J. Korean Math. Soc. **30** (1993), No. 2, pp. 385–397

ANALYTIC CLASSIFICATION OF PLANE CURVE SINGULARITIES DEFINED BY SOME HOMOGENEOUS POLYNOMIALS

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0. Introduction

Let $V = \{(z,y) : f(z,y) = 0\}$ be an analytic subvariety of a polydisc near the origin in $\mathbf{C}^2$ where f is a homogeneous polynomical and square-free. We know that any homogeneous polynomial with two variables which is square-free can be written as $z^n + a_{n-1}yz^{n-1} + \cdots + a_1y^{n-1}z + y^n$ where $a_1, \cdots, a_{n-1}$ are constant by a suitable nonsingular linear change of coordinates in $\mathbf{C}^2$. Here we assume that f has the following form : (1) $f = z^n + a_iy^{n-i}z^i + \cdots + a_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2i+3)$. (2) either $f = z^3 + ay^2z + y^3$ or $f = z^4 + ay^3z + y^4$. If $g = z^n + b_jy^{n-j}z^j + \cdots + b_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2j+3)$, then in section 1 we show by the elementary method that f is analytically equivalent to g if and only if there is a unit ω with $\omega^n = 1$ such that $b_k = a_k\omega^k$ for each $k = 1, 2, \cdots, i = j$. In section 2 we prove that all homogeneous polynomials of degree three each of which is square-free are analytically equivalent and that if $f = z^4 + ay^3z + y^4$ and $g = z^4 + by^3z + y^4$ where f and g are square-free, then f and g are analytically equivalent if and only if $a^4 = b^4$. Moreover, we give examples with which we understand the condition that $n \geq 5$ and $n \geq 2i+3$.

1. Analytic classification of plane curve singularities defined by $f = z^n + a_iy^{n-i}z^i + \cdots + a_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2i+3)$

DEFINITION 1.1. Let $V = \{(z,y) : f(z,y) = 0\}$ and $W = \{(z,y) : g(z,y) = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f, g are holomorphic and square-free near the origin in $\mathbf{C}^2$. V and W are said to be analytically equivalent if there exists a

Received June 12, 1992. Revised January 21, 1993.
Supported in part by the Korean Ministry of Education, 1990.

germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(O) = O$ where U_1 and U_2 are open subsets containing the origin in $\mathbf{C}^2$. In this case we call $f(z,y)$ and $g(z,y)$ analytically equivalent near the origin and denote this relation by $f \approx g$. Note by [3] that $f \approx g$ if and only if $f(Az + By, Cz + Dy) = ug(z,y)$ for $u \neq 0$ and $AD - BC \neq 0$ whenever f and g are homogeneous.

Before proving the main result, we need the following Lemma.

LEMMA 1.2. *Recall the notation* ${}_nC_k = \binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$. *Then*

$$D = \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-1}C_1 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-1}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_k & {}_{n+1}C_k & \cdots & {}_{n+k-1}C_k \end{vmatrix}$$

$$= \begin{vmatrix} {}_nC_1 & {}_nC_0 & 0 & \cdots & 0 \\ {}_nC_2 & {}_nC_1 & {}_nC_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_nC_{k-2} & {}_nC_{k-3} & \cdots & {}_nC_0 \\ {}_nC_k & {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 \end{vmatrix}$$

$$= (-1)^{k(k-1)/2} \begin{vmatrix} 0 & \cdots & 0 & {}_{n+k-2}C_0 & {}_{n+k-1}C_1 \\ 0 & \cdots & {}_{n+k-3}C_0 & {}_{n+k-2}C_1 & {}_{n+k-1}C_2 \\ \vdots & & \vdots & \vdots & \vdots \\ {}_nC_0 & \cdots & {}_{n+k-3}C_{k-3} & {}_{n+k-2}C_{k-2} & {}_{n+k-1}C_{k-1} \\ {}_nC_1 & \cdots & {}_{n+k-3}C_{k-2} & {}_{n+k-2}C_{k-1} & {}_{n+k-1}C_k \end{vmatrix}$$

$$= {}_{n+k-1}C_k.$$

Proof. See [4].

