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POLYGONAL KNOTS

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0. Introduction

In the 3-dimensional Euclidean space, every knot is ambient isotopic to spatial polygon, i.e., a simple closed curve obtained by joining a finitely many vertices with straight edges. The spatial polygons will be called the *polygonal knots.*

The minimal number of vertices (or equivalently, edges) of the family of polygonal knots ambient isotopic to a given knot is certainly a knot invariant. We call it the *polygon index.* In this work, we show that every nontrivial knot has polygon index not smaller than 6. Also we give some estimations of the polygon indices of some knots.

1. Pentagons are unknotted

The polygon index of a knot k will be denoted by $P(k)$. For a polygonal knot k, the number of vertices (or equivalently, edges) will be called the *polygon number* of k, and denote by $p(k)$. Given n points $P_1, P_2, \ldots, P_n$ in $\mathbf{R}^3$, let $\overline{P_1P_2\ldots P_n}$ denote the convex hull of the set $\{P_1, P_2, \ldots P_n\}$.

PROPOSITION 1. *If $P(k) \leq 5$, then k is unknotted.*

Proof. Let k be a polygonal knot. We will show that k is unknotted if $p(k) \leq 5$.

Suppose $p(k) = 3$. Then k is a triangle, and hence unknotted.

Suppose $p(k) = 4$. Then k is a spatial 4-gon. If A, B, C and D are vertices of k in that order, then k is the boundary of the disc $\overline{ABC} \cup \overline{ACD}$ or $\overline{ABD} \cup \overline{BCD}$. Hence k is unknotted.

Suppose $p(k) = 5$. Let A, B, C, D and E be the vertices of k in that order. We may assume that no three cyclically consecutive vertices are

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on one line. If all the five vertices are on one plane, then k is a Jorden curve, and hence unknotted. If all but one vertex, say A, are on one plane, then k is the boundary of the disc $\overline{ABC} \cup \overline{ACD} \cup \overline{ADE}$. Note that the disc is the cone on $\overline{BC} \cup \overline{CD} \cup \overline{DE}$ with apex at A.

Now suppose that A, B, C, D and E are in general position. If $\overline{DE}$ does not intersect the plane determined by $\overline{ABC}$, then k is the boundary of the disc $\overline{ABC} \cup \overline{ACD} \cup \overline{ADE}$. If $\overline{DE}$ intersects the plane determined by $\overline{ABC}$, let F be the intersection point. If the line BF separates A and C, then k is the boundary of the disc $\overline{AEF} \cup \overline{ABF} \cup \overline{BCF} \cup \overline{CDF}$. Note also that the disc is the cone on $\overline{AE} \cup \overline{AB} \cup \overline{BC} \cup \overline{CD}$ with apex at F. Suppose the line BF does not separate A and C. Let F' be the midpoint of $\overline{DF}$ if $\angle FBA < \angle FBC$ or the midpoint of $\overline{EF}$ if $\angle FBA > \angle FBC$. Then k is the boundary of the disc $\overline{AEF'} \cup \overline{ABF'} \cup \overline{BCF'} \cup \overline{CDF'}$. Again the disc is the cone on $\overline{AE} \cup \overline{AB} \cup \overline{BC} \cup \overline{CD}$ with apex at F'. □

COROLLARY 2. *The trefoil knot has polygon index 6.*

Proof.

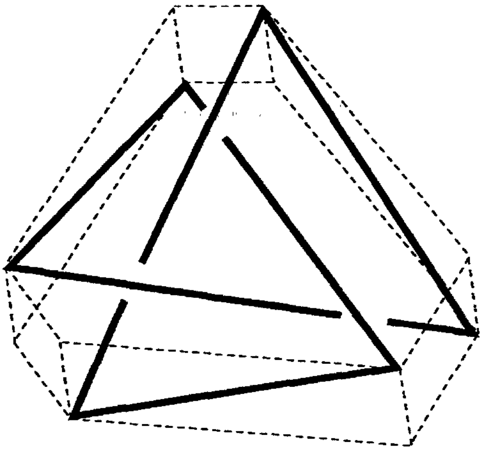

FIGURE 1.

COROLLARY 3. *k is a nontrivial knot if and only if $P(k) \geq 6$.*

2. The superbridge index

For a knot k, let $S(k)$ denote the superbridge index of k. See [1] for the definition of the superbridge index. Given a pair of relatively prime integers r, s, let $T_{r,s}$ denote the (r,s)-torus knot.

LEMMA 4. *For any knot* $k, 2S(k) \leq P(k)$.

