J. Korean Math. Soc. 30 (1993), No. 2, pp. 399–412

ON PREDUALS OF OPERATOR ALGEBRAS AND COMMON NONCYCLIC VECTORS I

HAN SOO KIM[1)] AND HAE GYU KIM[2)]

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert spaces and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A ***dual algebra*** is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = tr(AL), A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x \quad \text{for all} \quad u \in \mathcal{H}.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbf{A}_{m,n}$ (to be defined in section 2) were defined by H.Bercovici, C.Foias and C.Pearcy in [3]. Also these classes are closely related to the study of the theory of dual algebras. Especially, C.Apostol, H.Bercovici, C.Foias and C.Pearcy [1] established property $X_{\theta,\gamma}$ (to be defined in section 2), and researched

Received June 18, 1992.
Supported by the Basic Science Research Institute Program, Ministry of Education, 1992[1)] and TGRC–KOSEF grant[2)].

a relationship with the class $\mathbf{A}_{\aleph_0}$(to be defined in section 2). In 1991, B.Chevreau and C.Pearcy [6] studied for the first time common non-cyclic vectors for families of opeators (to be defined in section 2) in order to solve the invariant subspace problem of bounded operators whose spectral radius is one.
In a sequel to this study, in this paper we obtain a main theorem with a technique different from that used in [6] which gives better information than the results of the generalization of [6].

2. Preliminaries

The notation and terminology employed herein agree with those in [4], [5] and [9]. Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.* We denote by $\mathbf{A} = \mathbf{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Sz.-Nagy-Foias functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry (see [4, Theorem 4.1]).

DEFINITION 2.1. ([8]) Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let m and n be any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if $m \times n$ system of simultaneous equations of the form

$$(3) \qquad [x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution consisting of a pair of sequences $\{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n}$ of vectors from $\mathcal{H}$. For brief notation, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$. Furthermore, if m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbf{A}_{m,n} = \mathbf{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbf{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$.

DEFINITION 2.2. ([1]) Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and θ is a nonnegative real number. We denote by $\mathcal{X}_\theta(\mathcal{A})$ the set of all $[L]$

in $Q_{\mathcal{A}}$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of vectors from $\mathcal{H}$ satisfying

(a) $\limsup_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \leq \theta$,
(b) $\|x_i\| \leq 1, \|y_i\| \leq 1, 1 \leq i < \infty$, and
(c) $\|[x_i \otimes z]\| + \|[z \otimes x_i]\| + \|[y_i \otimes z]\| + \|[z \otimes y_i]\| \to 0$,

for all z in $\mathcal{H}$. For $0 \leq \theta < \gamma < +\infty$, the dual algebra $\mathcal{A}$ is said to have property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathcal{X}_\theta(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_{\mathcal{A}}$:

$$\overline{aco}(\mathcal{X}_\theta(\mathcal{A})) \supset \{[L] \in Q_{\mathcal{A}} : \|[L]\| \leq \gamma\} = B_{0,\gamma}. \tag{4}$$

DEFINITION 2.3. ([6]) If $\{T_\alpha\}_{\alpha\in A}$ is a family of operators in $\mathcal{L}(\mathcal{H})$ and x is a nonzero vector in $\mathcal{H}$ such that for each $\alpha \in A$,

$$\mathcal{M}_\alpha = \bigvee_{n=0}^{\infty} T_\alpha^n x \neq \mathcal{H}, \tag{5}$$

then x is said to be a *common noncyclic vector* for the family $\{T_\alpha\}_{\alpha\in A}$.

3. Main results

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^*\}^n$ converges to zero in the strong operator topology and is denoted by,as usual,$C_{0\cdot} = (C_{\cdot 0})^*$. The following theorems show that, for operators in $\mathbf{A}_{\aleph_0}$ and $C_{\cdot 0}, m \times n \times l$ systems of simultaneous equations can be solved with reasonable estimates on the distance from the initial data to the solution.