THEOREM 1.3. *Let* $V = \{(z,y) : f = z^n + a_i y^{n-i} z^i + \cdots + a_1 y^{n-1} z + y^n = 0\}$ *and* $W = \{(z,y) : g = z^n + b_j y^{n-j} z^j + \cdots + b_1 y^{n-1} z + y^n = 0\}$ *be analytic subvarieties of a polydisc near the origin in* $\mathbf{C}^2$ *where* f *and* g *are homogeneous polynomials and square-free, and* $n \geq 2i + 3$,

$n \geq 2j+3$ and $n \geq 5$. Then $f \approx g$ if and only if there is a unit ω with $\omega^n = 1$ such that $b_k = a_k\omega^k$ for $k = 1, 2, \cdots, i = j$.

Proof. Assume that $f \approx g$. Then we know that $f(Az+By, Cz+Dy) = (Az+By)^n + a_i(Cz+Dy)^{n-i}(Az+By)^i + a_{i-1}(Cz+Dy)^{n-i+1} \times (Az+By)^{i-1} + \cdots + a_1(Cz+Dy)^{n-1}(Az+By) + (Cz+Dy)^n = ug(z,y)$ for a nonzero constant u where $AD-BC \neq 0$. Because $n-(i+2) \geq i+1$ and i and j may be viewed as same integers, coefficients of the following monomials $yz^{n-1}, y^2z^{n-2}, \cdots, y^{i+2}z^{n-(i+2)}$ in the polynomial $f(Az+By, Cz+Dy)$ are zero. Let us write down these coefficients in detail as follows :

([1])

$$yz^{n-1} : \binom{n}{1}A^{n-1}B + a_i \sum_{k+l=1}\binom{n-i}{k}\binom{i}{l}C^{n-i-k}D^kA^{i-l}B^l$$
$$+ \cdots + a_1 \sum_{k+l=1}\binom{n-1}{k}\binom{1}{l}C^{n-1-k}D^kA^{1-l}B^l + \binom{n}{1}C^{n-1}D = 0.$$

([2])

$$y^2z^{n-2} : \binom{n}{2}A^{n-2}B^2 + a_i \sum_{k+l=2}\binom{n-i}{k}\binom{i}{l}C^{n-i-k}D^kA^{i-l}B^l$$
$$+ \cdots + a_1 \sum_{k+l=2}\binom{n-1}{k}\binom{1}{l}C^{n-1-k}D^kA^{1-l}B^l + \binom{n}{2}C^{n-2}D^2$$
$$= 0.$$

$\cdots$

([$i+2$])

$$y^{i+2}z^{n-(i+2)} : \binom{n}{i+2}A^{n-(i+2)}B^{i+2} + a_i \sum_{k+l=i+2}\binom{n-i}{k}\binom{i}{l}\times$$
$$C^{n-i-k}D^kA^{i-l}B^l + \cdots + a_1 \sum_{k+l=i+2}\binom{n-1}{k}\binom{1}{l}C^{n-1-k}D^kA^{1-l}B^l$$
$$+ \binom{n}{i+2}C^{n-(i+2)}D^{i+2} = 0.$$

Considering $1, a_i, a_{i-1}, \cdots, a_1, 1$ as a nontrivial solution of the above $[i+2]$-homogeneous equations, then we get an $(i+2)\times(i+2)$ square matrix Δ consisting of coefficients of $1, a_i, a_{i-1}, \cdots, a_1, 1$ in these equations whose determinant $|\Delta|$ must be zero. Now write down the determinant $|\Delta|$:

$0 = |\Delta| = |(\alpha_{pq})|$ where

(i) $\alpha_{pq} = \sum_{k+l=p} \binom{n-i-2+q}{k}\binom{i+2-q}{l} C^{n-i-2+q-k} D^k A^{i+2-q-l} B^l$
with $1 \le p \le i+2$ and $2 \le q \le i+1$,

(ii) $\alpha_{p1} = \binom{n}{p} A^{n-p} B^p$ and $\alpha_{p,i+2} = \binom{n}{p} C^{n-p} D^p$
with $1 \le p \le i+2$.

