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SMALL ISOMORPHISMS BETWEEN FUNCTION SPACES

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1. Introduction

For a locally compact Hausdorff set X, we denote by $C_0(X)$ the Banach space of all continuous complex valued functions defined on X which vanish at infinity, equipped with the usual sup-norm. In case X is compact, we write $C(X)$ instead of $C_0(X)$. A well-known Banach-Stone theorem states that the function spaces $C(X)$ and $C(Y)$ are linearly isometric if and only if X and Y are homeomorphic. D. Amir [1] and M. Cambern [3] independently generalized this theorem by proving that if $C_0(X)$ and $C_0(Y)$ are linearly isomorphic under an isomorphism T satisfying $\|T\|\|T^{-1}\| < 2$, then X and Y must also be homeomorphic.

Amir-Cambern theorem has been generalized to the Banach spaces of continuous E- valued functions, this is: if T is a linear isomorphism from $C_0(X,E)$ onto $C_0(Y,E)$ such that $\|T\|\|T^{-1}\| < 1+\epsilon$, where the Banach space E satisfies certain geometric conditions, then X and Y are homeomorphic [2,4,5]. In this paper we have a generalization of another version of the Banach-Stone theorem under some geometric assumptions on the Banach spaces.

2. Preliminaries

We use the standard Banach space terminology. For a Banach space E we denote by E_1 the closed unit ball of E and by E^* the dual space. An extremely regular subspace A of $C_0(X)$ means a closed subspace such that if for any $x_0 \in X$, any real number ϵ with $0 < \epsilon < 1$ and any neighborhood V of x_0 there is a function $f \in A$ such that $1 = \|f\| = f(x_0)$ and $|f(x)| < \epsilon$ for every $x \in X \backslash V$. Throughout

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this paper, let A and B be extremely regular subspaces of $C_0(X)$ and $C_0(Y)$, respectively. Let E, F be Banach spaces. We define

$$d_{B-M}(E,F) = \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism from } E \text{ onto } F\},$$

$$\lambda(E) = \inf\{\sup\{\|e_1 + \lambda e_2\| : |\lambda| = 1\} : e_1, e_2 \in E, \|e_1\| = \|e_2\| = 1\}$$

and

$$\mu(E) = \sup\{\inf\{\|e_1 + \lambda e_2\| : |\lambda| = 1\} : e_1, e_2 \in E, \|e_1\| = \|e_2\| = 1\}.$$

We denote by $A\check{\otimes}E$ the complete injective tensor product of A and E. $C_0(X)\check{\otimes}E$ can be naturally identified with $C_0(X,E)$ and $C_0(X,E)$ as a subspace of $C_0(X \times E_1^*)$ or $C_0(X \otimes E_1^*)$, where E_1^* is taken with the weak * topology. For an $f \in C_0(X,E)$ the obvious element of $C_0(X \times E_1^*)$ which corressponds to f is denoted by the same symbol and so

$$e^*(f(x)) = f(x, e^*) \text{ for } x \in X, e^* \in E_1^*.$$

A net $(f_\gamma)_{\gamma \in \Gamma}$ in $A\check{\otimes}E$ (respectively A) peaks at a point $(x,e) \in X \times E_1$ (at $x \in X$) means that if $\|f_\gamma\| \to 1, \|f_\gamma(x) - e\| \to 0$ (respectively $|f_\gamma(x) - 1| \to 0$) and $\|f_\gamma(\cdot)\| \to 0$ uniformly off any neighborhood of x.

For $F \in (A\check{\otimes}E)^*$, $\mu \sim F$ means that μ is a Borel measure on $X \times E_1^*$ which is a norm preserving extension of F to $C_0(X \times E_1^*)$.

The following propositions are seen in [5].

PROPOSITION 2.1. *Assume $\mu \sim F \in (A\check{\otimes}E)^*$. Then for any $x \in X$ there is an $e^* \in E^*$ such that $\mu|_{\{x\}\times E_1^*} \sim \delta_x \otimes e^*$.*

PROPOSITION 2.2. *Assume $F \in (A\check{\otimes}E)^*$, $\mu_i \sim F$ and $\mu_i|_{\{x\}\times E_1^*} \sim \delta_x \times e_i^*$, $i = 1, 2$. Then $e_1^* = e_2^*$.*

PROPOSITION 2.3. *Assume that $\mu \sim F$ and that X_0 is a Borel subset of X. Then*

$$var(\mu|_{X_0 \times E_1^*}) = \|\mu|_{X_0 \times E_1^*}\|,$$

where the norm is taken in $(A\check{\otimes}E)^*$.

