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THE DIRICHLET EIGENVALUE ESTIMATE ON A COMPACT RIEMANNIAN MANIFOLD

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§1. Introduction

Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. We consider the following Dirichlet eigenvalue problem on M of the equation

$$(1.1) \qquad \begin{aligned} \Delta u &= -\lambda u \ \text{ in } \ M \\ u &\equiv 0 \ \text{ on } \ \partial M. \end{aligned}$$

It is well known that the set of eigenvalues $\{\lambda_k\}$ of (1.1) can be arranged in a nondecreasing order as follows:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots$$

and $\lambda_1 \int u^2 \leq \int |\nabla u|^2$ for all $u \in \overset{\circ}{H}{}_1^2(M)$. For application, it is important to estimate the lower bound of λ_1. In case that M is a submanifold of some manifold N, the Ricci curvature of N has the positive lower bound in M and the average curvature of ∂M is nonnegative, Sperb showed a lower bound of λ_1 in [2]. For compact Riemannian manifold with nonconvex boundary, the first Neumann eigenvalue is estimated by R.Chen[1]. Our purpose in this paper is to estimate the lower bound of λ_1 on compact Riemannian manifold with nonconvex boundary.

DEFINITION. Let ∂M be the boundary of a compact Riemannian manifold M. Then ∂M satisfies the "interior rolling ε-ball" condition if for each point $p \in \partial M$, there is a geodesic ball $B_q(\frac{\varepsilon}{2})$, centered at $q \in M$ with radius $\frac{\varepsilon}{2}$, such that $p = \overline{B_q(\frac{\varepsilon}{2})} \cap \partial M$ and $B_q(\frac{\varepsilon}{2}) \subset M$.

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THEOREM 1.1. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. Let ∂M satisfy "the interior rolling ϵ-ball" condition. Let R, K and H be nonnegative constants such that the Ricci curvature of M is bounded below by $-R$, the mean curvature of ∂M is bounded below by $-K$ and the second fundamental form elements of ∂M is bounded below by $-H$. If u is a solution of the equation*

$$\begin{aligned} \Delta u + \lambda_1 u = & \text{ in } M \\ u \equiv 0 & \text{ on } \partial M, \end{aligned} \tag{1.2}$$

where λ_1 is the first Dirichlet eigenvalue. Then

$$\lambda_1 \geq \frac{1}{\sqrt{1+2(n-1)\varepsilon K}} \frac{(1-\alpha^2)}{(n-1)\rho^2}(1+B)\exp(-(1+B)),$$

where

$$\begin{aligned} B =& \{1 + \frac{(n-1)\rho^2}{1-\alpha^2}C\}^{\frac{1}{2}}, \\ C =& \frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1+2(n-1)\varepsilon K\}^{\frac{1}{2}}R \\ &+ (n-1)K\{\frac{1}{\varepsilon} + 2(n-1)(1+3H)\}, \quad 0 < \alpha \leq \frac{1}{2}. \end{aligned}$$

and ρ is the radius of the largest geodesic ball contained in M and the upper bound of ε is given by (2.11) and (2.12).

In §2, we shall give a gradient estimate which is essential in proof of Theorem 1.1. In §3, we shall give a proof of Theorem 1.1.

§2 A gradient estimate

LEMMA 2.1.[2]. *Let S^{n-1} be a hypersurface of the Riemannian manifold M and Δ_s the Laplacian in the induced metric of S^{n-1}. At any point of S^{n-1}, the following relation holds:*

$$\Delta u = \Delta_s u + (n-1)K_0 \frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2},$$

where K_0 is the mean curvature, ν is the outward normal vector.

THEOREM 2.2.. *Let M and ∂M satisfy the hypothesis of Theorem 1.1. If u is a solution of the equation (1.2), then*

$$\frac{|\nabla u|^2}{(\beta-u)^2}(x) \leq \frac{(n-1)}{(1-\alpha^2)}\left[\frac{(2n-3)^2+\alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1+2(n-1)\varepsilon K\}^{\frac{1}{2}}R + \frac{2\beta}{\beta - supu}\{1+2(n-1)\varepsilon K\}^{\frac{1}{2}}\lambda_1 + C_1\right]$$

where $C_1 = (n-1)\,K\,\{\frac{1}{\varepsilon}+2(n-1)(1+3H)\}$, and $\beta > supu$.

