega] = \omega$. Let $f_0 : \mathbf{S} \to \mathbf{S}_0$ be a homeomorphism so that $[f_0] \in \mathcal{T}^\#(\mathbf{S})$. Assume that $[f_0] = \omega_0$ and denote by $B_\epsilon(\omega_0) \subset \mathcal{T}^\#(\mathbf{S})$ the set of elements in $\mathcal{T}^\#(\mathbf{S})$ with $\|\omega - \omega_0\| < \epsilon$.

Define, for any $\omega = \phi_\omega(z)dz^2 \in \mathcal{T}^\#(\mathbf{S}) \setminus \{0\}$, a metric on $\mathbf{S}$ by

$$ds^2_\omega := \lambda^2(z)\,|dz + \Psi_\omega(z)d\bar{z}|^2\,, \tag{2.3}$$

where

$$\Psi_\omega(z) = \|\omega\| \frac{\overline{\phi_\omega(z)}}{|\phi_\omega(z)|},$$

and λ^2 is a smooth real-valued $(1,1)$–form. The metric (2.3) defines a new conformal structure on $\mathbf{S}$, which will be denoted $\mathbf{S}_\omega = (\mathbf{S}, ds^2_\omega)$.

Suppose we have two metrics $ds^2_{\omega_1}$ and $ds^2_{\omega_2}$ on $\mathbf{S}$. Then the identity map on $\mathbf{S}$ induces a quasiconformal mapping f between $\mathbf{S}_{\omega_1}$ and $\mathbf{S}_{\omega_2}$. Let w_1 and w_2 be local coordinates on $\mathbf{S}_{\omega_1}$ and $\mathbf{S}_{\omega_2}$ respectively. Then we get $ds^2_{\omega_1} = C_1^2|dw_1|^2$ and $ds^2_{\omega_2} = C_2^2|dw_2|^2$. We claim the dilatation of f in terms of these metrics may be written

$$K_f(w_1) = \sqrt{\sup \frac{ds^2_{\omega_2}}{ds^2_{\omega_1}} \Big/ \inf \frac{ds^2_{\omega_2}}{ds^2_{\omega_1}}},$$

where supremum and infimum are taken over all directions at w_1.

LEMMA 2.2. **(Garsia [6])** *If $[f_\omega] \in \overline{B}_\epsilon(\omega_0)$ and if there is a quasiconformal mapping $\chi : \mathbf{S}_\omega \to \mathbf{S}_{\omega'}$, whose dilatation K_χ satisfies*

(1) *$K_\chi \le K_0$, where K_0 is a dilatation of the extremal quasiconformal map from $\mathbf{S}_\omega$ to $\mathbf{S}_{\omega'}$ homotopic to the identity on $\mathbf{S}$,*
(2) *$K_\chi \le 1 + \delta$ except on $A \subset P$ and*
(3) *area $A \le \eta$,*

then there is a constant $b = b(K_0, \delta, \eta)$ so that

$$\|\omega' - \omega\| \le b(K_0, \delta, \eta).$$

Further, if K_0 is bounded as $(\delta, \eta) \to (0,0)$, then $b(K_0, \delta, \eta) \to 0$.

Proof. See Garsia [6, p. 100 ff].

2.3. The Deformation Function h for Bordered Surfaces

Let $\mathbf{S}$ be a subsurface of a compact Riemann surface in $\mathbf{M}$, its boundary consists of m isolated points p_i $(i = 1, \cdots, m)$ and n analytic boundary curves γ_j $(j = 1, \cdots, n)$, where $2g - 2 + m + n > 0$ so that $\widetilde{\mathbf{S}} = \Delta$. Let $\mathbf{S}_0$ be any Riemann surface structure on $\mathbf{S}$.

As in Section 2.1, we get the double $\mathbf{S}^d$ of $\mathbf{S}$ and construct a doubly sheeted covering $\widehat{\mathbf{S}^d}$ of $\mathbf{S}^d$ and give fundamental domain P in Δ to $\widehat{\mathbf{S}^d}$.