Let $T : A\check{\otimes}E \to B\check{\otimes}F$ be a linear map such that $\|f\| \le \|Tf\|$ for $f \in A\check{\otimes}E$ and $\|T\| < 1+\epsilon$. We fix $e \in \partial E_1$ and denote by T_e the map from A into $B\check{\otimes}F$ defined by $T_e(f) = T(f \otimes e)$ for $f \in A$. Evidently T_e is norm non-decreasing and $\|T_e\| \le \|T\|$. Let M be such that

$$\max\left(\frac{1-\epsilon+\|T\|}{2}, \frac{1+\epsilon}{\lambda(F)}\right) < M < 1.$$

For any $x \in X$ we put

$$S_{x,e} = \{y \in Y : \exists f^* \in \partial F_1^* |T_e^*(\delta_y \otimes f^*)(\{x\})| \ge M\},$$

$$\tilde{S}_{x,e} = \{y \in Y : \exists f^* \in F_1^* \; \forall f \in A |f^*(T_e(f)(y)) - f(x)| \le \epsilon\|f\|\},$$

$$S_x = \{y \in Y : \exists f^* \in \partial F_1^* \|T^*(\delta_y \otimes f^*)|_{\{x\}\times E_1^*}\| \ge M\}.$$

For any sequence $(f_n)_{n=1}^\infty$ from A which peaks at x we put

$$A_{n,e} = \{y \otimes f^* \in Y \otimes F_1^* : |T_e f_n(y \otimes f^*)| \ge M\},$$

$$A_{\infty,e} = \cap_{n=1}^\infty \cup_{j=n}^\infty A_{j,e},$$

$$A_{x,e} = \{y \in Y : \exists f^* \in F_1^* \; \exists (f_n)_{n=1}^\infty \subset A \text{ peaking at } x \text{ with } y \otimes f^* \in A_{\infty,e}\}.$$

The following Lemmas 2.4 and 2.5 are seen in [5].

LEMMA 2.4. $\forall_x \in X,\ A_{x,e} \ne \emptyset$.

LEMMA 2.5. $\forall_x \in X,\ A_{x,e} = S_{x,e}$.

LEMMA 2.6. *If* $x_1 \ne x_2 \in X$ *then* $S_{x_1,e_1} \cap S_{x_1,e_2} = \emptyset$ *for any* $e_1, e_2 \in \partial E_1$.

Proof. Assume $S_{x_1,e_1} \cap S_{x_1,e_2} = \emptyset$. Then there are $y_0 \in Y$ and sequences $(f_n^1)_{n=1}^\infty, (f_n^2)_{n=1}^\infty$ in A peaking at x_1 and x_2, respectively, such that

$$\|T(f_n^i \otimes e_i)(y_0)\| \ge M \text{ for } n \in N,\ i = 1,2. \tag{1}$$

We have $\lim_n \|f_n^1 \otimes e_1 + \lambda f_n^2 \otimes e_2\| = 1$ for $|\lambda| = 1$ and

$$\limsup_n \|T(f_n^1 \otimes e_1)(y_0) + \lambda T(f_n^2 \otimes e_2)(y_0)\| \le \|T\| < 1+\epsilon. \tag{2}$$

By (1) and (2) we get $\lambda(F) \le (1+\epsilon)/M$, which contradicts the definition of M.

LEMMA 2.7. *For any* $x \in X$, $S_x = \cup_{e \in \partial E_1} S_{x,e}$.

Proof. Choose $y \in S_x$. Then there is an $f^* \in \partial F_1^*$ such that $\|T^*(\delta_y \otimes f^*)|_{\{x\}\times E_1^*}\| \geq M$. By the Propositions 2.1 and 2.2 there is an $e^* \in E^*$ such that $\|T^*(\delta_y \otimes f^*)|_{\{x\}\times E_1^*}\| = \|\delta_x \otimes e^*\| \geq M$ and there is an $e \in \partial E_1$ such that $e^*(e) \geq M$. Let $(f_n)_{n=1}^{\infty}$ in A be peaking at x such that $1 = |f_n(x)|$. Then $\lim_n |T^*(\delta_y \otimes f^*)(f_n \otimes e)| \geq M$ and $|T_e^*(\delta_y \otimes f^*)(\{x\})| \geq M$. Therefore $y \in S_{x,e}$.

