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REMARKS ON VARIATIONAL INEQUALITIES AND GENERALIZED QUASI–VARIATIONAL INEQUALITIES

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1. Introduction

The problem of solving quasi-variational inequality (in short, QVI) was originally introduced by Bensoussan and Lions in 1973 (see [**3**]) in connection with some stochastic impulse control problems. Problems of QVI arise in the applications such as free-boundary problems, complementarity problems, mathematical economics and management. For a paper in which quasi-variational inequalities are studied, see Mosco [**12**].

Inspired by earlier works, some authors, e.g., Ding, Shih and Tan considered an abstract generalized quasi-variational inequality problem (in short, GQVI) from a theoretical standpoint. Shih and Tan [**14**, Theorems 3 and 4] obtained existence theorems for GQVI and Kim [**9**] gave a generalization of Shih and Tan [**14**, Theorem 4] in a real Hausdorff locally convex topological vector space.

The aim of this paper is to unify and extend the above results to non-compact setting. We first give a more general formulation of Shih and Tan [**14**, Theorem 3], then we give a theorem which simultaneously generalizes Shih and Tan [**14**, Theorem 4] and Kim [**9**]. In the second part of this paper, we generalize and combine Bae, Kim and Tan [**2**, Theorem 4] and Tan [**16**, Theorem 3]. Our proofs are based on 'net argument', which gives another refined proofs for the results of the authors.

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2. Preliminaries

Let 2^X be the set of all nonempty subsets of X for a topological space X.

Let X and Y be topological spaces and $D : X \to 2^Y$ a multifunction. We denote by $D(X)$ the image of D, namely, the set $\bigcup\{D(x)|x \in X\}$.

Let E be a real Hausdorff topological vector space, and E^* its topological dual. A multifunction $D : X \to 2^E$ is said to be *upper hemicontinuous* if for each $f \in E^*$ and for any real α, the set $\{x \in X|\sup_{y\in D(x)} f(y) < \alpha\}$ is open in X. It is well known that an upper semicontinuous multifunction is upper hemicontinuous. From now on, X is always assumed to be a nonempty convex subset of E.

A multifunction $T : X \to 2^{E^*}$ is said to be *monotone* on X if for each x and y in X, each u in $T(x)$, and each w in $T(y)$, $\langle w-u, y-x\rangle \geq 0$, and *semi-monotone* on X if for any $x, y \in X$,

$$\inf_{u\in T(x)} \langle u, y-x\rangle \leq \inf_{w\in T(y)} \langle w, y-x\rangle.$$

It is clear that a monotone multifunction T is semi-monotone. But the following example shows that the converse is not true.

$$T : R \to R, \quad T(x) = \begin{cases} [0,\infty) & \text{if } x \geq 0 \\ (-\infty, 0] & \text{if } x < 0. \end{cases}$$

Let $cc(X)\,[kc(X)]$ denote the set of all nonempty closed [compact] convex subsets of E contained in X.

We denote by *co* and $-$ the convex hull and closure, respectively, with respect to E.

The *inward set of* X *at* $x \in E, I_X(x)$ is defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X-x).$$

3. A generalization of QVI of Shih and Tan

We begin this section with the following theorem.

THEOREM 1. (Ding and Tan [5, Theorem 1]) Let X be a convex subset in a Hausdorff topological vector space E. Let ϕ and ψ be two real valued functions on $X \times X$ having the following properties:

(1.1) $\psi(x,x) \leq 0$ for all $x \in X$;

(1.2) for each fixed $x \in X$, $\phi(x,y)$ is a lower semicontinuous function of y on K' for each nonempty compact subset K' of X;

(1.3) for each fixed $y \in X$, the set $\{x \in X | \psi(x,y) > 0\}$ contains the convex hull of the set $\{x \in X | \phi(x,y) > 0\}$;

(1.4) there is a nonempty compact convex subset L of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, there exists an $x \in co(L \cup \{y\})$ with $\phi(x,y) > 0$. Then there exists a $\hat{y} \in K$ such that $\phi(x,\hat{y}) \leq 0$ for all $x \in X$.