THEOREM 3.1. *Suppose that nonzero distinct operators T_1 and T_2 are in $\mathbf{A}_{\aleph_0}$ and $C_{\cdot 0}$, that $\epsilon > 0$ and that $\left\{[L_{ij}^{(k)}]_{T_k}\right\} \subset Q_{T_k}$ for each $k = 1,2,\ \ 1 \leq i,j \leq \infty$. Suppose also that sequences of vectors $\left\{x_i^{(k)}\right\}_{i=1,k=1}^{\infty,2}, \left\{y_j^{(k)}\right\}_{j=1,k=1}^{\infty,2}$ and finite sequence $\{z_1, \cdots, z_l\}$ from $\mathcal{H}$*

are given. Then there exist sequences of vectors $\{\hat{x}_i\}_{i=1}^{\infty}$ *and* $\left\{\hat{y}_j^{(k)}\right\}_{j=1,k=1}^{\infty,2}$ *in* $\mathcal{H}$ *satisfying the following conditions: for* $k = 1, 2$,

$$(6) \quad \|[L_{ij}^{(k)}]_{T_k} - [\hat{x}_i \otimes \hat{y}_j^{(k)}]_{T_k}\| \le \epsilon \quad \|[L_{ij}^{(k)}]_{T_k} - [x_i^{(2)} \otimes y_j^{(k)}]_{T_k}\|,$$

$$(7) \quad \max\left\{\|x_i^{(2)} - \hat{x}_i\|, \|y_j^{(k)} - \hat{y}_j^{(k)}\|\right\} \le \left\{\epsilon + (1+\epsilon)\sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k} - [x_i^{(2)} \otimes y_j^{(k)}]_{T_k}\|\right\}^{1/2},$$

$$(8) \quad \|\hat{x}_i\| \le \left(\|x_i^{(2)}\|^2 + 2\epsilon + \sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k} - [x_i^{(2)} \otimes y_j^{(k)}]_{T_k}\|\right)^{1/2},$$

$$(9) \quad \|\hat{y}_j^{(k)}\| \le \left(\|y_j^{(k)}\|^2 + \|[L_{ij}^{(k)}]_{T_k} - [x_i^{(2)} \otimes y_j^{(k)}]_{T_k}\| + \epsilon\right)^{1/2},$$

(10)

$$\|[(x_i^{(2)} - \hat{x}_i) \otimes z_t]_{T_k}\| < \epsilon, \|[z_t \otimes (x_i^{(2)} - \hat{x}_i)]_{T_k}\| < \epsilon,$$
$$\|[(y_j^{(k)} - \hat{y}_j^{(k)}) \otimes z_t]_{T_k}\| < \epsilon, \|[z_t \otimes (y_j^{(k)} - \hat{y}_j^{(k)})]_{T_k}\| < \epsilon,$$
$$1 \le t \le l.$$

Proof. If $[x_i^{(2)} \otimes y_j{}^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}$, then we set $x_i^{(1)} = x_i^{(2)} = \hat{x}_i, y_j^{(1)} = \hat{y}_j{}^{(1)}$ and $y_j^{(2)} = \hat{y}_j{}^{(2)}$ for each i and j. Thus we may assume that $\|[x_i^{(2)} \otimes y_j{}^{(k)}]_{T_k} - [L_{ij}^{(k)}]_{T_k}\| = \delta_k > 0, \quad k = 1, 2$. And, we choose $\eta > 0$ so small that $\eta < \epsilon/2$. Let $[K_{ij}^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k} - [x_i^{(2)} \otimes y_j{}^{(k)}]_{T_k}$. Since $T_k \in \mathbf{A}_{\aleph_0}, k = 1, 2$, by [4, Definition 2.7 and Theorem 6.3], $\mathcal{A}_{T_k}$ has property $X_{0,1}$. Hence $\overline{aco}(\mathcal{X}_0(\mathcal{A}_{T_k}))$ contains the closed ball with radius 1 centered at the origin in Q_{T_k}. Thus we can choose elements

$[(L_{ij}^{(k)})_1]_{T_k}, \cdots, [(L_{ij}^{(k)})_l]_{T_k}$ in $\mathcal{X}_0(\mathcal{A}_{T_k})$ and scalars $\alpha_1{}^{(k)}, \cdots, \alpha_l{}^{(k)}$ satisfying, for each $k = 1, 2$,

$$\|[K_{ij}^{(k)}]_{T_k} - \sum_{n=1}^{l} \alpha_n{}^{(k)} [(L_{ij}^{(k)})_n]_{T_k}\| < \delta_k \eta \tag{11}$$

$$\text{and} \quad \sum_{n=1}^{l} |\alpha_n{}^{(k)}| \leq \delta_k.$$

Furthermore, by definition of $\mathcal{X}_0(\mathcal{A}_{T_k})$, for each fixed i, j and $n, 1 \leq n \leq l$, there exist sequences $\left\{x_{i,p}^{(n,k)}\right\}_{p=1,k=1}^{\infty,2}$ and $\left\{y_{j,p}^{(n,k)}\right\}_{p=1,k=1}^{\infty,2}$ in $\mathcal{H}$ satisfying