Conversely if $y \in S_{x,e}$ then there is an $f^* \in \partial F_1^*$ such that $|T_e^*(\delta_y \otimes f^*)(\{x\})| \geq M$. Since $\|T^*(\delta_y \otimes f^*)|_{\{x\}\times E_1^*}\| = \|\delta_x \otimes e^*\|$, for any sequence $(f_n)_{n=1}^{\infty}$ in A which is peaking at x we get

$$\begin{aligned} M &\leq \lim_n |T_e^*(\delta_y \otimes f^*)(f_n)| \\ &= \lim_n |T^*(\delta_y \otimes f^*)(f_n \otimes e)| \\ &= \lim_n \|(\delta_x \otimes e^*)(f_n \otimes e)\| \\ &= \|T^*(\delta_y \otimes f^*)|_{\{x\}\times E_1^*}\|. \end{aligned}$$

Therefore $y \in S_x$. This completes the proof.

LEMMA 2.8. *If* $x_1 \neq x_2 \in X$ *then* $S_{x_1} \cap S_{x_2} = \emptyset$.

Proof. Apply Lemma 2.6 and Lemma 2.7.

LEMMA 2.9. *For any* $x \in X$ *and* $f \in A\check{\otimes}E$ *which satisfy* $\|f(x)\| \neq 0$, *we have*

$$\|Tf(x)\| \geq \|f(x)\| - \epsilon\|f\|,\ y \in S_{x,e},\ \text{where } e = f(x)/\|f(x)\|. \tag{3}$$

Proof. For any $x \in X$ and for $f \in A\check{\otimes}E$, let $e = f(x)/\|f(x)\|$ and $y \in S_{x,e} \subset S_x$ where $e = \frac{f(x)}{\|f(x)\|}$. Let $(f_n)_{n=1}^{\infty}$ in A be peaking at x such that $1 = |f_n(x)|$ and let $f^* \in \partial F_1^*$ be given by the definition of $S_{x,e}$. This means

$$\begin{aligned} &T^*(\delta_y \otimes f^*) \sim \lambda\delta_x \otimes e^* + \Delta\mu \in C_0(X \times E_1^*)^*, \\ &\qquad \text{where } |\lambda| \geq M \text{ and } |\Delta\mu|(\{x\} \times E_1^*) = 0,\ \|e^*\| = 1, \\ &T_e^*(\delta_y \otimes f^*) \sim \lambda\delta_x + \Delta\mu' \in C_0(X)^*, \\ &\qquad \text{where } |\lambda| \geq M \text{ and } |\Delta\mu'|(\{x\}) = 0. \end{aligned}$$

Multiplying f^* by a suitable scalar of modulus one we can assume that λ is a positive real number. Since $|\Delta\mu|(\{x\} \times E_1^*) = 0$ we get that $\lim_n |\Delta\mu|(f_n \otimes \|f(x)\|e) = 0$ and by Proposition 2.3 and the definition of M we get

$$\begin{aligned} &\lim_n |f^*(T(f - f_n \otimes \|f(x)\|e)(y))| \\ &\quad \leq \lim_n |\lambda e^*((f - f_n \otimes \|f(x)\|e)(x))| \\ &\qquad + \lim_n |\Delta\mu(f - f_n \otimes \|f(x)\|e)| \\ &\quad \leq \|\Delta\mu\| \|f\| + \lim_n |\Delta\mu(f_n \otimes \|f(x)\|e)| \\ &\quad = \|\Delta\mu\| \|f\| \\ &\quad \leq (\|T\| - \lambda)\|f\|. \end{aligned} \tag{4}$$

Since $\Delta\mu'(\{x\}) = 0$, we know that $\lim_n \Delta\mu'(f_n) = 0$ and

$$\begin{aligned} &\lim_n |f^*(T(f_n \otimes \|f(x)\|e)(y)) - \|f(x)\|| \\ &= \lim_n |T_e^*(\delta_y \otimes f^*)|(\|(f(x)\|f_n) - \|f(x)\|| \\ &\leq \lim_n |\lambda\|f(x)\|f_n(x) - \|f(x)\|| + \lim_n |\Delta\mu'(\|f(x)\|f_n)| \\ &= \lim_n \|f(x)\||\lambda f_n(x) - 1| + \lim_n \|f(x)\||\Delta\mu'(f_n)| \\ &\leq (\|f(x)\||\lambda - 1)|. \end{aligned} \tag{5}$$

BY (4) and (5)

$$\begin{aligned} &|f^*(Tf(y)) - \|f(x)\|| \\ &\leq \lim_n |f^*(T(f - f_n \otimes \|f(x)\|e)(y))| \\ &\quad + \lim_n |f^*(T(f_n \otimes \|f(x)\|e)(y)) - \|f(x)\|| \\ &\leq \|f\|(|\lambda - 1| + \|T\| - \lambda) \\ &\leq \|f\|(\|T\| + (1 - 2M)) \\ &= \|f\|(\|T\| - 1 + 2(1 - M)) \\ &\leq \epsilon\|f\|. \end{aligned}$$

Therefore $\|Tf(y)\| \geq \|f(x)\| - \epsilon\|f\|$.