Then we claim that $|\Delta| = tA^{n-(i+2)}B[C^{n-(i+2)}D]^{i+1}(AD-BC)^{{}_{i+2}C_2}$ for some nonzero constant t. Note that $\sum_{k+l=j} \binom{n-i}{k}\binom{i}{l} = \binom{n}{j}$ for a given nonnegative integer j. We know that each element in the first column of Δ has $A^{n-(i+2)}B$ as common factor. Now we are going to prove that any elementary signed product from Δ has $[C^{n-(i+2)}D]^{i+1}$ as common divisor. Consider the degree of C of each element in Δ as follows :

$$\begin{pmatrix} 0 & n-(i+1) & \dots & n-2 & n-1 \\ 0 & n-(i+2) & \dots & n-3 & n-2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n-(i+i+2) & \dots & n-(i+3) & n-(i+2) \end{pmatrix}$$

So the degree of C for each elementary signed product from Δ is greater than or equal to the following number : $n-(2i+2)+n-(2i+1)+\cdots+n-(i+2)+0+1+\cdots+i = (i+1)(n-i-2)$. Similarly, we can prove that the degree of D for each elementary signed product from Δ is greater than or equal to the integer $(i+1)$.

Now it is enough to prove that $|\Delta| / \{A^{n-i-2}B(C^{n-i-2}D)^{i+1}\} = t(AD-BC)^{{}_{i+2}C_2}$ for a nonzero constant t. To show this, divide each element in the first column of Δ by $A^{n-i-2}B$ and each element in the remaining columns of Δ by $C^{n-i-2}D$. Let the matrix got in this way from Δ be $P = P(A,B,C,D)$. Then it suffices to prove that the determinant $|P| = t(AD-BC)^{{}_{i+2}C_2}$. Note that each elementary signed product from $|P|$ has the same degree $(i+2)(i+1)$ since $n(i+2)-(n-i-1)-(n-i-1)(i+1) = (i+2)(i+1)$. Let

$Q(A,B) = P(A,B,A,B)$. Note that $|Q(A,A)| = |P(A,A,A,A)| = 0$ and $|Q(A,-A)| = |P(A,-A,-A,A)| = 0$ because any two column vectors in $Q(A,A)$ or $\Delta(A,A,A,A)$ are same and any two column vectors in $Q(A,-A)$ or $\Delta(A,-A,-A,A)$ are same up to a sign. Therefore $|Q(A,B)| = (A^2-B^2)^{{}_{i+2}C_2}Q_1(A,B)$. But since the degree of each elementary signed product from $Q(A,B)$ is always $(i+2)(i+1)$, $Q_1(A,B)$ must be a constant, say t. Therefore $|\Delta| = A^{n-(i+2)}B(C^{n-(i+2)}D)^{i+1}$ $[P_1(A,B,C,D)]^{{}_{i+2}C_2}$ where $P_1(A,B,C,D)$ is a homogeneous polynomial of degree 2.

We claim that $P_1(A,B,C,D) = m(AD-BC)$ for a nonzero constant m. Let $P_1(A,B,C,D) = s_1A^2+s_2AB+s_3AC+s_4AD+s_5B^2+s_6BC+s_7BD+s_8C^2+s_9CD+s_{10}D^2$.

First we want to prove that $s_1 = s_5 = s_8 = s_{10} = 0$. To prove $s_1 = 0$, let $\alpha_0 = \text{Max}\{\alpha : cA^\alpha B^\beta C^\gamma D^\delta$ is an any nonzero elementary signed product from $\Delta\}$. Considering each elementary signed product from Δ, we see easily that $\alpha_0 = n-1+i+(i-1)+\cdots+1+0 = n-1+i(i+1)/2$. But if $P_1(A,B,C,D)$ contains a nonzero term s_1A^2 in its expansion, the highest degree of A among all elementary signed products from Δ would be an integer $n-(i+2)+(i+2)(i+1)$. Note that $n-(i+2)+(i+2)(i+1)-\alpha_0 = n-(i+2)+(i+2)(i+1)-[n-1+i(i+1)/2] = (i+1)(i+2)/2 > 0$ for $i \geq 0$. So $s_1 = 0$. Similarly, we can get $s_5 = s_8 = s_{10} = 0$.

Next, we prove that $s_2 = s_3 = s_7 = s_9 = 0$. To prove $s_3 = 0$, let $a_0 = \text{Max}\{\alpha+\gamma : cA^\alpha B^\beta C^\gamma D^\delta$ is an any nonzero elementary signed product from $\Delta\}$. If $P_1(A,B,C,D)$ contains a nonzero term s_3AC in its expansion, $a_0 = n-(i+2)+(n-(i+2))(i+1)+(i+2)(i+1) = (n-1)(i+2)$. In fact, a_0 is equal to an integer $(n-1)+(n-2)+\cdots+(n-i-2) = (2n-i-3)(i+2)/2$, looking at all elements in Δ. Note that $(n-1)(i+2)-(2n-i-3)(i+2)/2 = (i+1)(i+2)/2 > 0$ for $i \geq 0$. So $s_3 = 0$. Similarly, we can get $s_2 = s_7 = s_9 = 0$.