Proof. It is easy to see that the number of maxima of the height function of a polygonal knot k with respect to any axis is not greater than $p(k)/2$. Therefore $S(k) \leq p(k)/2$, and hence $2S(k) \leq P(k)$. □

COROLLARY 5. *Let r, s be relatively prime positive integers satisfying $2 \leq r < s$. Then*

$$P(T_{r,s}) \geq 2\min\{2r, s\}.$$

Proof. This follows from the fact that $S(T_{r,s}) = \min\{2r, s\}$. See [1,Theorem B]. □

3. Upper bounds for the polygon index

THEOREM 6. *If s is odd and if $s \geq 3$, then $P(T_{2,s}) \leq s + 3$.*

Proof. Since $T_{2,3}$ is the trefoil knot, the theorem holds for $s = 3$, by Corollary 2. For $s = 5, 7, 9, \ldots$, we construct $T_{2,s}$ as follows.

For $i \geq 1$, let $a_i = \frac{1}{2}i(i-1)$ and let

$$x_i = \cos\frac{4i-3-s}{s}\pi,$$
$$y_i = \sin\frac{4i-3-s}{s}\pi.$$

If $s = 4n + 1$, for some positive integer n, define

$$A_i = \begin{cases} (x_i, \ y_i, \ a_{2i-1}) & 1 \leq i \leq n+1 \\ (x_{i-1}, y_{i-1}, -a_{4n+5-2i}) & n+2 \leq i \leq 2n+2 \\ (x_{i-1}, y_{i-1}, a_{2i-4n-4}) & 2n+3 \leq i \leq 3n+2 \\ (x_{i-2}, y_{i-2}, 2a_{2n+2}) & i = 3n+3 \\ (x_{i-2}, y_{i-2}, -2a_{2n+2}) & i = 3n+4 \\ (x_{i-3}, y_{i-3}, -a_{8n+10-2i}) & 3n+5 \leq i \leq 4n+4, \end{cases}$$

and if $s = 4n + 3$, for some positive integer n, define

$$A_i = \begin{cases} (x_i, \quad y_i, \quad a_{2i-1}) & 1 \leq i \leq n+1 \\ (x_{n+1}, y_{n+1}, 2a_{2n+3}) & i = n+2 \\ (x_{n+2}, y_{n+2}, -2a_{2n+3}) & i = n+3 \\ (x_{i-2}, y_{i-2}, -a_{4n+9-2i}) & n+4 \leq i \leq 2n+4 \\ (x_{i-2}, y_{i-2}, a_{2i-4n-8}) & 2n+5 \leq i \leq 3n+5 \\ (x_{i-3}, y_{i-3}, -a_{8n+14-2i}) & 3n+6 \leq i \leq 4n+6. \end{cases}$$

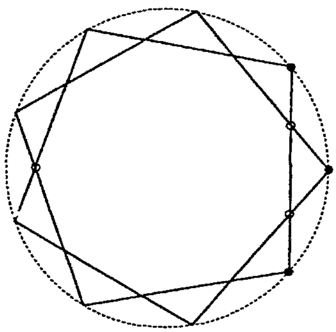

FIGURE 2.

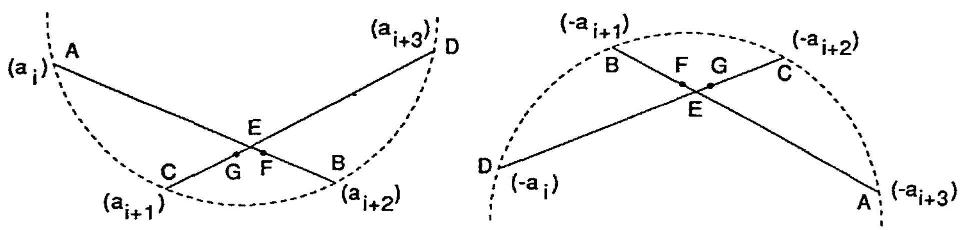

FIGURE 3.

Then the projection of the $(s + 3)$-gon $A_1 A_2 \ldots A_{s+3}$ into the xy-plane is a regular s-gonal star as in Figure 2. In Figure 2, at each thick vertex there are two vertices overlapped. It is clear that every crossing, except three (which are circled in Figure 2), in the projection is as in Figure 3, where the labels in parentheses represent the z-coordinates. Choose points F and G on $\overline{AB}$ and $\overline{CD}$, respectively, so that $\overline{FB} = \frac{1}{4}\overline{AB}$ and $\overline{CG} = \frac{1}{4}\overline{CD}$. Then the intersection $\overline{AB} \cap \overline{CD}$ is the point $E = \overline{AF} \cap \overline{GD}$. Noitce that F and G have the same z-coordinate $\pm\frac{1}{4}(2i^3 + 4i + 3)$. Therefore $\overline{CD}$ crosses over $\overline{AB}$ at E. With similar computations on the three exceptional crossings of Figure 2, we can see that projection is an alternating diagram so that it represents a copy of $T_{2,s}$. □

COROLLARY 7. $P(T_{2,5}) = 8$.