(12)

$$\|[x_{i,p}^{(n,k)} \otimes z]_{T_k}\| \xrightarrow{p} 0, \|[z \otimes x_{i,p}^{(n,k)}]_{T_k}\| \xrightarrow{p} 0,$$
$$\|[y_{j,p}^{(n,k)} \otimes z]_{T_k}\| \xrightarrow{p} 0, \|[z \otimes y_{j,p}^{(n,k)}]_{T_k}\| \xrightarrow{p} 0, \forall z \in \mathcal{H},$$
$$\|x_{i,p}^{(n,k)}\| \leq 1, \quad \|y_{j,p}^{(n,k)}\| \leq 1, \quad 1 \leq p \leq \infty,$$

and

$$\limsup_{p \to \infty} \|[(L_{ij}^{(k)})_n]_{T_k} - [x_{i,p}^{(n,k)} \otimes y_{j,p}^{(n,k)}]_{T_k}\| = 0, \quad k = 1, 2.$$

Thus there exists an integer p_0 sufficiently large that

$$\|[(L_{ij}^{(k)})_n]_{T_k} - [x_{i,p}^{(n,k)} \otimes y_{j,p}^{(n,k)}]_{T_k}\| < \eta, \tag{13}$$
$$p \geq p_0, 1 \leq n \leq l, k = 1, 2.$$

Thus, by combining (11) and (13) we obtain

$$\|[K_{ij}^{(k)}]_{T_k} - \sum_{n=1}^{l} \alpha_n{}^{(k)} [x_{i,p_n}^{(n,k)} \otimes y_{j,p_n}^{(n,k)}]_{T_k}\| \leq 2\eta\delta_k \tag{14}$$

for $p_n \geq p_0, 1 \leq n \leq l, k = 1, 2$. Choose $\beta_n{}^{(k)}$ so that $(\beta_n{}^{(k)})^2 = \alpha_n^{(k)}, 1 \leq n \leq l, k = 1, 2$. We set

$$\hat{x}_i = x_i^{(2)} + \sum_{k=1}^{2} \sum_{n=1}^{l} \beta_n^{(k)} x_{i,p_n}^{(n,k)}, \tag{15}$$

$$\hat{y}_j^{(k)} = y_j^{(k)} + \sum_{n=1}^{l} \overline{\beta_n^{(k)}} y_{j,p_n}^{(n,k)},$$

where the increasing sequence $\{p_1, \cdots, p_l\}$ consists of positive integers greater than p_0 to be chosen, one-by-one, in the order indicated. From (11) and (15) we obtain

(16)
$$\begin{aligned}
&\|\hat{x}_i - x_i^{(2)}\|^2 \\
&\leq \sum_{k=1}^{2}\left\{\delta_k + \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} |\beta_{n_1}^{(k)}\beta_{n_2}^{(k)}| \cdot |(x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)})|\right\} \\
&+ \sum_{n_1,n_2=1}^{l}\sum_{k_1,k_2=1}^{2} |\beta_{n_1}^{(k_1)}\beta_{n_2}^{(k_2)}| \cdot |(x_{i,p_{n_1}}^{(n_1,k_1)}, x_{i,p_{n_2}}^{(n_2,k_2)})|
\end{aligned}$$

(17)
$$\begin{aligned}
\|\hat{x}_i\|^2 &= \|x_i^{(2)} + (\hat{x}_i - x_i^{(2)})\|^2 \\
&\leq \|x_i^{(2)}\|^2 + \sum_{k=1}^{2}\delta_k \\
&\quad + \sum_{k=1}^{2}\sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} |\beta_{n_1}^{(k)}\beta_{n_2}^{(k)}| \cdot |(x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)})| \\
&\quad + 2\sum_{k=1}^{2}\sum_{n=1}^{l} |\beta_n^{(k)}| \cdot |(x_{i,p_n}^{(n,k)}, x_i^{(2)})| \\
&\quad + \sum_{n_1,n_2=1}^{l}\sum_{k_1,k_2=1}^{2} |\beta_{n_1}^{(k_1)}\beta_{n_2}^{(k_2)}| \cdot |(x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)})|
\end{aligned}$$

and similar upper bounds for $\|\hat{y}_j^{(k)} - y_j^{(k)}\|^2$ and $\|\hat{y}_j^{(k)}\|^2$. Moreover, again from (15), we obtain