Therefore $P_1(A,B,C,D) = s_4AD+s_6BC$. Then $s_4 = -s_6$ because $|P(A,B,A,B)| = t(A^2-B^2)^{{}_{i+2}C_2}$ and $|Q(A,A)| = |Q(A,-A)| = 0$. Thus we proved that $|\Delta| = tA^{n-i-2}B(C^{n-i-2}D)^{i+1}(AD-BC)^{{}_{i+2}C_2}$ for some constant t.

To prove $t \neq 0$, consider the term rB^d for a nonzero coefficient r in $|\Delta(1,B,1,1)|$ where $\Delta = \Delta(A,B,C,D)$ and d is the degree of $|\Delta(1,B,1,1)|$ as a polynomial of B. To find rB^d, write down elements of Δ only whose degree of B is the maximum on each column as below:

$$\begin{pmatrix} * & * & \cdots & {}_{n-1}C_0B^1 & {}_nC_1B^0 \\ \vdots & \vdots & & \vdots & \vdots \\ * & {}_{n-i}C_0B^i & \cdots & {}_{n-1}C_{i-1}B^1 & {}_nC_iB^0 \\ * & {}_{n-i}C_1B^i & \cdots & {}_{n-1}C_iB^1 & {}_nC_{i+1}B^0 \\ {}_nC_{i+2}B^{i+2} & {}_{n-i}C_2B^i & \cdots & {}_{n-1}C_{i+1}B^1 & {}_nC_{i+2}B^0 \end{pmatrix}$$

Then we see that rB^d is equal to

$$(-1)^{i+3}{}_nC_{i+2}B^{i+2+i(i+1)/2}\begin{vmatrix} 0 & 0 & \cdots & {}_nC_1 \\ \vdots & \vdots & & \vdots \\ 0 & {}_{n-i+1}C_0 & \cdots & {}_nC_{i-1} \\ {}_{n-i}C_0 & {}_{n-i+1}C_1 & \cdots & {}_nC_i \\ {}_{n-i}C_1 & {}_{n-i+1}C_2 & \cdots & {}_nC_{i+1} \end{vmatrix}$$

$= (-1)^{i+3}(-1)^{(i+1)i/2}{}_nC_{i+2}\cdot{}_nC_{i+1}\cdot B^{i+2+i(i+1)/2}$ by Lemma 1.2. But from $|\Delta| = |\Delta(A,B,C,D)|$, $|\Delta(1,B,1,1)|$ is $tB(1-B)^{{}_{i+2}C_2}$. Thus $tB(-B)^{{}_{i+2}C_2} = (-1)^{i+3}(-1)^{(i+1)i/2}{}_nC_{i+2}\cdot{}_nC_{i+1}B^{i+2+i(i+1)/2}$, and so $t = {}_nC_{i+2}\cdot{}_nC_{i+1}$. Therefore we get $|\Delta(A,B,C,D)| = 0$ if and only if $ABCD = 0$.

Claim that $|\Delta(A,B,C,D)| = 0$ if and only of $B = C = 0$ whenever $a_i \neq 0$ for some i $(2i+3 \leq n)$. It is enough to consider the following cases separately:

(a) $C = 0$: It suffices to check the coefficient of yz^{n-1} in the expansion of $f(Az+By, Cz+Dy)$. Then $AB = 0$ implies $B = 0$ since $AD - BC \neq 0$.

(b) $D = 0$: Check the coefficient of $y^{i+2}z^{n-(i+2)}$ in $f(Az+By, Cz+Dy)$. Since $l \leq i$, $AB = 0$ implies $A = 0$ since $AD - BC \neq 0$. Looking at the coefficient of yz^{n-1}, then $A = D = 0$ implies $C^{n-1}Ba_1 = 0$. Since $AD - BC = -BC \neq 0$, $a_1 = 0$. Next, apply the result $A = D = a_1 = 0$ to the coefficient of y^2z^{n-2}. Trivially $a_2 = 0$. Apply this technique in order to $a_3, \cdots, a_i$. Then we get easily that $a_1 = a_2 = \cdots = a_i = 0$. So $f(Az+By, Cz+Dy) = f(By, Cz) = (By)^n + (Cz)^n = ug(z,y) = u(z^n + b_iy^iz^{n-i} + \cdots + b_1y^{n-1}z + y^n)$ for a nonzero constant u implies that $B^n = C^n = u$ and $a_k = b_k = 0$ for $1 \leq k \leq i$ $(2i+3 \leq n)$.