Proof. Let $\psi(r)$ be a nonnegative C^2-function defined on $[0,\infty)$ such that,

$$\psi(r) = \begin{cases} \leq 2(n-1)\varepsilon K & \text{if } r\in[0,\frac{\varepsilon}{2}) \\ 2(n-1)\varepsilon K & \text{if } r\in[\varepsilon,\infty) \end{cases}$$

with $\psi(0)=0$, $\psi'(0)=4(n-1)\,K$, $0\leq\psi'(r)\leq 4(n-1)\,K$ and $\psi''\geq -\frac{2(n-1)K}{\varepsilon}$. Define $\phi(x) = \psi(r(x))$, where $r(x)$ denotes the distance function from boundary ∂M to $x\in M$. For $\beta > 1 = \sup u$, we define the function

$$G(x) = (1+\phi)^{\frac{1}{2}}\,\frac{|\nabla u|^2}{(\beta-u)^2} \quad \text{on } M.$$

By the compactness of M, there is a point $x_0\in M$ such that G achives its maximum. Suppose that x_0 is a boundary point of ∂M. At x_0, we may choose an orthonormal frame field $e_1, e_2, \ldots, e_n$ such that $e_n = \frac{\partial}{\partial\nu}$, where $\frac{\partial}{\partial\nu}$ is the unit outward normal vector. Then we have

$$\text{(2.1)}\qquad \frac{\partial G}{\partial\nu}(x_0) = \frac{1}{2}\frac{\partial\phi}{\partial\nu}\frac{|\nabla u|^2}{(\beta-u)^2} + 2\frac{\partial u}{\partial\nu}\frac{\partial^2 u}{\partial\nu^2}\frac{1}{(\beta-u)^2} + \frac{2|\nabla u|^2}{(\beta-u)^3}\frac{\partial u}{\partial\nu} \geq 0.$$

From (1.2), it is clear that

$$\Delta u(x_0) = -\lambda_1 u(x_0) = 0,$$
$$\Delta_{\partial M}\, u(x_0) = 0.$$

Let K_0 be the mean curvature at x_0. By Lemma 2.1, at x_0,

$$\Delta u = \Delta_{\partial M} u + (n-1) K_0 \frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2}.$$

Hence, we have

$$\frac{\partial^2 u}{\partial \nu^2} = -(n-1) K_0 \frac{\partial u}{\partial \nu} = (n-1) K_0 |\nabla u|, \quad \text{at} \quad x_0 \in \partial M. \tag{2.2}$$

Using (2.2), we obtain that

$$\begin{aligned}
\frac{\partial G}{\partial \nu}(x_0) &= \frac{1}{2}\frac{\partial \phi}{\partial \nu}\frac{|\nabla u|^2}{(\beta-u)^2} - 2(n-1) K_0 \frac{|\nabla u|^2}{(\beta-u)^2} - 2\frac{|\nabla u|^3}{(\beta-u)^3} \\
&\le \frac{|\nabla u|^2}{(\beta-u)^2}\left(\frac{1}{2}\frac{\partial \phi}{\partial \nu} + 2(n-1) K - 2\frac{|\nabla u|}{(\beta-u)}\right. \\
&= \frac{|\nabla u|^2}{(\beta-u)^2}\left(-2(n-1) K + 2(n-1) K - 2\frac{|\nabla u|}{(\beta-u)}\right) < 0.
\end{aligned}$$

This contradicts (2.1). Therefore x_0 is an interior point of M and hence, $\nabla G(x_0) = 0$ and $\Delta G(x_0) \le 0$. At x_0, we may choose an orthonormal frame field $\{e_i\}$ such that $u_1(x_0) = |\nabla u(x_0)|$. Since, for each i, $G_i(x_0) = 0$, we obtain that

$$\begin{aligned}
&\text{if } i \ne 1, \quad u_{1i} = -\frac{1}{4}(1+\phi)^{-1} \phi_i |\nabla u|, \\
&\text{if } i = 1, \quad u_{11} = -\frac{|\nabla u|^2}{(\beta-u)} - \frac{1}{4}(1+\phi)^{-1} \phi_1 |\nabla u|.
\end{aligned} \tag{2.3}$$