From now on, let us denote by E a real Hausdorff locally convex topological vector space. We present the following theorem which is a more general formulation of Shih and Tan [14, Theorem 3].

THEOREM 2. *Let X be a paracompact convex subset of E, L a nonempty compact convex subset of X and K a nonempty compact subset of X. Let E^* be the topological dual of E equipped with the strong topology. Let $S : X \to cc(X)$ be upper hemicontinuous and $T : X \to kc(E^*)$ upper semicontinuous. Assume that the set*

$$\Sigma := \left\{ y \in X \mid \sup_{x \in S(y)} \inf_{z \in T(y)} \langle z, y - x \rangle > 0 \right\}$$

is open in X. Assume further that for each $y \in X \backslash K$, there exists an $x \in S(y) \cap I_L(y)$ with $\inf_{z \in T(y)} \langle z, y - x \rangle > 0$. Then there exists a $\hat{y} \in X$ such that

(2.1) $\hat{y} \in S(\hat{y})$ and

(2.2) there is a $\hat{z} \in T(\hat{y})$ with $\langle \hat{z}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$.

Proof. We follow the method in [14], so we divide the proof into two steps:

Step 1. We assert that there exists a $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \inf_{z \in T(\hat{y})} \langle z, \hat{y} - x \rangle \leq 0.$$

Suppose that the assertion is false. Then for all $y \in X$, either $y \notin S(y)$ or there exists an $x \in S(y)$ such that $\inf_{z \in T(y)} \langle z, y - x \rangle > 0$. By the separation theorem for a Hausdorff locally convex space, whenever $y \notin S(y)$, there exists a $p \in E^*$ with $\langle p, y \rangle - \sup_{x \in S(y)} \langle p, x \rangle > 0$. For each $y \in X$, we set

$$\alpha(y) := \sup_{x \in S(y)} \inf_{z \in T(y)} \langle z, y - x \rangle.$$

Let $V_0 := \{y \in X | \alpha(y) > 0\}$, and for each $p \in E^*$, we set

$$V(p) := \left\{ y \in X | \langle p, y \rangle - \sup_{x \in S(y)} \langle p, x \rangle > 0 \right\}.$$

Since $\Sigma = V_0, X = V_0 \cup \bigcup_{p \in E^*} V(p)$. By hypothesis, V_0 is open in X. By the upper hemicontinuity of S, $V(p)$ is open in X for every $p \in E^*$. Since X is paracompact, there exists a continuous partition of unity $\{\beta_0, \beta_p\}_{p \in E^*}$ subordinate to the covering $\{V_0, V(p)\}_{p \in E^*}$. Thus for each 0 and $p \in E^*$, β_0 and β_p are nonnegative real continuous functions on X, with their supports supp $\beta_0 \subset V_0$ and supp $\beta_p \subset V(p)$. The family $\{\text{supp } \beta_0, \text{supp } \beta_p\}_{p \in E^*}$ is a locally finite covering of X and $\sum_{j \in E^* \cup \{0\}} \beta_j(x) = 1$ for all $x \in X$. Let us define $\phi : X \times X \to R$ by

$$\phi(x, y) := \beta_0(y) \inf_{z \in T(y)} \langle z, y - x \rangle + \sum_{p \in E^*} \beta_p(y) \langle p, y - x \rangle.$$

Clearly,

(2.3) $\phi(x, x) = 0$ for all $x \in X$.