THEOREM 8. *If r and s are relatively prime integers satisfying $2 \leq r < s$, then*

$$P(T_{r,s}) \leq r\, min\, \{n|n > \frac{2s}{r}\}.$$

Proof. Let $m = \min\{n|n > \frac{2s}{r}\}$. For $j = 0, 1, \ldots, mr-1$, define

$$P_j = \left(\cos\frac{2\pi j}{m}(L + 2\sec\frac{\pi}{m}\cos\frac{2\pi sj}{mr}),\right.$$
$$\left.\sin\frac{2\pi j}{m}(L + 2\sec\frac{\pi}{m}\cos\frac{2\pi sj}{mr}), \sin\frac{2\pi sj}{mr}\right).$$

We will show that the mr-gon $k = P_0P_1\ldots P_{mr-1}$ is a copy of $T_{r,s}$ if L is large enough. Let k_0 be the regular m-gon with vertices at

$$Q_j = \left(L\cos\frac{2\pi j}{m},\ L\sin\frac{2\pi j}{m},\ 0\right)$$

for $j = 0, 1, \ldots, m$. Let B_j be the set of r edges of k around $\overline{Q_jQ_{j+1}}$, for $j = 0, 1, \ldots, m-1$.

For $u = 0, 1, \ldots, r-1$, define

$$a_u(x,t) = \left(\cos(x + \frac{2\pi u}{r}),\ \sin(x + \frac{2\pi u}{r}),\ L - t\tan\frac{\pi}{m}\cos(x + \frac{2\pi u}{r})\right),$$
$$b_u(x,t) = \left(\cos(y + \frac{2\pi u}{r}),\ \sin(y + \frac{2\pi u}{r}),\ -L + t\tan\frac{\pi}{m}\cos(y + \frac{2\pi u}{r})\right),$$

where $0 \leq t \leq 1$ and $y = x + \frac{2\pi s}{mr}$. Let $l_u(x,t)$ denote the line segment joining $a_u(x,t)$ and $b_u(x,t)$ and let $B(x,t) = \{l_u(x,t)|0 \leq u \leq r-1\}$. It is not hard to see that there is a rigid motion of $\mathbf{R}^3$ mapping $B(\frac{2\pi s}{mr}j, 1)$ onto B_j. To show that k is a torus knot $T_{r,s}$, it is enough to show that $l_u(x,t)$ and $l_v(x,t)$ do not intersect for sufficiently large L. Let $d_{uv}(x,t)$ denote the distance between the line segments $l_u(x,t)$ and $l_v(x,t)$. Then

$$d_{uv}(x,t) = \frac{|\{(a_u(x,t) - b_u(x,t)) \times (a_v(x,t) - b_v(x,t))\} \cdot (a_u(x,t) - a_v(x,t))|}{\|(a_u(x,t) - b_u(x,t)) \times (a_v(x,t) - b_v(x,t))\|}.$$

It is a good exercise in Vector Calculus and Trigonometry to compute

$$\lim_{L\to\infty} d_{uv}(x,t) = 2 \cos \frac{\pi s}{mr} \sin \frac{\pi|u-v|}{r}.$$

Notice that this limit is independent of x and t, and is positive if $u \neq v$. □

COROLLARY 9. *For any positive integer* $r, P(T_{2r+1,3r+1})$ *is either* $6r+2$ *or* $6r+3$.

Proof. Notice that $\min\{2(2r+1), 3r+1\} = 3r+1$ and $\min\{n|n > 2(3r+1)/(2r+1)\} = 3$. According to Corollary 5 and Theorem 8, we have $6r+2 \leq P(T_{2r+1,3r+1}) \leq 6r+3$. □

THEOREM 10. *Let* W_s *be the Whitehead-double of the unknot having crossing number* s *for* $s \geq 3$. *Then* $P(W_s) \leq s+3$.

Proof. By Corollary 2, $P(W_3) = 6$. As mentioned in [2,Theorem 6], $P(W_4) = 7$, and $P(W_5) \geq 8$. Figure 4 shows the projection of the octagonal knot with vertices at $(5,8,0), (5,0,0), (1,7,8), (7,7,-5), (7,2,5), (3.5,4.5,-3), (6.5,4.5,1)$ and $(3,8,1)$, into the xy-plane.
Hence $P(W_5) = 8$.