(18)
$$\begin{aligned}
&\|[(\hat{x}_i - x_i^{(2)}) \otimes z_t]_{T_k}\| \\
&\leq \sum_{k=1}^{2}\sum_{n=1}^{l} |\beta_n^{(k)}| \, \|[x_{i,p_n}^{(n,k)} \otimes z_t]_{T_k}\|, \quad 1 \leq t \leq l,
\end{aligned}$$

and similar upper bounds for $\|[z_t \otimes (\hat{x}_i - x_i^{(2)})]_{T_k}\|$, $\|[(y_j^{(k)} - \hat{y}_j^{(k)}) \otimes z_t]_{T_k}\|$, $\|[z_t \otimes (y_j^{(k)} - \hat{y}_j^{(k)})]_{T_k}\|$.

Finally, from (14) and (15) we have, for each $k = 1, 2$,

$$\begin{aligned}(19)\qquad & \|[L_{ij}^{(k)}]_{T_k} - [\hat{x}_i \otimes \hat{y}_j^{(k)}]_{T_k}\| \\ & \le 2\delta_k \eta + \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} \mid \beta_{n_1}^{(k)} \beta_{n_2}^{(k)} \mid \|[x_{i,p_{n_1}}^{(n_1,k)} \otimes y_{j,p_{n_2}}^{(n_2,k)}]_{T_k}\| \\ & \quad + \sum_{n=1}^{l} \mid \beta_n^{(k)} \mid \|[x_i^{(2)} \otimes y_{j,p_n}^{(n,k)}]_{T_k}\| \\ & \quad + \sum_{n=1}^{l}\sum_{k_1=1}^{2} \mid \beta_n^{(k_1)} \mid \|[x_{i,p_n}^{(n,k_1)} \otimes y_j^{(k)}]_{T_k}\| \\ & \quad + \sum_{n_1,n_2=1}^{l} \mid \beta_{n_1}^{(k)} \beta_{n_2}^{(2)} \mid \|[x_{i,p_{n_1}}^{(n_1,2)} \otimes y_{j,p_{n_2}}^{(n_2,k)}]_{T_k}\|.\end{aligned}$$

Since $T_1, T_2 \in C_{\cdot 0}$ and since

$$(20)\qquad \lim_p \|[w \otimes x_{i,p}{}^{(n,k)}]_{T_k}\| = \lim_p \|[x_{i,p}{}^{(n,k)} \otimes w]_{T_k}\| = 0,$$

$1 \le n \le l, k = 1, 2$, for any fixed w in $\mathcal{H}$, we can choose the increasing sequence $\{p_1, \cdots, p_l\}$ so that the following are valid: for $k = 1, 2$,

$$(21)\qquad \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} \mid \beta_{n_1}^{(k)} \beta_{n_2}^{(k)} \mid \cdot \mid (x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)}) \mid \quad \le \epsilon\delta_k$$

$$(22)\qquad \sum_{n_1,n_2=1}^{l} \sum_{k_1,k_2=1}^{2} |\beta_{n_1}^{(k_1)} \beta_{n_2}^{(k_2)}| \cdot |(x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)})| < \epsilon$$

$$\begin{aligned}(23)\qquad & \sum_{k=1}^{2} \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} \mid \beta_{n_1}^{(k)} \beta_{n_2}^{(k)} \mid \cdot \mid (x_{i,p_{n_1}}^{(n_1,k)}, x_{i,p_{n_2}}^{(n_2,k)}) \mid \\ & \quad + 2\sum_{k=1}^{2}\sum_{n=1}^{l} \mid \beta_n^{(k)} \mid \cdot \mid (x_{i,p_n}^{(n,k)}, x_i^{(2)}) \mid \\ & \le \epsilon\end{aligned}$$

$$\sum_{k=1}^{2}\sum_{n=1}^{l} \mid \beta_n^{(k)} \mid \|[x_{i,p_n}^{(n,k)} \otimes z_t]_{T_k}\| \le \epsilon, 1 \le t \le l, \tag{24}$$