(2.4) For each $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on each compact subset K' of X. Indeed, consider the function

$$f(y) := \inf_{z \in T(y)} \langle z, y - x \rangle = - \sup_{z \in T(y)} \langle z, x - y \rangle.$$

Since E^* has the strong topology and K' is bounded, $W(z, y) = \langle z, x - y \rangle$ is continuous on $E^* \times K'$. In addition, T is upper semicontinuous and has a compact value for each $y \in X$. By Aubin and Cellina [1, Theorem 5, p.52], for each $x \in X$, $f(y) = -\sup_{z \in T(y)} \langle z, x - y \rangle$ is a lower semicontinuous function of y on K'. Hence, so is the function $y \mapsto \beta_0(y) \inf_{z\in T(y)} \langle z, y-x\rangle$ by Takahashi [15, Lemma 3]. Moreover, the function $y \mapsto \sum_{p\in E^*} \beta_p(y) \langle p, y-x\rangle$ is continuous on X. Thus $y \mapsto \phi(x,y)$ is lower semicontinuous on each compact set $K' \subset X$.

(2.5) For each $y \in X$, $\phi(x,y)$ is a concave function of x on X. Indeed, $x \mapsto \beta_0(y) \inf_{z\in T(y)} \langle z, y-x\rangle$ is concave and

$$x \mapsto \sum_{p\in E^*} \beta_p(y) \langle p, y-x\rangle$$

is affine, and hence $x \mapsto \phi(x,y)$ is concave.

(2.6) For each $y \in X\backslash K$, there is an $x \in co(L\cup\{y\})$ with $\phi(x,y) > 0$. Indeed, by hypotheses, there exists an $x_0 \in S(y) \cap I_L(y)$ such that $\beta_0(y) \inf_{z\in T(y)} \langle z, y-x_0\rangle \geq 0$. In case $\beta_o(y) = 0$, we may assume that

$$\phi(x_0,y) = \sum_{p\in E^*} \beta_p(y) \langle p, y-x_0\rangle = \sum_{i=1}^{n} \beta_{p_i}(y) \langle p_i, y-x_0\rangle$$

for some positive integer n $(\beta_{p_i}(y) \neq 0,\ \ i = 1,\dots,n)$. Since y belongs to $\bigcap_{i=1}^n V(p_i)$ and $x_0 \in S(y)$, $\sum_{i=1}^n \beta_{p_i}(y)\langle p_i, y-x_0\rangle > 0$. Hence, $\phi(x_0,y) > 0$. In case $\beta_0(y) > 0$, we can show that $\phi(x_0,y) > 0$ in a similar way. Note that $I_L(y) = I_{co(L\cup\{y\})}(y)$ because L is convex. Moreover, $I_{co(L\cup\{y\})}(y)$ is in fact the set

$$A := \{u \in E | u = y + r(v-y) \text{ for some } v \in co(L\cup\{y\}),\ r \geq 1\}.$$

Thus, we have an $x \in co(L\cup\{y\})$ such that $x_0 = y + r(x-y)$ with $r \geq 1$. Hence,

$$\begin{aligned}\phi(x,y) &= \phi(\frac{1}{r}\, x_0 + (1-\frac{1}{r})y, y) \\ &\geq \frac{1}{r}\,\phi(x_0,y) + (1-\frac{1}{r})\,\phi(y,y) \\ &= \frac{1}{r}\,\phi(x_0,y) > 0\end{aligned}$$

by the concavity of ϕ in its first variable. Therefore, for each $y \in X\backslash K$, there exists an $x \in co(L\cup\{y\})$ such that $\phi(x,y) > 0$. By Theorem 1 with $\phi = \psi$, there exists a $\hat{y} \in K$ such that

$$\phi(x,\hat{y}) \leq 0 \ \text{ for all } \ x \in X. \tag{2.7}$$

Now we show that there exists an $\hat{x} \in S(\hat{y})$ with $\phi(\hat{x}, \hat{y}) > 0$. If $\beta_0(\hat{y}) = 0$, then $\phi(x, \hat{y}) = \sum_{i=1}^{n} \beta_{p_i}(\hat{y}) \langle p_i, \hat{y} - x \rangle$ where p_i's are given elements of E^* and $\beta_{p_i}(\hat{y}) > 0$. Since $\hat{y} \in \bigcap_{i=1}^{n} V(p_i)$,