It is clear that

(2.4)

$$\begin{aligned}
\sum_{i,j=1}^{n} (u_{ji})^2 &\ge \sum_{i=1}^{n} u_{ii}^2 \ge u_{11}^2 + \frac{1}{n-1}(\Delta u - u_{11})^2 \\
&\ge u_{11}^2 + \frac{u_{11}^2}{2(n-1)} - \frac{(\Delta u)^2}{n-1} \\
&= \frac{2n-1}{2(n-1)}\left(\frac{|\nabla u|^4}{(\beta-u)^2} + \frac{1}{2}(1+\phi)^{-1}\phi_1 \frac{|\nabla u|^3}{(\beta-u)}\right. \\
&\quad \left. + \frac{1}{16}(1+\phi)^{-2}\phi_1^2 |\nabla u|^2\right) - \frac{1}{n-1}\lambda_1^2 u^2.
\end{aligned}$$

By using (2.3) and $u_{ijk} - u_{ikj} = \sum_{l=1}^{n} u_l R_{lijk}$, we have

(2.5)

$$
\begin{aligned}
\Delta G(x_0) = &-\frac{1}{4}(1+\phi)^{-\frac{3}{2}}\,|\nabla\phi|^2\,\frac{|\nabla u|^2}{(\beta-u)^2}+\frac{1}{2}(1+\phi)^{-\frac{1}{2}}\,\Delta\phi\,\frac{|\nabla u|^2}{(\beta-u)^2}\\
&+2(1+\phi)^{-\frac{1}{2}}\,\phi_1\,\frac{u_1}{(\beta-u)^2}\,(-\frac{|\nabla u|^2}{(\beta-u)}-\frac{1}{4}(1+\phi)^{-1}\,\phi_1|\nabla u|)\\
&+\sum_{i=2}^{n}2(1+\phi)^{-\frac{1}{2}}\,\phi_i\,\frac{u_1}{(\beta-u)^2}\,(-\frac{1}{4}(1+\phi)^{-1}\,\phi_i|\nabla u|)\\
&+2(1+\phi)^{-\frac{1}{2}}\phi_1\,\frac{|\nabla u|^3}{(\beta-u)^3}+\sum_{i,j=1}^{n}2(1+\phi)^{\frac{1}{2}}\,\frac{(u_{ji})^2}{(\beta-u)^2}\\
&+\sum_{j=1}^{n}2\frac{(1+\phi)^{\frac{1}{2}}}{(\beta-u)^2}\,(u_j(\Delta u)_j)+\sum_{i,j=1}^{n}2\frac{(1+\phi)^{\frac{1}{2}}}{(\beta-u)^2}\,u_ju_iR_{ij}\\
&+\frac{8(1+\phi)^{\frac{1}{2}}}{(\beta-u)^3}\,|\nabla u|^2\,(-\frac{|\nabla u|^2}{(\beta-u)}-\frac{1}{4}(1+\phi)^{-1}\,\phi_1|\nabla u|)\\
&+2(1+\phi)^{\frac{1}{2}}\,(\frac{-\lambda_1u|\nabla u|^2}{(\beta-u)^3})+6(1+\phi)^{\frac{1}{2}}\,\frac{|\nabla u|^4}{(\beta-u)^4}.
\end{aligned}
$$

Multiplying (2.5) by $(1+\phi)^{\frac{1}{2}}\,\frac{(\beta-u)^2}{|\nabla u|^2}$ and substituting (2.4), we have

$$
\begin{aligned}
0 \geq &\frac{(1+\phi)}{(n-1)}\,\frac{|\nabla u|^2}{(\beta-u)^2}+\frac{-(2n-3)}{2(n-1)}\,\phi_1\,\frac{|\nabla u|}{(\beta-u)}\\
&-\frac{2(1+\phi)}{(n-1)}\,\frac{\lambda_1^2u^2}{|\nabla u|^2}+\frac{(2n-1)}{16(n-1)}\,(1+\phi)^{-1}\,\phi_1^2\\
&-\frac{2\lambda_1u(1+\phi)}{(\beta-u)}-2R(1+\phi)-2\lambda_1(1+\phi)\\
&-\frac{3}{4}(1+\phi)^{-1}\,|\nabla\phi|^2+\frac{1}{2}\Delta\phi.
\end{aligned}
\tag{2.6}
$$