(c) $A = 0$: Since each element of the first column in the matrix is zero if $A = 0$, as in the beginning of the proof, consider $a_i, a_{i-1}, \cdots, a_1, 1$ as a nontrivial solution of the homogeneous equations $[1], [2], \cdots, [i+1]$ assuming that $A = 0$. Then we get an $(i+1) \times (i+1)$ square matrix $A_{i+2,1}(A, B, C, D)$ consisting of coefficients of $a_i, a_{i-1}, \cdots, a_1, 1$ from the equations $[1], [2], \cdots, [i+1]$. In fact, $A_{i+2,1}(A, B, C, D)$ is called a minor matrix of Δ by deleting the first column and the last row of Δ. Then $A_{i+2,1}(0, B, C, D) =$

$$\begin{pmatrix} 0 & 0 & \cdots & \binom{n}{1}C^{n-1}D \\ \vdots & \vdots & & \vdots \\ 0 & \binom{n-i+1}{0}C^{n-i+1}B^{i-1} & \cdots & \binom{n}{i-1}C^{n-i+1}D^{i-1} \\ \binom{n-i}{0}C^{n-i}B^i & \binom{n-i+1}{1}C^{n-i}DB^{i-1} & \cdots & \binom{n}{i}C^{n-i}D^i \\ \binom{n-i}{1}C^{n-i-1}DB^i & \binom{n-i+1}{2}C^{n-i-1}D^2B^{i-1} & \cdots & \binom{n}{i+1}C^{n-i-1}D^{i+1} \end{pmatrix}$$

Then $|A_{i+2,1}(0, B, C, D)| = B^{i(i+1)/2}C^{n(i+1)-(i+2)(i+1)/2}D^{i+1} \times$

$$\begin{vmatrix} 0 & 0 & \cdots & {}_nC_1 \\ \vdots & \vdots & & \vdots \\ 0 & {}_{n-i+1}C_0 & \cdots & {}_nC_{i-1} \\ {}_{n-i}C_0 & {}_{n-i+1}C_1 & \cdots & {}_nC_i \\ {}_{n-i}C_1 & {}_{n-i+1}C_2 & \cdots & {}_nC_{i+1} \end{vmatrix}$$

$= B^{i(i+1)/2}C^{n(i+1)-(i+2)(i+1)/2}D^{i+1} \cdot (-1)^{(i+1)i/2} {}_nC_{i+1}$ by Lemma 1.2. Since $|A_{i+2,1}(0, B, C, D)| = 0$, $A = 0$ and $AD - BC \neq 0$, D must be zero. From coefficients of $yz^{n-1}, y^2z^{n-2}, \cdots, y^iz^{n-i}$ in the homogeneous equations $[1], [2], \cdots, [i]$, then we have $BCa_1 = BCa_2 = \cdots = BCa_i = 0$ because $A = D = 0$. So we get $a_1 = a_2 = \cdots = a_i = 0$. Thus we have the same result as in the case (b).

(d) $B = 0$: Since $AD - BC \neq 0$, $AD \neq 0$. Just as in the case (c), note that each element of the first column of Δ is zero if $B = 0$. So by the similar method as in the case (c) it is enough to consider the minor matrix $A_{i+2,1}(A, B, C, D)$. Let us compute $A_{i+2,1}(A, 0, C, D)$. Then $A_{i+2,1}(A, 0, C, D) = (\beta_{pq})$ where

$$\beta_{pq} = \binom{n-i+q-1}{p} C^{n-i+p+q-1} D^p A^{i-(q-1)}$$

with $1 \le p, q \le i+1$.

Then $|A_{i+2,1}(A,0,C,D)| = A^{i(i+1)/2} D^{(i+1)(i+2)/2} C^{(i+1)(n-i-1)} \times$

$$\begin{vmatrix} {}_{n-i}C_1 & {}_{n-i+1}C_1 & \cdots & {}_nC_1 \\ {}_{n-i}C_2 & {}_{n-i+1}C_2 & \cdots & {}_nC_2 \\ \vdots & & & \\ {}_{n-i}C_{i+1} & {}_{n-i+1}C_{i+1} & \cdots & {}_nC_{i+1} \end{vmatrix}$$

$= {}_nC_{i+1} A^{i(i+1)/2} D^{(i+1)(i+2)/2} C^{(i+1)(n-i-1)}$ by Lemma 1.2. So $C = 0$ because $|A_{i+2,1}(A,0,C,D)| = 0$, $B = 0$ and $AD - BC \neq 0$.