$$\langle p_i, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \langle p_i, x \rangle \geq \langle p_i, \hat{x} \rangle$$

so that $\langle p_i, \hat{y} - \hat{x} \rangle > 0$ for all $\hat{x} \in S(\hat{y})$. Thus $\phi(\hat{x}, \hat{y}) > 0$ for all $\hat{x} \in S(\hat{y})$. If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0$ so that $\alpha(\hat{y}) > 0$. Hence, $\inf_{z \in T(\hat{y})} \langle z, \hat{y} - \hat{x} \rangle \geq \frac{1}{2}\, \alpha(\hat{y}) > 0$ for some $\hat{x} \in S(\hat{y})$. Using similar argument as above, we can also assert that $\phi(\hat{x}, \hat{y}) > 0$ for some $\hat{x} \in S(\hat{y})$. In any case, there exists an $\hat{x} \in S(\hat{y})$ with $\phi(\hat{x}, \hat{y}) > 0$, which contradicts (2.7). This proves Step 1.

Step 2. There exists a $\hat{z} \in T(\hat{y})$ such that $\langle \hat{z}, \hat{y} - x \rangle \leq 0$ for $x \in S(\hat{y})$. Indeed, define $F : S(\hat{y}) \times T(\hat{y}) \longrightarrow R$ by

$$F(x, z) := \langle z, \hat{y} - x \rangle.$$

Note that for each $x \in S(\hat{y})$, $z \mapsto F(x, z)$ is continuous and affine and for each $z \in T(\hat{y})$, $x \mapsto F(x, z)$ is affine. Thus by Kneser's minimax theorem [10], we have

$$\min_{z \in T(\hat{y})} \sup_{x \in S(\hat{y})} F(x, z) = \sup_{x \in S(\hat{y})} \min_{z \in T(\hat{y})} F(x, z)$$

Hence,

$$\min_{z \in T(\hat{y})} \sup_{x \in S(\hat{y})} F(x, z) = \min_{z \in T(\hat{y})} \sup_{x \in S(\hat{y})} \langle z, \hat{y} - x \rangle \leq 0$$

by Step 1. Since $T(\hat{y})$ is compact, there exists a $\hat{z} \in T(\hat{y})$ such that $\langle \hat{z}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. This completes our proof.

In order to generalize results of Kim [9] and Shih and Tan [14, Theorem 4], we need the following.

LEMMA 1. *Let E^* be equipped with the strong topology. Let X be a nonempty bounded subset of E and C a nonempty compact subset of E^*. Define $f : X \longrightarrow R$ by*

$$f(x) := \min_{z \in C} \langle z, x \rangle \quad \textit{for all} \quad x \in X.$$

Then f is weakly continuous on X.

Proof. This is immediate since the functions $\{\langle u, \cdot \rangle | u \in C\}$ are weakly equicontinuous.

THEOREM 3. *Let X, L, K, E^* and T be as in Theorem 2. In addition, let X be a bounded subset of $E, S : X \to kc(X)$ a continuous multifunction, and $T(X) \subset D$ for some compact subset D of E^*. Assume that for each $y \in X \backslash K$, there is an $x \in S(y) \cap I_L(y)$ with $\inf_{z \in T(y)} \langle z, y - x \rangle > 0$. Then the conclusion of Theorem 2 holds.*

Proof. By Theorem 2, it suffices to show that the set

$$\Lambda := \left\{ y \in X | \sup_{x \in S(y)} \inf_{z \in T(y)} \langle z, y - x \rangle \leq 0 \right\}$$

is closed in X. Let $(y_\alpha)_{\alpha \in \Gamma}$ be a net in Λ and $y_0 \in X$ such that $y_\alpha \to y_0$. Since $T(y_0)$ and $S(y_0)$ are compact, by Lemma 1, there is an $x_0 \in S(y_0)$ such that

$$\sup_{x \in S(y_0)} \inf_{z \in T(y_0)} \langle z, y_0 - x \rangle = \inf_{z \in T(y_0)} \langle z, y_0 - x_0 \rangle. \tag{3.1}$$