3. Results

THEOREM 3.1. *Let X be a locally compact metric space, Y a locally compact Hausdorff space and A, B extremely regular subspaces of $C_0(X)$ and $C_0(Y)$, respectively. Let E, F be Banach spaces and let $T : A\check{\otimes}E \to B\check{\otimes}F$ be a linear map such that $\|f\| \leq \|Tf\|$ for $f \in A\check{\otimes}E$ and $\|T\| < 1+\epsilon < \lambda(F)$. Then there is a subset Y_0 of Y and a continuous surjective map $\phi : Y_0 \to X$ such that for any $x \in X$ and $f \in A\check{\otimes}E$*

$$(*) \qquad \sup_{y\in\phi^{-1}(\{x\})} \|Tf(y)\| \geq \|f(x)\| - \epsilon\|f\|.$$

Proof. We define a function $\phi : \cup_{x\in X} S_x \subset Y \to X$ by $\phi(y) = x$ if $y \in S_x$. By Lemma 2.8 ϕ is well defined, by Lemmas 2.4, 2.5 and 2.7 ϕ is surjective and by Lemma 2.9 ϕ satisfies $(*)$. It remains to prove that ϕ is continuous. Assuming the contrary there are $x_n \in X, y_n \in Y$ and an open neighborhood V of x_0 such that $y_n \in S_{x_n} \to y_0 \in S_{x_0}$ and $x_n \in X\backslash\overline{V}$ for all $n \in N$. Fix $\delta > 0$. Since $y_0 \in S_{x_0} = \cup_{e\in\partial E_1} S_{x,e}$ there is an $f_1 \in A$ and an $e_0 \in \partial E_1$ such that $\|f_1\| \leq 1+\delta, f_1(x_0) = 1, |f_1(x)| \leq \delta$ for $x \in X\backslash V$ and $\|T_{e_0} f_1(y_0)\| > M - \delta$. Next since $T_{e_0} f_1$ is norm continuous and $y_n \to y_0$ there is an $n_0 \in N$ such that $\|T_{e_0} f_1(y_{n_0})\| > M - \delta$. Since $y_{n_0} \in S_{x_{n_0}} = \cup_{e\in\partial E_1} S_{x_{n_0},e}$ there is an $f_2 \in A$ and an $e_{n_0} \in \partial E_1$ such that $\|f_2\| \leq 1+\delta, f_2(x_{n_0}) = 1, |f_2(x)| \leq \delta$ for $x \in V$ and $\|T_{e_{n_0}} f_2(y_{n_0})\| > M - \delta$. We have

$$\|T_{e_0} f_1(y_{n_0})\| > M - \delta, \quad \|T_{e_{n_0}} f_2(y_{n_0})\| > M - \delta$$

and

$$\|T_{e_0} f_1(y_{n_0}) \pm \|T_{e_{n_0}} f_2(y_{n_0})\| \leq \|T\|\|f_1\otimes e_0 \pm f_2\otimes e_{n_0}\| \leq (1+\epsilon)(1+2\delta).$$

Hence, since δ is an arbitrary positive number we get $\lambda(F) \leq (1+\epsilon)/M$, which contradicts the definition of M and so ends the proof.

To prove Theorem 3.3 we need the following lemma, which is due to K.Jarosz [5].

LEMMA 3.2. *Let X, Y be locally compact spaces and E, F Banach spaces and $T : C_0(X,E) \to B\check{\otimes}F$ a linear map such that $\|f\| \leq \|Tf\|, \|T\| < 1+\epsilon < 4/(2+\mu(F^*))$. Then for any $e \in \partial E_1$, there is a subset $\tilde{Y}_e$ of Y and a continuous surjective map $\phi_e : \tilde{Y}_e \to X$ such that*

$$|f_y^*(T(f\otimes e)(y)) - f\circ\phi_e(y)| \leq \epsilon\|f\|, \quad f \in C_0(X),$$

where $Y_e \ni y \to f_y^*$ is a map from Y_e into $\partial f_1^*, \tilde{Y}_e = \cup_{x\in X}\tilde{S}_{x,e}$ and $\phi_e(y) = x$ if $y \in \tilde{S}_{x,e}$.