It is clear that

$$
\begin{aligned}
&\frac{\alpha^2(1+\phi)}{(n-1)}\,\frac{|\nabla u|^2}{(\beta-u)^2}-\frac{(2n-3)}{2(n-1)}\,\phi_1\,\frac{|\nabla u|}{(\beta-u)}\\
&\geq -\frac{1}{16}\,\frac{(2n-3)^2}{\alpha^2(n-1)}\,(1+\phi)^{-1}\,\phi_1^2.
\end{aligned}
\tag{2.7}
$$

Substituting (2.7) into (2.6), we have, for $0 < \alpha < 1$,

$$
\begin{aligned}
(2.8)\qquad 0 \geq & \frac{(1-\alpha^2)(1+\phi)\,|\nabla u|^2}{(n-1)\ \ (\beta-u)^2} \\
& - \frac{1}{16\alpha^2}\frac{(2n-3)^2}{(n-1)}(1+\phi)^{-1}\phi_1^2 \\
& - \frac{2(1+\phi)\lambda_1^2u^2}{(n-1)|\nabla u|^2} + \frac{(2n-1)}{16(n-1)}(1+\phi)^{-1}\phi_1^2 \\
& - \frac{2\lambda_1 u}{(\beta-u)}(1+\phi) - 2(1+\phi)(R+\lambda_1) \\
& - \frac{3}{4}(1+\phi)^{-1}\,|\nabla\phi|^2 + \frac{1}{2}\Delta\phi.
\end{aligned}
$$

Multiplying (2.8) by $\frac{|\nabla u|^2}{(\beta-u)^2}$, we obtain

(2.9)

$$
\begin{aligned}
0 \geq & \frac{(1-\alpha^2)}{(n-1)}G(x_0)^2 - G(x_0)\{\frac{(2n-3)^2}{16\alpha^2(n-1)}(1+\phi)^{-\frac{3}{2}}\phi_1^2 \\
& + \frac{2\lambda_1 u}{(\beta-u)}(1+\phi)^{\frac{1}{2}} - \frac{(2n-1)}{16(n-1)}(1+\phi)^{-\frac{3}{2}}\phi_1^2 \\
& + 2(1+\phi)^{\frac{1}{2}}(R+\lambda_1) + \frac{3}{4}(1+\phi)^{-\frac{3}{2}}\,|\nabla\phi|^2 \\
& - \frac{1}{2}\,\Delta\phi(1+\phi)^{-\frac{1}{2}}\} - \frac{2(1+\phi)\lambda_1^2u^2}{(\beta-u)^2(n-1)}.
\end{aligned}
$$

From (2.9), we obtain that

(2.10)

$$
\begin{aligned}
0 \geq & \frac{(1-\alpha^2)}{(n-1)}G(x_0)^2 \\
& - \{\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)}(1+\phi)^{-\frac{3}{2}}\,|\nabla\phi|^2 \\
& + 2(1+\phi)^{\frac{1}{2}}\,(R+\lambda_1) + \frac{2\lambda_1 u}{\beta-u}(1+\phi)^{\frac{1}{2}} \\
& - \frac{1}{2}\Delta\phi(1+\phi)^{-\frac{1}{2}}\}\,G(x_0) \\
& - \frac{2(1+\phi)\lambda_1^2u^2}{(\beta-u)^2(n-1)}, \quad \text{for } \ 0 < \alpha \leq \frac{1}{2}.
\end{aligned}
$$

To compute $\Delta\phi$, let $\partial M(\varepsilon)$ be the set $\{x \in M \,|r(x) \leq \varepsilon\}$, and k_ε be the upper bound of the sectional curvature in $\partial M(\varepsilon)$. We may choose ε to be small so that

$$\sqrt{k_\varepsilon}\tan(\varepsilon\sqrt{k_\varepsilon}) \leq \frac{H}{2} + \frac{1}{2} \tag{2.11}$$

$$\frac{H}{\sqrt{k_\varepsilon}}\tan(\varepsilon\sqrt{k_\varepsilon}) \leq \frac{1}{2}. \tag{2.12}$$

By using an indezx comparison theorem in Riemannian geomerty [3], one can show that if $x \in \partial M(\varepsilon)$, we have

$$\Delta r \geq -(n-1)\frac{H + \sqrt{k}_\varepsilon \tan(\varepsilon\sqrt{k}_\varepsilon)}{1 - \tan(\varepsilon\sqrt{k}_\varepsilon)H/\sqrt{k}_\varepsilon} \geq -(n-1)(3H+1).$$

Then we have

$$\begin{aligned} \Delta\phi &= \psi''|\nabla r|^2 + \psi'\Delta r \\ &\geq -\frac{2(n-1)K}{\varepsilon} - 4(n-1)^2K(3H+1). \end{aligned} \tag{2.13}$$

Let $C_1 = (n-1)K(\frac{1}{\varepsilon} + 2(n-1)(1+3H))$.