$$\begin{aligned} &\sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{l} \mid \beta_{n_1}^{(k)}\beta_{n_2}^{(k)} \mid \|[x_{i,p_{n_1}}^{(n_1,k)} \otimes y_{j,p_{n_2}}^{(n_2,k)}]_{T_k}\| \\ &+\sum_{n=1}^{l} \mid \beta_n^{(k)} \mid \|[x_i^{(2)} \otimes y_{j,p_n}^{(n,k)}]_{T_k}\| \\ &+\sum_{n=1}^{l}\sum_{k_1=1}^{2} \mid \beta_n^{(k_1)} \mid \|[x_{i,p_n}^{(n,k_1)} \otimes y_j^{(k)}]_{T_k}\| \\ &+ \sum_{n_1,n_2=1}^{l} \mid \beta_{n_1}^{(k)}\beta_{n_2}^{(2)} \mid \|[x_{i,p_{n_1}}^{(n_1,2)} \otimes y_{j,p_{n_2}}^{(n_2,k)}]_{T_k}\| \\ &\le (\epsilon - 2\eta)\delta_k \end{aligned} \tag{25}$$

and similar inequalities involving the $y_{j,p_n}^{(n,k)}$. Then we have,by (19) and (25),

$$\|[L_{ij}^{(k)}]_{T_k} - [\hat{x}_i \otimes \hat{y}_j^{(k)}]_{T_k}\| < \epsilon\delta_k, k = 1,2. \tag{26}$$

Therefore, we come to know, from (16)-(26), that $\hat{x}_i, \hat{y}_j^{(1)}$ and $\hat{y}_j^{(2)}$ satisfy (6)-(10).

THEOREM 3.2. *Suppose that nonzero distinct operators T_1 and T_2 are in $\mathbf{A}_{\aleph_0}$ and $C_{\cdot 0}$, that $\epsilon > 0$ and that $\left\{[L_{ij}^{(k)}]_{T_k}\right\} \subset Q_{T_k}$ for each $k = 1,2$, $1 \le i,j \le \infty$. Suppose also that a finite set of vectors $\{z_1,\cdots,z_l\}$ from $\mathcal{H}$ is given. Then there exist sequences of vectors $\{\hat{x}_i\}_{i=1}^{\infty}$ and $\left\{\hat{y}_j^{(k)}\right\}_{j=1,k=1}^{\infty,2}$ in $\mathcal{H}$ such that*

$$[L_{ij}^{(k)}]_{T_k} = [\hat{x}_i \otimes \hat{y}_j^{(k)}]_{T_k}, \quad 1 \le i,j \le \infty, k = 1,2, \tag{27}$$

$$\max\left\{\|\hat{x}_i\|, \|\hat{y}_j^{(1)}\|, \|\hat{y}_j^{(2)}\|\right\} \le (1+\epsilon)\left\{\sum_{k=1}^{2}\|[L_{ij}^{(k)}]_{T_k}\|\right\}^{1/2}, \tag{28}$$

and

$$\begin{gathered}\|[\hat{x}_i \otimes z_t]_{T_k}\| < \epsilon, \|[z_t \otimes \hat{x}_i]_{T_k}\| < \epsilon, \\ \|[\hat{y}_j^{(k)} \otimes z_t]_{T_k}\| < \epsilon, \|[z_t \otimes \hat{y}_j^{(k)}]_{T_k}\| < \epsilon, \\ 1 \le t \le l, k = 1, 2.\end{gathered} \tag{29}$$

Proof. We choose $\epsilon_1 > 0$ so small that $\epsilon_1 \le \epsilon$, $\epsilon_1 < 1$, and

$$\begin{aligned}&\left(\frac{1}{1-\epsilon_1}(\epsilon_1 + \sum_{k=1}^{2}\|[L_{ij}^{(k)}]_{T_k}\|)\right)^{1/2} \\ &\qquad < (1+\epsilon)\left\{\sum_{k=1}^{2}\|[L_{ij}^{(k)}]_{T_k}\|\right\}^{1/2}.\end{aligned} \tag{30}$$

We next define sequences $\{\hat{x}_{i,p}\}_{p=0}^{\infty}$ and $\left\{\hat{y}_{j,p}^{(k)}\right\}_{p=0,k=1}^{\infty,2}$ from $\mathcal{H}$ by induction. Set $\hat{x}_{i,-1} = \hat{y}_{j,-1}^{(k)} = \hat{x}_{i,0} = \hat{y}_{j,0}^{(k)} = 0$. And suppose that, for some nonnegative integer n, sequences $\{\hat{x}_{i,0}, \cdots, \hat{x}_{i,n}\}$ and $\left\{\hat{y}_{j,0}^{(k)}, \cdots, \hat{y}_{j,n}^{(k)}\right\}$ have been chosen to satisfy