THEOREM 3.3. *Let X be a locally compact space, Y a locally compact metric space, B an extremely regular subspace of $C_0(Y)$, E, F Banach spaces and $T : C_0(X,E) \to B\check{\otimes}F$ a surjective isomorphism such that $\|T\| < 1+\epsilon < min(\lambda(E), \frac{4}{2+\mu(F^*)})$ and $\|T^{-1}\| \leq 1$. Then there is a homeomorphism ϕ from Y onto X.*

Proof. We consider two possibilities:

(i) max(dim$E$, dim$F$)> 1,

(Ii) dim$E$ = dim$F$ = 1.

Assuming (i) we have $1+\epsilon < \min(\lambda(E), \frac{4}{2+\mu(F^*)}) \leq \sqrt{2}$. By Lemma 3.2, if $e \in \partial E_1$ there is a surjective map $\psi_e : \tilde{Y}_e \to X$ defined by $\psi_e(y) = x$ if $y \in \tilde{S}_{x,e}$ where $\tilde{Y}_e = \cup_{x\in X}\tilde{S}_{x,e}$. By Theorem 3.1 for $T' = \|T\|T^{-1}$ in place of T, we get a subset X_0 in X and a continuous map $\phi : X_0 \to Y$ such that $\phi(x) = y$ if $x \in S_y$. Since ϕ is continuous we can extend $\phi : X_0 \to Y$ to $\overline{\phi} : \overline{X}_0 \to Y$. Let $x \in X, e \in \partial E_1$ and let a net (f_γ) in $C_0(X)$ be peaking at x with $1 = \|f_\gamma(x)\|$. By Lemma 3.2, there is a $y_0 \in \tilde{S}_{x,e} = \psi_e^{-1}(\{x\})$ such that

$$\|(T(f_\gamma\otimes e)(y_0))\| \geq \|(f_\gamma\otimes e)(x)\| - \epsilon\|f_\gamma\otimes e\|. \tag{6}$$

By Lemma 2.9, for each γ there exists $x_\gamma \in S_{y_0}$ such that

$$\|T\|\|(f_\gamma\otimes e)(x_\gamma)\| \geq \|T(f_\gamma\otimes e)(y_0)\| - \epsilon\|T(f_\gamma\otimes e)\|. \tag{7}$$

By (6) and (7) we get

$$\|T\|\|(f_\gamma\otimes e)(x_\gamma)\| \geq \|(f_\gamma\otimes e)(x)\| - \epsilon\|f_\gamma\otimes e\| - \epsilon\|T(f_\gamma\otimes e)\|.$$

Since $\|f_\gamma\| \to 1$, we get $\lim_\gamma \|T\|\|(f_\gamma\otimes e)(x_\gamma)\| \geq 1-\epsilon(2+\epsilon) > 1 - (\sqrt{2}-1)(\sqrt{2}+1) = 0$. Hence we get $x_\gamma \to x$ and so $\psi_e\circ\overline{\phi}(x) = x$. Since

x is an arbitrary element of X, we have $X_0 = X$ so $\psi_e \circ \phi(x) = Id_X(x)$. Therefore S_y is a one-point set for all y in Y. Therefore $\phi : X \to Y$ is bijective and $\psi_e : Y \to X$ is bijective. Since ψ_e is continuous by Lemma 3.2 and ϕ is continuous by Theorem 3.1, $\phi : X \to Y$ is a homeomorphism.

Assuming (ii) we have $\epsilon < 1$ and by Lemma 3.2

$$|\epsilon(y)T(f)(y) - f \circ \phi(y)| \le \epsilon \|f\| \text{ for } f \in C_0(X), \tag{8}$$

where $|\epsilon(y)| \le 1$ for $y \in Y_0$. By the symmentry arguments and Theorem 3.1 we have also

$$|\epsilon'(x)\|T\|T^{-1}(g)(x) - g \circ \psi(x)| \le \epsilon \|g\| \text{ for } g \in B, \tag{9}$$

where $|\epsilon'(x)| = 1$. Let x_0, y_0, x_1 be such that $\phi(y_0) = x_0$ and $\psi(x_1) = y_0$. By (8) and (9) we get

$$|f(x_0) - \epsilon(y_0)\epsilon'(x_1)\|T\|f(x_1)| \le \epsilon(\|f\| + |\epsilon(y_0)|\|Tf\|), \quad f \in C_0(X).$$

By the regularity of A we get $x_0 = x_1$, and hence ϕ is a homemorphism.

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