Substituting (2.13) into (2.12), (2.12) becomes

$$\begin{aligned} 0 \geq &(\frac{1-\alpha^2}{n-1})\, G(x_0)^2 - \{\frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 \\ &+ 2(1+2(n-1)\varepsilon K)^{\frac{1}{2}}\, R \\ &+ \frac{2\beta}{\beta - \text{sup}u}(1 + 2(n-1)\,\varepsilon K)^{\frac{1}{2}}\,\lambda_1 + C_1\}\, G(x_0) \\ &- \frac{2(1+2(n-1)\,\varepsilon K)\,\lambda_1^2\,(\text{sup}u)^2}{(n-1)(\beta - \text{sup}u)^2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{|\nabla u|^2}{(\beta - u)^2}(x) &\leq G(x_0) \\ &\leq \frac{(n-1)}{1-\alpha^2}\Bigg[\frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 \\ &\quad + 2(1+2(n-1)\,\varepsilon K)^{\frac{1}{2}} R \\ &\quad + \frac{2\beta}{\beta - \text{sup}u}(1 + 2(n-1)\,\varepsilon K)^{\frac{1}{2}}\,\lambda_1 + C_1\Bigg]. \end{aligned}$$

REMARK. *By the "interior rolling ε-ball" condition, we can choose a geodesic from boundary to x_0 which has no focal point. Hence we can use the index comparison theorem.*

§3. Proof of Theorem 1.1

Proof. We may assume that $0 \leq u \leq 1$. From Theorem 2.2, we know that

$$\frac{|\nabla u|}{(\beta - u)} \leq \left(\frac{n-1}{1-\alpha^2}\right)^{\frac{1}{2}} \left[\frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} R + \frac{2\beta}{\beta - 1}\{\frac{1}{\varepsilon} + 2(n-1)\varepsilon K)^{\frac{1}{2}} \lambda_1 + C_1\right]^{\frac{1}{2}}.$$

Let x_M be a point in M where u assumes its maximum, and let $\overline{x}$ be a point on ∂M nearest to x_M as geodesic distance. Let ρ be the radius of the largest geodesic ball contained in M. Integrating (1), we have

$$\log\frac{\beta}{\beta-1} \leq (\frac{n-1}{1-\alpha^2})^{\frac{1}{2}} \rho \left[\frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} R + \frac{2\beta}{\beta - 1}\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} \lambda_1 + C_1\right]^{\frac{1}{2}}.$$

Hence we obtain the first Dirichlet eigenvalue

$$\lambda_1 \geq \frac{1}{\sqrt{1 + 2(n-1)\varepsilon K}} \frac{\beta - 1}{2\beta} \left\{(\frac{1-\alpha^2}{n-1}) \frac{1}{\rho^2} (\log\frac{\beta}{\beta-1})^2 - C\right\},$$

where

$$C = \frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} R + C_1.$$

Let

$$f(\beta) = \frac{1}{2\sqrt{1+2(n-1)\varepsilon K}} \frac{\beta-1}{\beta} \left\{ \frac{(1-\alpha^2)}{(n-1)\rho^2} (\log \frac{\beta}{\beta-1})^2 - C \right\},$$

Then $f(\beta)$ has a maximum at $\frac{\beta}{\beta-1} = e^{1+\sqrt{1+\frac{(n-1)}{1-\alpha^2}\rho^2 C}}$. Hence

$$\lambda_1 \geq \frac{1}{\sqrt{1+2(n-1)\varepsilon K}} \frac{(1-\alpha^2)}{(n-1)\rho^2}(1+B)\exp(-(1+B)),$$

where

$$B = \left\{1 + \tfrac{(n-1)\rho^2}{1-\alpha^2} C\right\}^{\frac{1}{2}},$$

$$C = \frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2}(n-1)K^2 + 2\{1+2(n-1)\varepsilon K\}^{\frac{1}{2}} R + C_1$$

and

$$C_1 = (n-1)K\{\tfrac{1}{\varepsilon} + 2(n-1)(1+3H)\}.$$

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