Yet we make the following agreement, because of the homogeneity of the Riemann surface, that through possible distinction we move m distinguished points in one half of $\mathbf{S}^d$ into the other half so that one half of $\mathbf{S}^d$ has $2m$ distinguished points and the other none. We may then think that $\widehat{\mathbf{S}^d}$ originated from $\mathbf{S}^d$ which is of type $(2g+n-1, 2m)$: the distinguished point p_i are linked on $\mathbf{S}^d$ by curves and those curves are mutually disjoint and disjoint from γ_j so that each p_i is either beginning or ending point. Two copies $(\mathbf{S}^d)_i$, $i = 1, 2$, of $\mathbf{S}^d$, are cut along these curves and they are glued crosswise together along these curves. Let Λ be the union of the curves on P which corresponds to these cuts and γ_j. Let O' and O be open neighborhoods of Λ with the following properties;

(a) $\overline{O} \subset O'$.

(b) area $O' < \frac{1}{2}\eta$.

(c) $P - O$ corresponds to the compact subdomain of $\widehat{\mathbf{S}^d}$ which decomposes into 4 connected components (2 if $n = 0$), each of them correspond to compactification $\mathbf{S}_1$ of $\mathbf{S}$.

To this $\mathbf{S}_1$, the compactification of $\mathbf{S}$, we characterize ϵ on $\mathbf{S}_1$. And only on this $\mathbf{S}_1$, $\mathbf{S}$ will be deformed. We identify the surface $\mathbf{S}_1$ with the subdomain P_1 of P.

In the next chapter, we will construct a deformation function h on P_1. We will change h so that it vanishes in the set $O \cap P_1$ and stays unchanged outside the set $O' \cap P_1$.

2.4. The Proof of the Theorem 1.2

Once we have got the deformation function h on P_1, then we have $\mathbf{S}_1^{\omega}$. The natural structure of the deformed surface $\mathbf{S}_1^{\omega}$ will be trans-

planted and it will be extended onto whole $\widehat{\mathbf{S}^d}$ by means of anticonformal mapping of Δ onto itself and suitable transformation from G, the cover group of Δ over $\mathbf{S}^d$, whereby all symmetric properties of $\widehat{\mathbf{S}^d}$ will be transfered onto the deformed surface $\widehat{\mathbf{S}^d}^\omega$.

Because of the symmetry of the surface $\widehat{\mathbf{S}^d}^\omega$, its module belongs to the space $\mathcal{T}^\#(\mathbf{S}) \simeq Q_1^{sym}(\mathbf{S}^d) \simeq \mathbf{B}^{6g-6+2m+3n}$ (open unit ball).

The existence of the families of deformations This only needs a characterization of ϵ on $\mathbf{S}_1$. But it is an easy consequence of the definition of exponential map and the property of the map β. This only requires the existence of a nowhere vanishing section of the normal bundle of $\mathbf{S}_1$.

The existence of the conformal model First denote $\widehat{\mathbf{S}^d}$ by R and $\widehat{\mathbf{S}_0^d}$ by R_0. Let, for convenience, $\phi_0 : R \to R_0$ be a homeomorphism such that $\omega_0' = [\phi_0] \in \mathcal{T}(R)$ is a base point. We fix a map $h : R \times \overline{B}_\epsilon(\omega_0') \to (-\epsilon, \epsilon)$ so that h is a $\mathcal{C}^\infty$-function on R for each fixed ω. Actually this h has been extended from h on $\mathbf{S}_1$. Denote by $[R^\omega] = [(R^\omega, \phi^\omega)]$ the conformal equivalence class of the surface R^ω as a marked surface (R^ω, ϕ^ω). We then define a map Ξ of $\overline{B}_\epsilon(\omega_0')$ to $\mathcal{T}(R) \subset \mathcal{T}^\#(\mathbf{S})$, as noted in the Theorem 2.1, by

$$\begin{aligned} \Xi : \overline{B}_\epsilon(\omega_0') &\longrightarrow \mathcal{T}(R) \subset \mathcal{T}^\#(\mathbf{S}) \\ \omega &\longmapsto [\phi^\omega]. \end{aligned}$$

Here the surface R^ω is the ϵ-normal deformation of R defined by the map

(2.4)

$$\begin{aligned} \phi^\omega := R_{h(\cdot,\omega)} : \quad R &\longrightarrow \mathbf{M} \subset \mathbf{R}^m \\ X(z) &\longmapsto \alpha_{X(z)}(1) = \beta(X(z), h(X(z))) \\ &= X(z) + h(X(z), \omega)\tilde{\Gamma}(X(z)) + O(h^2), \end{aligned}$$

where X is a local coordinate for R. Then, as a result of Brouwer's fixed point theorem, we will have proved the existence of the conformal model if we can prove that, given $[\phi_0] = \omega_0'$ and $\epsilon > 0$, for ω in the

closed ball $\overline{B}_\epsilon(\omega_0') \subset \mathcal{T}(R)$, there is a family of deformations R^ω of R depending on parameters $\omega \in \overline{B}_\epsilon(\omega_0')$ so that the following is true.