In the case of (a) and (d), that is, $B = C = 0$, by [3] $f(Az+By, Cz+Dy) = f(Az, Dy) = (Az)^n + a_i(Dy)^{n-i}(Az)^i + \cdots + a_1(Dy)^{n-1}Az + (Dy)^n = ug(z,y) = u(z^n + b_iy^{n-i}z^i + \cdots + b_1y^{n-1}z + y^n)$ for some nonzero constant u. Thus we get : $A^n = u$, $D^n = u$, $D^{n-k}A^ka_k = ub_k (2k+3 \le n)$. Since $(A/D)^n = 1$, put $\omega = A/D$. Also $D^{n-k}A^ka_k = ub_k$ implies that $(A/D)^ka_k = b_k$. Thus we get $b_k = a_k\omega^k$ for $k = 1, 2, \cdots, i$ $(2i+3 \le n)$.

In the case of (b) and (c), that is, $A = D = 0$, there is nothing to prove.

Conversely, suppose that there exists a unit ω with $\omega^n = 1$ such that $b_k = a_k\omega^k$ for $k = 1, 2, \cdots, i = j$. Define the map ψ by $\psi(z,y) = (\omega z, y)$. Then $f \circ \psi(z,y) = z^n + a_iy^{n-i}(\omega z)^i + \cdots + a_1y^{n-1}(\omega z) + y^n = z^n + b_iy^{n-i}z^i + \cdots + b_1y^{n-1}z + y^n = g(z,y)$. Thus the theorem is proved.

COROLLARY 1.4. *Let f and g be defined as in the Theorem 1.3. If $f \approx g$ and $f(Az+By, Cz+Dy) = ug(z,y)$ for some nonzero constant u, then either $B = C = 0$, or $A = D = 0$ and $a_k = b_k = 0$ for $1 \le k \le i = j$ with $n \ge 2i+3$.*

2. Analytic classification of plane curve singularities defined by $z^3 + ay^2z + y^3$ or $z^4 + ay^3z + y^4$

THEOREM 2.1. *Let $V = \{(z,y) : f = z^3 + ay^2z + y^3 = 0\}$ be an analytic subvariety of a polydisc near the origin in $\mathbf{C}^2$ where f is square-free. Then any f is analytically equivalent each other for any number a.*

Proof. We know that any homogeneous polynomial with two variables of degree three which is square-free can be written into $f = z^3 + ay^2z + y^3$ by a nonsingular linear change of coordinate at the origin, and also this f can be transformed into $u(z^3 + \alpha yz^2 + \beta y^2 z) = uz(z^2 + \alpha yz + \beta y^2)$ for a nonzero constant u by another linear change of coordinates. Note that $uz(z^2 + \alpha yz + \beta y^2) = uz(z^2 + 2\alpha_1 y_1 z + y_1^2)$ by a linear change of coordinates and that this polynomial becomes $uz((1-\alpha_1^2)z^2 + (y_1 - \alpha_1 z)^2) = u_1 z_1(z_1^2 + y_2^2)$ for a nonzero constant u_1 where $z_1 = (1-\alpha_1^2)^{1/2}z$, $y_2 = y_1 - \alpha_1 z$ and $\alpha_1 \neq 1$. Thus $f \approx z(z^2+y^2)$.

THEOREM 2.2. *Let $V = \{(z,y) : f = z^4 + \alpha y^3 z + y^4 = 0\}$ and $W = \{(z,y) : g = z^4 + \beta y^3 z + y^4 = 0\}$ be analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f and g are square-free. Then $f \approx g$ if and only if $\alpha^4 = \beta^4$.*