By the lower semicontinuity of S, we obtain a net $(x_\alpha)_{\alpha \in \Gamma}$ in X such that for each $\alpha \in \Gamma$, $x_\alpha \in S(y_\alpha)$ and $x_\alpha \to x_0$. Since $y_\alpha \in \Lambda$,

$$\sup_{x \in S(y_\alpha)} \inf_{z \in T(y_\alpha)} \langle z, y_\alpha - x \rangle \leq 0,$$

so,

$$\inf_{z \in T(y_\alpha)} \langle z, y_\alpha - x_\alpha \rangle \leq 0.$$

From the compactness of $T(y_\alpha)$, we can choose a $z_\alpha \in T(y_\alpha)$ satisfying

$$\inf_{z \in T(y_\alpha)} \langle z, y_\alpha - x_\alpha \rangle = \langle z_\alpha, y_\alpha - x_\alpha \rangle \leq 0. \tag{3.2}$$

By hypothesis, there exists a subnet $(z_{\alpha'})_{\alpha' \in \Gamma'}$ of $(z_\alpha)_{\alpha \in \Gamma}$ and a $z_0 \in D$ such that $z_{\alpha'} \to z_0$. Since E^* has the strong topology and X is bounded, we have $\langle z_{\alpha'}, y_{\alpha'} - x_{\alpha'} \rangle \to \langle z_0, y_0 - x_0 \rangle$, and hence,

$$\langle z_0, y_0 - x_0 \rangle \leq 0 \tag{3.3}$$

from (3.2). Note that the graph of T is colsed in $X \times E^*$, so $z_0 \in T(y_0)$. Therefore,

$$\begin{aligned} \sup_{x \in S(y_0)} \inf_{z \in T(y_0)} \langle z, y_0 - x \rangle &= \inf_{z \in T(y_0)} \langle z, y_0 - x_0 \rangle \\ &\leq \langle z_0, y_0 - x_0 \rangle \\ &\leq 0 \end{aligned}$$

by (3.1) and (3.3). This shows that $y_0 \in \Lambda$, hence, Λ is closed in X, as desired. The proof is complete.

As direct consequences of Theorem 3, we obtain two corollaries as follows.

COROLLARY 2. *Let X be a nonempty compact convex subset of E. Let $S : X \longrightarrow cc(X)$ be continuous and $T : X \longrightarrow kc(E^*)$ upper semicontinuous. Then there exists a point $\hat{y} \in X$ such that*

(1) *$\hat{y} \in S(\hat{y})$ and*
(2) *there exists a $\hat{z} \in T(\hat{y})$ with $\langle \hat{z}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$.*

Proof. Set $X = L = K$ in Theorem 3. Since $T(X)$ is compact, all the conditions of Theorem 3 are obviously satisfied. This completes our proof.

COROLLARY 3. *Let E be a normed vector space and E^* its dual space with the usual norm topology. Let X be a convex subset of E, L a nonempty compact convex subset of X, and K a nonempty compact subset of X. Let $S : X \longrightarrow kc(X)$ be a continuous map and $T : X \longrightarrow kc(E^*)$ an upper semicontinuous map. Assume that for each $y \in X \backslash K$, there is an $x \in S(y) \cap I_L(y)$ with $\inf_{z \in T(y)} \langle z, y - x \rangle > 0$. Then there exists a point $\hat{y} \in X$ such that*

(1) *$\hat{y} \in S(\hat{y})$ and*
(2) *there exists a $\hat{z} \in T(\hat{y})$ with $\langle \hat{z}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$.*

Proof. In fact, the boundedness of X is needed to assure that of the set $\{y_\alpha\} \cup \{y_0\}$ in Theorem 3. Moreover, we need the assumption $T(X) \subset D$ so as to guarantee that $\bigcup_{\alpha \in \Gamma} T(y_\alpha)$ is contained in a compact set. Since E is a normed vector space, we have only to consider a sequence $\{y_n\}$ converging to y_0. Clearly, the set $\{y_n\} \cup \{y_0\}$ is compact. Also $\bigcup_{i=0}^{\infty} T(y_i)$ is compact. In addition, X is a metric space, so X is paracompact. Repeating the same argument as in Theorerm 3, we have the desired conclusion.