$$\begin{gathered}\|[L_{ij}^{(k)}]_{T_k} - [\hat{x}_{i,p} \otimes \hat{y}_{j,p}^{(k)}]_{T_k}\| \le \epsilon_1^p \|[L_{ij}^{(k)}]_{T_k}\|, \\ 0 \le p \le n, k = 1, 2,\end{gathered} \tag{31}$$

$$\begin{gathered}\max\left\{\|\hat{x}_{i,p} - \hat{x}_{i,p-1}\|, \|\hat{y}_{j,p}^{(k)} - \hat{y}_{j,p-1}^{(k)}\|\right\} \\ \le \left\{\epsilon_1 + (1+\epsilon_1)\sum_{k=1}^{2}\|[L_{ij}^{(k)}]_{T_k}\|\right\}^{1/2} \epsilon_1^{(p-1)/2}, \\ 0 \le p \le n, k = 1, 2,\end{gathered} \tag{32}$$

(33)
$$\max\left\{\|\hat{x}_{i,p}\|, \|\hat{y}^{(k)}_{j,p}\|\right\} \leq \Big(\sum_{k=1}^{2}\sum_{t=0}^{p-1}\left\{\|[L^{(k)}_{ij}]_{T_k}\|\epsilon_1^t + \epsilon_1^{t+1}\right\}\Big)^{1/2},$$
$$0 \leq p \leq n, k = 1, 2,$$

and

(34)
$$\begin{aligned}
&\|[(\hat{x}_{i,p} - \hat{x}_{i,p-1}) \otimes z_t]_{T_k}\| < \epsilon_1/2^p,\\
&\|[z_t \otimes (\hat{x}_{i,p} - \hat{x}_{i,p-1})]_{T_k}\| < \epsilon_1/2^p,\\
&\|[(\hat{y}^{(k)}_{j,p} - \hat{y}^{(k)}_{j,p-1}) \otimes z_t]_{T_k}\| < \epsilon_1/2^p,\\
&\|[z_t \otimes (\hat{y}^{(k)}_{j,p} - \hat{y}^{(k)}_{j,p-1})]_{T_k}\| < \epsilon_1/2^p,\\
&1 \leq t \leq l, 0 \leq p \leq n, k = 1, 2.
\end{aligned}$$

Then we may apply Theorem 3.1 (with $\left\{x^{(1)}_i\right\}_{i=1}^{\infty}$, $\left\{x^{(2)}_i\right\}_{i=1}^{\infty}$, $\left\{y^{(k)}_j\right\}_{j=1,k=1}^{\infty,2}$ and ϵ of that theorem taken to be $\{\hat{x}_{i,0}\}_{i=1}^{\infty}$, $\{\hat{x}_{i,n}\}_{i=1}^{\infty}$, $\left\{\hat{y}^{(k)}_{j,n}\right\}_{j=1,k=1}^{\infty,2}$ and $\epsilon_2 = \min\left\{{\epsilon_1}^{n+1}/2, \epsilon_1/2^{n+1}\right\}$, respectively) to conclude the existence of vectors $\hat{x}_{i,n+1}$ and $\hat{y}^{(k)}_{j,n+1}$ in $\mathcal{H}$ such that

(35)
$$\begin{aligned}
&\|[L^{(k)}_{ij}]_{T_k} - [\hat{x}_{i,n+1} \otimes \hat{y}^{(k)}_{j,n+1}]_{T_k}\|\\
&\leq \epsilon_1\|[L^{(k)}_{ij}]_{T_k} - [\hat{x}_{i,n} \otimes \hat{y}^{(k)}_{j,n}]_{T_k}\|\\
&\leq {\epsilon_1}^{n+1}\|[L^{(k)}_{ij}]_{T_k}\|
\end{aligned}$$

(36)
$$\begin{aligned}
&\max\left\{\|\hat{x}_{i,n+1} - \hat{x}_{i,n}\|, \|\hat{y}^{(k)}_{j,n+1} - \hat{y}^{(k)}_{j,n}\|\right\}\\
&\leq \left\{\epsilon_2 + \sum_{k=1}^{2}(1+\epsilon_2)\|[L^{(k)}_{ij}]_{T_k} - [\hat{x}_{i,n} \otimes \hat{y}^{(k)}_{j,n}]_{T_k}\|\right\}^{1/2}\\
&\leq \left\{\epsilon_1 + (1+\epsilon_1)\sum_{k=1}^{2}\|[L^{(k)}_{ij}]_{T_k}\|\right\}^{1/2} \epsilon_1^{n/2}
\end{aligned}$$