Proof. Assume that $f \approx g$. Then $f(Az + By, Cz + Dy) = (Az + By)^4 + \alpha(Cz + Dy)^3(Az + By) + (Cz + Dy)^4 = (A^4 + \alpha AC^3 + C^4)z^4 + (4A^3B + (3C^2DA + C^3B)\alpha + 4C^3D)yz^3 + (6A^2B^2 + (3CD^2A + 3C^2DB)\alpha + 6C^2D^2)y^2z^2 + (4AB^3 + (D^3A + 3CD^2B)\alpha + 4CD^3)y^3z + (B^4 + \alpha BD^3 + D^4)y^4 = u(z^4 + \beta y^3 z + y^4)$ for a nonzero constant u by [3].
So we have

$$A^4 + AC^3\alpha + C^4 = u \tag{1}$$

$$4A^3B + (3C^2DA + C^3B)\alpha + 4C^3D = 0 \tag{2}$$

$$6A^2B^2 + (3CD^2A + 3C^2DB)\alpha + 6C^2D^2 = 0 \tag{3}$$

$$4AB^3 + (D^3A + 3CD^2B)\alpha + 4CD^3 = u\beta \tag{4}$$

$$B^4 + BD^3\alpha + D^4 = u \tag{5}$$

Subtracting the equation (5) from the equation (1), we get

$$(A^4 - B^4) + (AC^3 - BD^3)\alpha + C^4 - D^4 = 0 \tag{6}$$

Now consider the following two cases : (i) $ABCD\alpha = 0$ and (ii) $ABCD\alpha \neq 0$.

(i) Let $ABCD\alpha = 0$. Then $A = 0,\ B = 0,\ C = 0,\ D = 0$ or $\alpha = 0$.

(a) $A = 0$: From equations (2) and (3), we get

$$C^3B\alpha + 4C^3D = 0,$$
$$3C^2DB\alpha + 6C^2D^2 = 0.$$

These two equations give $6C^3D^2 = 0$ and so $D = 0$ since $AD - BC \neq 0$. From (2), $C^3B\alpha = 0$ implies $\alpha = 0$. From (1) and (5), $C^4 = u = B^4$ and (4) implies $\beta = 0$. Thus $\alpha^4 = \beta^4 = 0$.

(b) $D = 0$: By (3), $AB = 0$ and so $A = 0$. Then we get the same result as in the case (a).

(c) $C = 0$: By (2), $AB = 0$ and so $B = 0$. By (1), (4) and (5), $A^4 = D^4 = u$ and $D^3A\alpha = u\beta$. Thus $A\alpha = D\beta$ and so $\alpha^4 = \beta^4$.

(d) $B = 0$: From (2) and (3), we get

$$\begin{aligned} 3C^2DA\alpha + 4C^3D &= 0, \\ 3CD^2A\alpha + 6C^2D^2 &= 0. \end{aligned}$$

These two equations give $2C^3D^2 = 0$ and so $C = 0$. Then we get the same result as in the case (c).

(e) $\alpha = 0$: From (2) and (3), we get

$$0 = \begin{vmatrix} 4A^3B & 4C^3D \\ 6A^2B^2 & 6C^2D^2 \end{vmatrix} = 24A^2BC^2D \begin{vmatrix} A & C \\ B & D \end{vmatrix}.$$

So $ABCD = 0$. Then we get the same result as in the case (a), (b), (c) or (d).

(ii) Hereafter we assume that $ABCD\alpha \neq 0$.

Then from (2), (3) and (6) which are considered homogeneous equations, we get

$$\begin{aligned} 0 &= \begin{vmatrix} 4A^3B & 3C^2DA + C^3B & 4C^3D \\ 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ A^4 - B^4 & AC^3 - BD^3 & C^4 - D^4 \end{vmatrix} \\ &= 6C(AD - BC)^3(A^2C^3 - BD^2(2AD + BC)). \end{aligned}$$

Since $ABCD \neq 0$, $2AD + BC \neq 0$. So

$$BC + 2AD = A^2C^3/(BD^2) \tag{7}$$

From (2) and (3), we get

$$\begin{aligned} C^2(3AD + BC)\alpha &= -4(A^3B + C^3D) \quad \text{and} \\ CD(AD + BC)\alpha &= -2(A^2B^2 + C^2D^2), \end{aligned}$$

which by eliminating α, give $0 = 2D(AD + BC)(A^3B + C^3D) - C(3AD+BC)(A^2B^2+C^2D^2) = (A^2B(2AD+BC)-C^3D^2)(AD-BC)$. Thus