REMARK. Corollary 2 is due to Kim [9] and Corollary 3 is a generalization of Shih and Tan [14, Theorem 4] to noncompact setting. Observe that our proof is a simple one for Kim's result [9].

4. A generalization of two results for variational inequalities

In this section, by combining two results of Tan [**16**, Theorem 3] and Bae, Kim and Tan [**2**, Theorem 4], we generalize them at the time.

THEOREM 4. *Let E^* be equipped with the strong topology. Let X be a nonempty convex subset of E and $T : X \longrightarrow 2^{E^*}$ be semi-monotone such that for each $x \in X$, $T(x)$ is (strongly) compact and for each one-dimensional flat $N \subset E$, $T|_{N\cap X}$ is upper semicontinuous from the relative topology of X to the strong topology of E^*. Let $M : X \longrightarrow 2^{E^*}$ be monotone such that for each one-dimensional flat $N \subset E, M|_{N\cap X}$ is lower semicontinuous from the relative topology of X to the weak star topology of E^*. Suppose that there exist a nonempty weakly compact convex subset L of X and a nonempty weakly compact subset K of X such that for each $y \in X\backslash K$, there exist a point $x \in co(L \cup \{y\})$ and an $f \in M(x)$ such that $\inf_{w\in T(x)} \langle w + f, y - x\rangle > 0$. Then there exists a $\hat{y} \in K$ such that*

$$\sup_{f\in M(\hat{y})} \inf_{w\in T(\hat{y})} \langle w + f, \hat{y} - x\rangle \leq 0 \quad \textit{for all} \quad x \in I_X(\hat{y}).$$

Proof. We divide the proof into three steps.

Step 1. There exist a $\hat{y} \in K$ such that

$$\sup_{f\in M(x)} \inf_{w\in T(x)} \langle w + f, \hat{y} - x\rangle \leq 0 \quad \text{for all} \quad x \in X.$$

Indeed, we define $\phi, \psi : X \times X \longrightarrow R$ by

$$\phi(x, y) := \sup_{f\in M(x)} \inf_{w\in T(x)} \langle w + f, y - x\rangle,$$
$$\psi(x, y) := \inf_{f\in M(y)} \inf_{w\in T(y)} \langle w + f, y - x\rangle$$

for all $x, y \in X$. Then we have the following.

(4.1) We have $\phi(x,y) \le \psi(x,y)$ for all $x, y \in X$, and $\psi(x,x) = 0$ for all $x \in X$. Indeed,

$$\begin{aligned}\phi(x,y) &= \sup_{f\in M(x)} \inf_{w\in T(x)} [\langle w, y-x\rangle + \langle f, y-x\rangle] \\ &= \sup_{f\in M(x)} [\langle f, y-x\rangle + \inf_{w\in T(x)} \langle w, y-x\rangle] \\ &= \sup_{f\in M(x)} \langle f, y-x\rangle + \inf_{w\in T(x)} \langle w, y-x\rangle \\ &\le \inf_{f\in M(y)} \langle f, y-x\rangle + \inf_{w\in T(y)} \langle w, y-x\rangle\end{aligned}$$

by the monotonicity of M and the semi-monotonicity of T. On the other hand,

$$\begin{aligned}\psi(x,y) &= \inf_{f\in M(y)} \inf_{w\in T(y)} [\langle w, y-x\rangle + \langle f, y-x\rangle] \\ &= \inf_{f\in M(y)} [\langle f, y-x\rangle + \inf_{w\in T(y)} \langle w, y-x\rangle] \\ &= \inf_{f\in M(y)} \langle f, y-x\rangle + \inf_{w\in T(y)} \langle w, y-x\rangle.\end{aligned}$$

Hence $\phi(x,y) \le \psi(x,y)$ for all $x, y \in X$.