$$
\begin{aligned}
(37)\qquad &\max\left\{\|\hat{x}_{i,n+1}\|, \|\hat{y}^{(k)}_{j,n+1}\|\right\} \\
&\leq \left\{\max\left\{\|\hat{x}_{i,n}\|^2, \|\hat{y}^{(k)}_{j,n}\|^2\right\} + 2\epsilon_2 + (A)\right\}^{1/2} \\
&\leq [\sum_{k=1}^{2}\sum_{t=0}^{n-1}\left\{\|[L^{(k)}_{ij}]_{T_k}\|\epsilon_1^t + \epsilon_1^{t+1}\right\} + \epsilon_1{}^{n+1} + (B)]^{1/2} \\
&= [\sum_{k=1}^{2}\sum_{t=0}^{n}\left\{\|[L^{(k)}_{ij}]_{T_k}\|\epsilon_1^t + \epsilon_1^{t+1}\right\}]^{1/2}
\end{aligned}
$$

where

$$
\begin{aligned}
(A) &= \sum_{k=1}^{2}\|[L^{(k)}_{ij}]_{T_k} - [\hat{x}_{i,n}\otimes\hat{y}^{(k)}_{j,n}]_{T_k}\| \\
(B) &= \sum_{k=1}^{2}\|[L^{(k)}_{ij}]_{T_k}\|\epsilon_1{}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
(38)\qquad &\|[(\hat{x}_{i,n+1} - \hat{x}_{i,n})\otimes z_t]_{T_k}\| < \epsilon_2 = \epsilon_1/2^{n+1}, \\
&\|[z_t\otimes(\hat{x}_{i,n+1} - \hat{x}_{i,n})]_{T_k}\| < \epsilon_1/2^{n+1}, \\
&\|[(\hat{y}^{(k)}_{j,n+1} - \hat{y}^{(k)}_{j,n})\otimes z_t]_{T_k}\| < \epsilon_1/2^{n+1}, \\
&\|[z_t\otimes(\hat{y}^{(k)}_{j,n+1} - \hat{y}^{(k)}_{j,n})]_{T_k}\| < \epsilon_1/2^{n+1}, \\
&1 \leq t \leq l, k = 1, 2.
\end{aligned}
$$

Thus, by induction, there exist sequences $\{\hat{x}_{i,p}\}_{p=0}^{\infty}$ and $\left\{\hat{y}^{(k)}_{j,p}\right\}_{p=0,k=1}^{\infty,2}$ in $\mathcal{H}$ that satisfy $(31) - (34)$ for $p = 0, 1, 2, \cdots$. Since $\epsilon_1 < 1$, it follows easily from (32) that these sequences are Cauchy, and hence converge; we set $\hat{x}_i = \lim_p \hat{x}_{i,p}$ and $\hat{y}^{(k)}_j = \lim_p \hat{y}^{(k)}_{j,p}$. It is obvious from (31) and the inequality $\|[u\otimes v]\| \leq \|u\|\|v\|$ that (27) is satisfied. Furthermore, for any $1 \le t \le l$, we have from (34) that

$$\begin{aligned} \|[\hat{x}_i \otimes z_t]_{T_k}\| &= \|[(\hat{x}_i - \hat{x}_{i,0}) \otimes z_t]_{T_k}\| \\ &= \|[\sum_{p=1}^{\infty}(\hat{x}_{i,p} - \hat{x}_{i,p-1}) \otimes z_t]_{T_k}\| \\ &\le \sum_{p=1}^{\infty} \|[(\hat{x}_{i,p} - \hat{x}_{i,p-1}) \otimes z_t]_{T_k}\| \\ &\le \sum_{p=1}^{\infty} \frac{\epsilon_1}{2^p} = \epsilon_1 < \epsilon, \end{aligned}$$

and the other inequalities in (29) are proved similarly. Finally, to see that (28) is valid, we use (30) and (33):

$$\begin{aligned} \|\hat{x}_i\| &= \lim_p \|\hat{x}_{i,p+1}\| \le \lim_p [\sum_{t=0}^{p}\sum_{k=1}^{2} \left\{ \|[L_{ij}^{(k)}]_{T_k}\|\epsilon_1^t + {\epsilon_1}^{t+1} \right\}]^{1/2} \\ &= [\left\{ \sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k}\|(1/(1-\epsilon_1)) \right\} + \epsilon_1/(1-\epsilon_1)]^{1/2} \\ &= \left\{ (\sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k}\| + \epsilon_1)/(1-\epsilon_1) \right\}^{1/2} \\ &\le (1+\epsilon) \left\{ \sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k}\| \right\}^{1/2}. \end{aligned}$$

Hence

$$\|\hat{x}_i\| \le (1+\epsilon) \left\{ \sum_{k=1}^{2} \|[L_{ij}^{(k)}]_{T_k}\| \right\}^{1/2},$$

and the corresponding inequality for $\|\hat{y}_j^{(1)}\|$ and $\|\hat{y}_j^{(2)}\|$ is obtained similarly. Thus the proof is complete.