LEMMA 2.3. **[Dependence of R^ω on Parameters ω]** *In the above notation,*

(1) $\Xi : \omega \longmapsto [\phi^\omega]$ *is continuous in* $\overline{B}_\epsilon(\omega_0')$

(2) $\|[\phi^\omega] - [id_\omega]\| \leq \epsilon, \forall \omega \in \overline{B}_\epsilon(\omega_0')$, *where* $id_\omega : R \to R_\omega$ *is the set-theoretic identity map.*

Garsia's Continuity Lemma 2.2 implies that the family $\{R^\omega\}$ satisfies property (1) if the coefficients of $(d\phi^\omega)^2$ depend continuously on $(z, \omega) \in \widetilde{R} \times \overline{B}_\epsilon(\omega_0')$. We give an explicit formula for the functions $h(\cdot, \omega)$ in Section 3.2; from the formulas it follows directly that this property is satisfied (see Lemma 3.6).

To prove property (2), we let $\chi = \phi^\omega \circ (id_\omega)^{-1} : R_\omega \to R^\omega$, then its dilatation K_χ satisfies

$$K_\chi^2 = \frac{\sup \frac{(d\phi^\omega)^2}{ds_\omega^2}}{\inf \frac{(d\phi^\omega)^2}{ds_\omega^2}}, \tag{2.5}$$

where both the supremum and infimum are taken over all direction and ds_ω^2 is defined in (2.3). The computation of K_χ is given in Lemma 3.6. The set A is the union of O' and the extension of A in (3.14) to whole of $\widehat{\mathbf{S}^d}$ via above process. We determine the constants δ, η in Section 3.2 for $\mathbf{S}_1$. And extend them to whole $\widehat{\mathbf{S}^d}$. Then use them to get $b(K_0, \delta, \eta) \leq \epsilon$ in Garsia's Continuity Lemma 2.2. Then application of Garsia's Continuity Lemma gives property (2).

By Lemma 2.2, the function Ξ satisfies the hypotheses of Brouwer's fixed point theorem. Therefore there is a point $\omega_1 \in \overline{B}_\epsilon(\omega_0')$ so that

$$\Xi(\omega_1) = [\phi^{\omega_1}] = \omega_0' = [\phi_0], \quad \text{where} \quad \phi_0 : R \to R_0,$$

i.e., for this $\omega_1 \in \overline{B}_\epsilon(\omega_0')$, the deformed surface R^{ω_1} can be mapped conformally onto R_0 by a mapping homotopic to $\phi_0 \circ (\phi^\omega)^{-1}$.

Therefore we obtain the following conformal mapping $\hat{g}$,

$$\hat{g} : \widehat{\mathbf{S}^d}^{\omega_1} \to \widehat{\mathbf{S}_0^d},$$

that is, the proof of the existence of the surface $\widehat{\mathbf{S}^d}^{\omega_1}$ for some $\omega_1 \in \mathcal{T}^{\#}(\mathbf{S})$ in the neighborhood of ω_0'. This mapping $\widehat{g}$ induces a conformal mapping g (see Ahlfors [3]),

$$g : \mathbf{S}^{\omega_1} \to \mathbf{S}_0.$$

Therefore the resulting surface $\mathbf{S}^{\omega_1}$ from $\mathbf{S}$ via deformation in the direction of the normal is the desired conformal model of the given surface $\mathbf{S}_0$. This completes the proof of the theorem for the bordered surfaces if we construct a deformation function h for compact surface $\mathbf{S}_1$.