$$2AD + BC = C^3D^2/(A^2B) \tag{8}$$

From (7) and (8), $A^2C^3/(BD)^2 = C^3D^2/(A^2B)$ and so we get

$$A^4 = D^4 \tag{9}$$

From (2), (3) and (4) we are going to compute α as follows : Let

$$\Delta = \begin{pmatrix} 4A^3B & 3C^2DA + C^3B & 4C^3D \\ 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ 4AB^3 & D^3A + 3CD^2B & 4CD^3 \end{pmatrix}, \text{ and then}$$

$$|\Delta| = 24ABC^2D^2(AD - BC)^3.$$

$$\alpha = \frac{1}{|\Delta|} \begin{vmatrix} 4A^3B & 0 & 4C^3D \\ 6A^2B^2 & 0 & 6C^2D^2 \\ 4AB^3 & u\beta & 4CD^3 \end{vmatrix} = \frac{-u\beta A}{D(AD - BC)^2} \tag{10}$$

Again, from (3), (4) and (5), we want to compute α as follows. Let

$$\Delta' = \begin{pmatrix} 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ 4AB^3 & D^3A + 3CD^2B & 4CD^3 \\ B^4 & BD^3 & D^4 \end{pmatrix}$$

Then $|\Delta'| = 6B^2D^4(AD - BC)^3$ and so

$$\alpha = \frac{1}{|\Delta'|} \begin{vmatrix} 6A^2B^2 & 0 & 6C^2D^2 \\ 4AB^3 & u\beta & 4CD^3 \\ B^4 & u & D^4 \end{vmatrix} \tag{11}$$

$$= \frac{u}{D^2(AD - BC)^3} \cdot ((AD + BC)\beta - 4AC)$$

From (10) and (11), we get

$$\frac{-u\beta A}{D(AD - BC)^2} = \frac{u}{D^2(AD - BC)^3} \cdot ((AD + BC)\beta - 4AC)$$

Thus

$$\beta = \frac{4AC}{2AD + BC} \tag{12}$$

Eliminating the first terms from the equations (2) and (3), we get $[6B(3C^2DA+C^3B)-4A(3CD^2A+3C^2DB)]\alpha+24BC^3D-24AC^2D^2 = 0$. Simplifying the above, we have

$$\alpha = -\frac{4CD}{2AD + BC} \tag{13}$$

From (12) and (13), $\beta/A = -\alpha/D$. Since $A^4 = D^4$ by (9), we get $\beta^4 = \alpha^4$.

Now, conversely, if $\alpha^4 = \beta^4 \neq 0$, then $f(\beta z, \alpha y) = \beta^4 z^4 + \alpha(\alpha^3 y^3)\beta z + \alpha^4 y^4 = \beta^4(z^4 + \beta y^3 z + y^4) = \beta^4 g(z, y)$. If $\alpha = \beta = 0$, then there is nothing to prove.

COROLLARY 2.3. *Let f and g be defined as in Theorem 2.2. If $f \approx g$ and $f(Az + By, Cz + Dy) = ug(z, y)$ for a nonzero constant u, then $ABCD$ may not be zero.*

Proof. It is enough to show that there is such an example with $ABCD \neq 0$. Let $f(z, y) = z^4 - 4e^{\pi i/4}y^3 z + y^4$ and $A = 1$, $B = e^{3\pi i/4}$, $C = e^{\pi i/4}$ and $D = 1$. Then $AD - BC = 2 \neq 0$ and $f(Az + By, Cz + Dy) = 4(z^4 + 4e^{\pi i/4}y^3 z + y^4) = 4g(z, y)$ by tedius computations. Note that $ABCD \neq 0$.

Finally we are going to give an example which is a help to understand the condition for restriction on the degree of homgeneous polynomials in Theorem 1.3 as follows :

Let $V = \{(z, y) : f(z, y) = z^5 + 10y^3 z^2 + 5y^4 z + y^5 = 0\}$. By a linear transformation $T : (z, y) \mapsto (y, z - y)$,

$$\begin{aligned}(f \circ T)(z, y) &= f(y, z - y) \\ &= y^5 + 10(z - y)^3 y^2 + 5(z - y)^4 y + (z - y)^5 \\ &= z^5 - 10y^3 z^2 + 15y^4 z - 5y^5.\end{aligned}$$

By another linear transformation $S : (z, y) \mapsto (z, -5^{-1/5}y)$, $f \circ T$ will be $g(z, y) = z^5 + 10 \cdot 5^{-3/5}y^3 z^2 + 15 \cdot 5^{-4/5}y^4 z + y^5$. Note that without the condition in Theorem 1.3, $f \approx g$.

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