COROLLARY 3.3. *Suppose T_1 and T_2 are in $\mathbf{A}_{\aleph_0} \cap C_{\cdot 0}, n \in \mathbf{N}$, and $\left\{[L_{ij}^{(k)}]\right\}_{1\le i,j\le n}$ is a doubly indexed sequence of elements in Q_{T_k}, for each $k = 1,2$. Then the set of vectors $(x_1, \cdots, x_n)$ in $\tilde{\mathcal{H}}_n$ for which there exist vectors $(y_1^{(k)}, \cdots, y_n^{(k)}), k = 1,2$, in $\tilde{\mathcal{H}}_n$ satisfying (27) is dense in $\tilde{\mathcal{H}}_n$.*

Proof. Let $\tilde{x}_o = (x_1^o, \cdots, x_n^o)$ be an arbitrary vector in $\tilde{\mathcal{H}}_n$, let w be a positive number, and use as initial data in theorem 3.1 and theorem 3.2 the vectors $(wx_1^o, \cdots, wx_n^o)$ and $(0, \cdots, 0)$ in $\tilde{\mathcal{H}}_n$. Then, according to that theorem, there exists a solution $\tilde{x}_w = (x_1^w, \cdots, x_n^w), \tilde{y}_w^{(k)} = (y_{1,k}^w, \cdots, y_{n,k}^w)$ of (27) such that

$$\begin{aligned} &\|x_i^w - wx_i^o\| < \{\epsilon + 2(1+\epsilon)\delta\}^{1/2}, \\ &\|y_{j,k}^w - 0\| < \{\epsilon + 2(1+\epsilon)\delta\}^{1/2}, \\ &\qquad 1 \le i,j \le n, k = 1,2, \end{aligned} \tag{39}$$

where δ is any fixed positive number that exceeds $\max_{i,j,k} \|[L_{ij}^{(k)}]_{T_k}\|$. Thus, since for every $w > 0$, the pair $(1/w)\tilde{x}_w, w\tilde{y}_w^{(k)}$ is also a solution of (27), and since $\|(1/w)\tilde{x}_w - \tilde{x}_o\| \to 0$ by (39), the result follows. In fact, to obtain $\|(1/w)\tilde{x}_w - \tilde{x}_o\| < \epsilon$, it suffices to take $w = [n\{\epsilon + 2(1+\epsilon)\delta\}^{1/2}]/\epsilon$, in which case the vector $w\tilde{y}_w^{(k)}$ satisfies $\|w\tilde{y}_w^{(k)}\| < n^2\{\epsilon + 2(1+\epsilon)\delta\}/\epsilon$, for all $k = 1,2$.

References

1. C. Apostol, H. Bercovici, C. Foias and C. Pearcy, *Invariant subspaces, dilation theory, and structure of the predual of a dual algebra. I*, Funct. Anal. **63** (1985), 369–404.
2. H. Bercovici, B.Chevreau, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. II*, Math. Z. **187** (1984), 97–103.
3. H. Bercovici, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I*, Michigan Math. J. **30** (1983), 335–354.
4. ———, *Dual algebras with applications to invariant subspaces and dilation theory*, C.B.M.S. Regional Conference Series, no. 56, A.M.S., Providence, R. I. (1985).

5. A. Brown and C. Pearcy, *Introduction to operator theory. I*, Elements of functional analysis, Springer-Verlag, New York, 1977.
6. B. Chevreau and C. Pearcy, *On common noncyclic vectors for families of operators*, Houston J. of Math. **17** (1991), 637–650.
7. G. Exner and P. Sullivan, *Normal operators and the classes* $\mathbf{A}_n$, J. Operator Theory **19** (1988), 81–94.
8. I. Jung, *Dual operator algebras and the classes* $\mathbf{A}_{m,n}$ *I*, J. Operator Theory (to appear).
9. B. Sz-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, North Holland Akademiai Kiado, Amsterdam and Budapest, 1970.

Department of Mathematics
Kyungpook National University
Daegu 702-701, Korea