3. Deformation of a compact surface

Let $\mathbf{S}$ be a compact Riemann surface of genus $g \geq 1$. But procedure actually needed here is that of a compact surface of $g > 1$ since we will construct a deformation function on $\mathbf{S}_1$ which has a fundamental domain P_1 (different from a parallelogram) in Δ as in the previous section. We use the same notations $\widetilde{\mathbf{S}}$, P, X and G as before. We may assume that ∂P has measure zero and P is compact in $\widetilde{\mathbf{S}}$ since $\mathbf{S}$ is compact (see Lehner [12, p. 203-205]). We may identify $\mathcal{T}(\mathbf{S})$ with $Q_1(\mathbf{S})$ which can be deduced from $\mathcal{T}^{\#}(\mathbf{S})$ by letting $m = n = 0$. We will roughly sketch the procedure. Refer to Ko [8] for details.

3.1. The Metric ds^2_ω

The metric of the $\epsilon-$normal deformation $\mathbf{S}_h$, defined by the map $\mathcal{S}_h$ in equation (1.3), of $\mathbf{S}$ satisfies the equation.

$$(d\mathcal{S}_h)^2 = \lambda^2(z)|dz|^2 + (dh)^2 + o(h)|dz|^2. \tag{3.1}$$

Here we use the uniformizing variable $z \in \tilde{\mathbf{S}}$ for $(dX)^2 = \lambda^2|dz|^2$. Let $\chi : \mathbf{S}_\omega \to \mathbf{S}_h$ be a mapping of $\mathbf{S}_\omega$ onto $\mathbf{S}_h$. Let ds^2_ω, given by (2.3), and $(d\mathcal{S}_h)^2$, be metrics for $\mathbf{S}_\omega$ and $\mathbf{S}_h$ respectively. We want to show that the dilatation K_χ of χ satisfies the hypotheses of Garsia's Continuity

Lemma 2.2. We derived an expression for K_χ in equation (2.5). It will be helpful to split ds^2_ω into the form given in (3.1). Let

$$\Pi(\omega) = \left\{ z | z \in \widetilde{\mathbf{S}}, \Im\phi_\omega(z) \neq 0 \right\}. \tag{3.2}$$

The metric ds^2_ω is smooth on the set $\Pi(\omega)$. Let

$$\gamma_\omega := (1 - \|\Psi_\omega\|_\infty)^{-2} = (1 - \|\omega\|)^{-2}. \tag{3.3}$$

Then define the following real valued functions on $\Pi(\omega)$.

(3.4)
$$\alpha^2_\omega := 2\gamma_\omega(\|\Psi_\omega\|_\infty + \Re\Psi_\omega(z)) = 2\gamma_\omega(\|\omega\| + \Re\Psi_\omega(z)),$$
$$\beta^2_\omega := 2\gamma_\omega(\|\Psi_\omega\|_\infty - \Re\Psi_\omega(z)) = 2\gamma_\omega(\|\omega\| - \Re\Psi_\omega(z)).$$

On each connected component of $\Pi(\omega)$, choose continuous (real) branches of $\alpha_\omega, \beta_\omega$ so that

$$\operatorname{sgn}(\alpha_\omega\beta_\omega) = \operatorname{sgn}(\Im\Psi_\omega(z)) \quad \text{and} \quad \beta_\omega > 0. \tag{3.5}$$

Since $dz^2 = dx^2 - dy^2 + 2idxdy$ and $d\bar{z}^2 = dx^2 - dy^2 - 2idxdy$, we get

$$\gamma_\omega ds^2_\omega = \lambda^2(z)\left(|dz|^2 + (\alpha_\omega dx + \beta_\omega dy)^2\right). \tag{3.6}$$

3.2. The Deformation Function h

To complete the Theorem 1.2, we need to describe a deformation function $h : \mathbf{S} \to (-\epsilon, \epsilon)$ which satisfies the following properties :

1. h is $\mathcal{C}^\infty$.
2. $\|h\|_\infty < \epsilon$.
3. $(dh)^2$ is proportional to $(\alpha_\omega dx + \beta_\omega dy)^2$ in view of equations (3.1) and (3.6).

We would like to define a function h satisfying condition 3 except on a sufficiently small set. Condition 3 suggests that we express $(dh)^2$ in terms of α_ω and β_ω. In higher genera ($g > 1$), α_ω and β_ω must be non-constant functions of z. The definition of h will come as a solution of a differential equation in which α_ω, β_ω and their derivatives appear as coefficients. In order to get a $\mathcal{C}^\infty$ solution, we need α_ω, β_ω to be smooth on all of P. Also they, together with their derivatives, must change as little as possible. For h to be well-defined on $\mathbf{S}$, it is convenient that it be zero in a neighborhood of the edges of P but remains smooth.