(4.2) For each fixed $x \in X, \phi(x,y)$ is a weakly lower semicontinuous function of y on each nonempty weakly compact subset A of X. Indeed, since $A - x$ is weakly bounded, and hence bounded in E and $T(x)$ is (strongly) compact, Lemma 1 shows that the function $g(y) := \inf_{w\in T(x)} \langle w, y-x\rangle$ is weakly continuous on A. Clearly, the function $h(y) := \sup_{f\in M(x)} \langle f, y-x\rangle$ is weakly lower semicontinuous on A. Therefore, $\phi(x,y) = g(y) + h(y)$ is a weakly lower semicontinuous function of y on A.

(4.3) For each fixed $y \in X, \psi(x,y)$ is a concave function of x on X. Indeed, since

$$\psi(x,y) = \inf_{f\in M(y)} \langle f, y-x\rangle + \inf_{w\in T(y)} \langle w, y-x\rangle,$$

obviously, $\psi(x,y)$ is a concave function of x on X.

(4.4) For each $y \in X \backslash K$, by hypothesis, there exist an $x \in co(L \cup \{y\})$ and an $f \in M(x)$ such that $\inf_{w \in T(x)} \langle w + f, y - x \rangle > 0$. Thus

$$\sup_{f \in M(x)} \inf_{w \in T(x)} \langle w + f, y - x \rangle > 0,$$

namely, $\phi(x, y) > 0$. Now we equip E with the weak topology. Then all the conditions of Theorem 1 are satisfied so that there exists a $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, namely,

$$\sup_{f \in M(x)} \inf_{w \in T(x)} \langle w + f, \hat{y} - x \rangle \leq 0$$

for all $x \in X$. This proves Step 1.

Step 2. We assert that $\sup_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \langle w + f, \hat{y} - x \rangle \leq 0$ for all $x \in X$.

Fix $x \in X$ arbitrarily. For each $t \in [0, 1]$, let $z_t = tx + (1 - t)\hat{y} = \hat{y} - t(\hat{y} - x)$. Since X is convex, $z_t \in X$ for all $t \in [0, 1]$. By Step 1,

$$\sup_{f \in M(z_t)} \inf_{w \in T(z_t)} \langle w + f, \hat{y} - z_t \rangle \leq 0 \quad \text{for all} \quad t \in [0, 1],$$

so that

$$t \sup_{f \in M(z_t)} \inf_{w \in T(z_t)} \langle w + f, \hat{y} - x \rangle \leq 0 \quad \text{for all} \quad t \in [0, 1],$$

and hence,

$$\sup_{f \in M(z_t)} \inf_{w \in T(z_t)} \langle w + f, \hat{y} - x \rangle \leq 0 \quad \text{for all} \quad t \in (0, 1], \tag{4.5}$$

Let $[\hat{y}, x]$ be the line segment $\{z \in X | z = (1 - t)\hat{y} + tx \ 0 \leq t \leq 1\}$. Observe that by giving the reverse order to the set $(0, 1]$, we may regard $\{z_t\}_{0 < t \leq 1}$ as a net converging to $\hat{y}$ in the line segment $[\hat{y}, x]$. Fix an $\hat{f} \in M(\hat{y})$. Since M is lower semicontinuous on $[\hat{y}, x]$ and $z_t \to \hat{y}$, for each $t \in (0, 1]$ there exists an $f_t \in M(z_t)$ satisfying $f_t \to \hat{f}$ in the weak star sense. Let $w_t \in T(z_t)$ be a point such that