But $\alpha_\omega, \beta_\omega$ are not yet defined on

$$N_\omega = \{z | z \in \Delta, \Im\phi_\omega(z) = 0\},$$

where $\phi_\omega(z)$ is the holomorphic function (since $\mathbf{S}$ is compact) of z used to define ds^2_ω.

To define them on N_ω, we need several lemmas.

3.2.1. Preparatory lemmas

We consider $\phi_\omega(z)$ as a function of $(z,\omega) \in \Delta \times Q_1(\mathbf{S})$, therefore we denote it by $\phi(z,\omega)$ and list its properties here :

(a) $\phi(z,\omega)$ and $\phi'(z,\omega) := \partial\phi(z,\omega)/\partial z$ are real-analytic on $\Delta \times (Q_1(\mathbf{S}) \setminus \{0\})$,

(b) For each fixed $\omega \in Q_1(\mathbf{S}) \setminus \{0\}$, $\phi(z,\omega)$ is holomorphic and non-constant in Δ.

LEMMA 3.1. *Let $K \subset \Delta$ be compact and $\omega \in Q_1(\mathbf{S}) \setminus \{0\}$. Then the set*

$$\Gamma(a,\omega) := \{z \in K \mid \phi(z,\omega) = a\},$$

varies continuously with ω.

Proof. The set $\Gamma(a,\omega)$ for fixed a and ω is discrete in Δ. For fixed choice of global coordinate z in Δ, ω uniquely defines $\phi_\omega(z)$. Let $\widetilde{\Phi} : Q_1(\mathbf{S}) \to \mathcal{O}(\Delta)$ be defined by $\omega \longmapsto \phi_\omega(z)$, then $\widetilde{\Phi}$ is a linear isomorphism. Therefore $D\widetilde{\Phi}$ is an isomorphism and this implies that $\det D\widetilde{\Phi} \neq 0$. By the Implicit Function Theorem, each zero of the function $\phi(z,\omega) - a$ varies continuously with ω. Since K is compact, there are only finitely many zeroes.

LEMMA 3.2. *Let $K \subset \Delta$ be compact and $\omega \in Q_1(\mathbf{S}) \setminus \{0\}$, then the set*

$$N_\omega \cap K := \{z \in K \mid \Im \phi_\omega(z) = 0\}$$

with Hausdorff topology (given by $d(A,B) = \inf_{x \in A, y \in B} d(x,y)$, where A and B are non-empty subsets of Δ) varies continuously with ω.

Proof. The same argument as in the previous Lemma works with $\Im \phi_\omega(z)$. In fact, this is a real-analytic version of the previous Lemma.

LEMMA 3.3. *Let $K \subset \Delta$ and $M \subset Q_1(\mathbf{S}) \setminus \{0\}$ be compact. Then for every $\eta > 0$, there exists a $\delta > 0$ such that if*

$$N_\omega(\delta) := \{z \mid \text{distance}\,(z, N_\omega) < \delta\}$$

is the tubular neighborhood of N_ω of width δ, then the area of $K \cap N_\omega(\delta)$ is $< \eta$, for each $\omega \in M$.

Proof. As a neighborhood of a finite number of analytic curves, $K \cap N_\omega(\delta)$ has area which tends to zero with δ. By outer measurability, the required δ exists.

LEMMA 3.4. *Suppose $K \subset \Delta$ and $M \subset Q_1(\mathbf{S}) \setminus \{0\}$ are compact. Then for a given $\delta > 0$, there exists a finite number of real numbers $x_1, \cdots, x_n$ such that if*

$$U(x_i, \omega, \delta) = \bigcup_{z_{ij} \in \phi_\omega^{-1}(x_i) \cap K} \{z \mid |z - z_{ij}| < \delta\},$$

then

$$K \cap N_\omega = \{z \in K \mid \Im \phi_\omega(z) = 0\} \subset \cup_{i=1}^n U(x_i, \omega, \delta), \quad \text{for all} \quad \omega \in M.$$

Proof. This can be obtained by combining the Lemma 3.2 and the compactness of the sets K and M.

3.2.2. Auxiliary functions

Suppose $\eta < \frac{1}{2}$ is a fixed number. Define a non-negative C^∞ function $\mu(z)$ by

$$\mu(z) = \begin{cases} 0 & \text{on a neighborhood } U_1 \text{ of } \overline{\Delta - P} \\ 1 & \text{outside a neighborhood } U \supset U_1, \end{cases}$$

where

$$\text{area}(P \cap U) < \eta/4.$$

And let $\tau(x)$ be a C^∞-function for $x \geq 0$ so that $0 \leq \tau(x) \leq 1$ and

$$\tau(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ 1 & \text{for } x \geq 4. \end{cases}$$

By Lemma 3.3, for any compact $M \subset Q_1(\mathbf{S}) \setminus \{0\}$, we can choose a $\delta' > 0$ such that

$$\text{area}(N_\omega(\delta') \cap P) < \eta/4, \quad \text{for each } \omega \in M.$$

Let $\delta = \frac{1}{3} \min(\delta', \text{ distance } (\partial P, \Delta - U_1))$.

By Lemma 3.4, there exist suitable real numbers $x_1, \cdots, x_n$ such that

$$\cup_{i=1}^n U(x_i, \omega, \delta) \supset P \cap N_\omega, \quad \text{for all } \omega \in M. \tag{3.7}$$

For $\omega \in M$ fixed,

$$\Gamma_\omega = \{z_{ij} \in P \mid \phi_\omega(z_{ij}) \in \cup_{i=1}^n \{x_i\}\}.$$

Then set

$$\mu_\eta(z, \omega) := \mu(z) \prod_{z_{ij} \in \Gamma_\omega} \tau\left(\frac{|z - z_{ij}|^2}{\delta^2}\right).$$

Analogously, we can construct a C^∞ function $\mu_{\eta'}(z, \omega)$ in Δ such that

$$\mu_{\eta'}(z, \omega) = \begin{cases} 0 & \text{in neighborhood of } (\Delta - P) \cup N_\omega \\ 1 & \text{if } \mu_\eta(z, \omega) \neq 0. \end{cases}$$

3.2.3. Construction of h

For α_ω and β_ω as in equation (3.4), let

$$\tilde{\alpha_\omega} = \mu_{\eta'} \cdot \alpha_\omega \text{ and } \tilde{\beta}_\omega = \mu_{\eta'} \cdot \beta_\omega + 1 - \mu_{\eta'} \text{ for } z \in \Delta \setminus N_\omega$$

and

$$\tilde{\alpha}_\omega = 0 \quad \text{and} \quad \tilde{\beta}_\omega = 1 \text{ for } z \in N_\omega.$$

Let

$$a = -\frac{\tilde{\alpha_\omega}}{\tilde{\beta}_\omega}, \quad b = \left(\frac{\partial \tilde{\beta}_\omega}{\partial x} - \frac{\partial \tilde{\alpha_\omega}}{\partial y}\right) \cdot \frac{1}{\tilde{\beta}_\omega}$$

and for (x, y) and $(x, y_0) \in \Delta$, set

$$y^* = y - y_0.$$

For each pair (y_0, ω), the ordinary differential equation

$$\frac{dy^*}{dx}(x, y_0, \omega) = a(x, y^* + y_0, \omega), \text{ where } y^*(0, y_0, \omega) = 0 \tag{3.8}$$

has exactly one solution. Moreover $y^*(x, y_0, \omega)$ is continuous in $\Delta \times M$ and it is differentiable in x and y_0 The same is true for the function

$$u^*(x, y_0, \omega) = \int_0^x b(t, y_0 + y^*(t, y_0, \omega), \omega) dt. \tag{3.9}$$

The mapping $g : \Delta \times M \to \Delta \times M$, given by

$$\begin{aligned} g(x, y_0, \omega) &= (x, y_0 + y^*(x, y_0, \omega), \omega) \\ &= (x, y(x, y_0, \omega), \omega), \end{aligned} \tag{3.10}$$

is continuous in $\Delta \times M$ and bijective, therefore it is a homeomorphism by the invariance of domain. Then we have

LEMMA 3.5. *The function*

$$u = u^* \circ g^{-1} : \Delta \times M \to \mathbf{R}^1$$

is the solution of equation

$$\begin{aligned} &\frac{\partial u}{\partial x}(x, y, \omega) + a(x, y, \omega) \frac{\partial u}{\partial y}(x, y, \omega) = b(x, y, \omega), \\ &\qquad u(0, y, \omega) = 0 \quad \text{for} \quad (0, y, \omega) \in \Delta \times M. \end{aligned} \tag{3.11}$$

and is C^∞ in x and y for a fixed $\omega \in M$.