$$\langle w_t, \hat{y} - x \rangle = \inf_{w \in T(z_t)} \langle w, \hat{y} - x \rangle. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\begin{aligned}\langle f_t, \hat{y} - x\rangle + \langle w_t, \hat{y} - x\rangle &\leq \sup_{f \in M(z_t)} \langle f, \hat{y} - x\rangle + \inf_{w \in T(z_t)} \langle w, \hat{y} - x\rangle \\ &= \sup_{f \in M(z_t)} \inf_{w \in T(z_t)} \langle w + f, \hat{y} - x\rangle \\ &\leq 0. \end{aligned} \tag{4.7}$$

Since T is upper semicontinuous on the compact set $[\hat{y}, x]$, the image of T, namely, $\bigcup_{t \in [0,1]} T(z_t)$ is a (strongly) compact subset of E^*, and hence, there exists a subnet $\{w_{t'}\}_{t' \in I}$ of $\{w_t\}_{0<t\leq 1}$ and $\hat{w} \in \bigcup_{t\in[0,1]} T(z_t)$ satisfying $w_{t'} \to \hat{w}$. Since $f_{t'} + w_{t'} \to \hat{f} + \hat{w}$ in the weak star sense,

$$\langle f_{t'} + w_{t'}, \hat{y} - x\rangle \to \langle \hat{f} + \hat{w}, \hat{y} - x\rangle.$$

Hence, by (4.7) we have

$$\langle f_{t'} + w_{t'}, \hat{y} - x\rangle \to \langle \hat{f} + \hat{w}, \hat{y} - x\rangle \leq 0. \tag{4.8}$$

Recall that the graph of T on $[\hat{y}, x]$ is closed and $z_{t'} \to \hat{y}$. Then, clearly, $\hat{w} \in T(\hat{y})$. Thus, by (4.8), we have

$$\begin{aligned}\langle \hat{f}, \hat{y} - x\rangle + \inf_{w \in T(\hat{y})} \langle w, \hat{y} - x\rangle &\leq \langle \hat{f}, \hat{y} - x\rangle + \langle \hat{w}, \hat{y} - x\rangle \\ &= \langle \hat{f} + \hat{w}, \hat{y} - x\rangle \\ &\leq 0.\end{aligned}$$

Since $\hat{f}$ is arbitrary, we have

$$\sup_{f \in M(\hat{y})} [\langle f, \hat{y} - x\rangle + \inf_{w \in T(\hat{y})} \langle w, \hat{y} - x\rangle] \leq 0.$$

Therefore, we have

$$\begin{aligned}&\sup_{f \in M(\hat{y})} [\langle f, \hat{y} - x\rangle + \inf_{w \in T(\hat{y})} \langle w, \hat{y} - x\rangle] \\ &\qquad = \sup_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} \langle w + f, \hat{y} - x\rangle \leq 0\end{aligned}$$

for all $x \in X$, as desired.

Step 3. We have $\sup_{f\in M(\hat{y})} \inf_{w\in T(\hat{y})} \langle w + f, \hat{y} - x\rangle \leq 0$ for all $x \in I_X(\hat{y})$.

Let $x \in I_X(\hat{y})$; then $x = \hat{y} + r(u - \hat{y})$ for some $u \in X$ and $r > 0$. Thus $\hat{y} - x = r(\hat{y} - u)$ so that by Step 2,

$$\sup_{f\in M(\hat{y})} \inf_{w\in T(\hat{y})} \langle w + f, \hat{y} - x\rangle = r \sup_{f\in M(\hat{y})} \inf_{w\in T(\hat{y})} \langle w + f, \hat{y} - u\rangle \leq 0.$$

This completes the proof.

REMARK. 1. For $M = 0$, Theorem 4 is due to Bae, Kim and Tan [**2**, Theorem 4].

2. For $T = 0$, Theorem 4 is a generalization of Tan [**16**, Theorem 3].

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