Proof. This can be found in Ko [8, p. 61-63].

From the preceeding arguments u is continuous and bounded on $\Delta \times M$, that is, for some constant u_0,

$$|u(x,y,\omega)| \leq u_0 \text{ for all } (x,y,\omega) \in \Delta \times M.$$

If we let $\varrho = e^{u_0-u} \cdot \left(\tilde{\alpha_\omega} dx + \tilde{\beta_\omega} dy\right)$, then ϱ is closed in Δ, hence it is exact.

Therefore, there is a function k which is continuous on $\Delta \times M$ and it is differentiable in x and y such that

$$\varrho = dk(x,y,\omega).$$

As a final auxiliary function, we define a real-valued function $\nu_\eta(x)$ for $\eta < \frac{1}{2}$ as follow.

(1) $|\dot{\nu_\eta}(x)| \leq 1$,

(2) $\nu_\eta(x) = \begin{cases} x & \text{for } \eta - 1 \leq x \leq 1 - \eta \\ 2 - x & \text{for } 1 + \eta \leq x \leq 3 - \eta, \end{cases}$

(3) $\nu_\eta(x+4) = \nu_\eta(x)$.

Let N_M be a number depending on M and let ϵ be the constant characterized in Section 2.4. For

$$N > N_M + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|,$$

let

(3.12)

$$h(x,y,\omega,N) = \frac{1}{N} \lambda(x,y) \mu_\eta(x,y,\omega) e^{u(x,y,\omega)-u_0} \cdot \nu_\eta(N \cdot k(x,y,\omega)).$$

Then h is a C^∞–function on P and continuous in $\omega \in M$. For ω fixed, we obtain

$$dh^2 = \lambda^2 \cdot \mu_\eta^2 \cdot \dot{\nu_\eta}^2 \cdot (\tilde{\alpha_\omega} dx + \tilde{\beta_\omega} dy)^2 + o(\frac{1}{N})|dz|^2. \tag{3.13}$$

We still want to show the area of

$$A = \{(x,y) | (x,y) \in P,\ \mu_\eta(x,y,\omega) \cdot \dot{\nu_\eta}^2(N \cdot k(x,y,\omega)) \neq 1\} \tag{3.14}$$

can be made arbitrarily small.

For this, for some k_M depending on M, we get

$$\text{area}\, A < \frac{\eta}{2} + k_M \cdot \eta. \tag{3.15}$$

Above computations are not so difficult. See Ko [9] for details.

3.3. Comparison of the Metrics $(dS^\omega)^2$ and ds^2_ω

Recall that the deformed surface $\mathbf{S}^\omega := \mathbf{S}_{h(\cdot,\omega)}$ is defined by

$$S^\omega(x,y) = X(x,y) + h(x,y;\omega,N)\tilde{\Gamma}(X(x,y)) + r(h(x,y)^2).$$

Then for K^2_χ, we will get

LEMMA 3.6. *Assume that $h(x,y,\omega,N)$ is given by (3.12) and that supremum and infimum are taken over all directions at a point z. Then the metric of the deformed surface $\mathbf{S}^\omega := \mathbf{S}_{h(\cdot,\omega)}$, defined by the map $S^\omega(x,y)$ as given above, satisfies the relations :*

(1) $\left(\sup((dS^{\omega_m})^2/(dS^\omega)^2)\right)/\left(\inf((dS^{\omega_m})^2/(dS^\omega)^2)\right) \to 1$ as $\omega_m \to \omega$.

(2) $K^2_\chi = \left(\sup((dS^\omega)^2/ds^2_\omega)\right)/\left(\inf((dS^\omega)^2/ds^2_\omega)\right)$

$$\leq \begin{cases} 1 + c_1(\eta;N) & \text{on } P - A \\ 4\gamma_\omega + c_2(\eta;N) & \text{on } A \end{cases}$$

if $\omega \in M$, where the constant c_1 can be made arbitrarily small for each fixed η and for sufficiently large N. c_2 is some constant which is not necessarily small. The area of A is given by (3.15).

Proof. See Ko [8, p. 71-73].

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