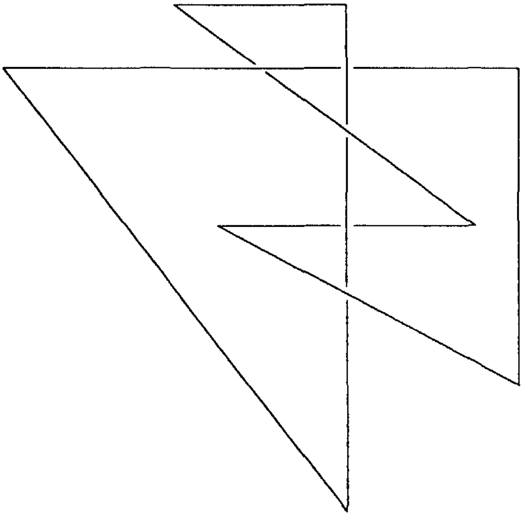

FIGURE 4.

For $s = 6, 7, 8, \ldots$, we construct W_s as follows. For $n \geq 1$, let $a_i = \frac{1}{2}i(i-1)$ and let $C_i = (\cos \frac{2i-s}{s-1}\pi, \sin \frac{2i-s}{s-1}\pi)$. If $s = 4n$ for some integer $n \geq 2$, define

$$A_i = \begin{cases} (C_{2i-1}, & a_{2i-1}) & \text{if } 1 \leq i \leq n \\ (\frac{1}{4}(C_{2n-1} + 3C_{2n+1}), & a_{2n+1}) & \text{if } i = n+1 \\ (C_{4n+4-2i}, & a_{4n+4-2i}) & \text{if } n+2 \leq i \leq 2n+1 \\ (C_{4n+4-2i}, & -a_{2i-4n-3}) & \text{if } 2n+2 \leq i \leq 3n+1 \\ (\frac{1}{4}(C_{2-2n} + 2C_{1-2n} + C_{-2n}), & a_{2n+1}) & \text{if } i = 3n+2 \\ ((0,0), & a_{2n+1}) & \text{if } i = 3n+3 \\ (\frac{1}{2}(C_{3-2n} + C_{1-2n}), & -a_{2n}) & \text{if } i = 3n+4 \\ (C_{2i-8n-7}, & -a_{8n+8-2i}) & \text{if } 3n+5 \leq i \leq 4n+3. \end{cases}$$

If $s = 4n+1$ for some integer $n \geq 2$, define

$$A_i = \begin{cases} (C_{2i-1}, & a_{2i-1}) & \text{if } \quad 1 \leq i \leq n \\ (\frac{1}{4}(C_{2n-1} + 3C_{2n+1}), & a_{2n+1}) & \text{if } \quad i = n+1 \\ (C_{4n+4-2i}, & a_{4n+4-2i}) & \text{if } \; n+2 \leq i \leq 2n+1 \\ (C_{4n+4-2i}, & -a_{2i-4n-3}) & \text{if } 2n+2 \leq i \leq 3n+1 \\ (\frac{1}{2}(C_{2-2n} + C_{-2n}), & -a_{2n+1}) & \text{if } \quad i = 3n+2 \\ ((0,0), & a_{2n+1}) & \text{if } \quad i = 3n+3 \\ (\frac{1}{4}(C_{1-2n} + 2C_{-2n} + C_{-1-2n}), & a_{2n+1}) & \text{if } \quad i = 3n+4 \\ (C_{2i-8n-9}, & -a_{8n+10-2i}) & \text{if } 3n+5 \leq i \leq 4n+4. \end{cases}$$

If $s = 4n + 2$ for some positive integer n, define

$$A_i = \begin{cases} (C_{2i-1}, & a_{2i-1}) & \text{if } \quad 1 \leq i \leq n \\ (\frac{1}{4}(C_{2n-1} + 3C_{2n+1}), & a_{2n+1}) & \text{if } \quad i = n+1 \\ (C_{4n+4-2i}, & a_{4n+4-2i}) & \text{if } \; n+2 \leq i \leq 2n+1 \\ (C_{4n+4-2i}, & -a_{2i-4n-3}) & \text{if } 2n+2 \leq i \leq 3n+2 \\ (\frac{1}{4}(C_{-2n} + 2C_{-1-2n} + C_{-2-2n}), & a_{2n+1}) & \text{if } \quad i = 3n+3 \\ ((0,0), & a_{2n+1}) & \text{if } \quad i = 3n+4 \\ (\frac{1}{2}(C_{1-2n} + C_{-1-2n}), & -a_{2n+2}) & \text{if } \quad i = 3n+5 \\ (C_{2i-8n-11}, & -a_{8n+12-2i}) & \text{if } 3n+5 \leq i \leq 4n+5. \end{cases}$$

If $s = 4n + 3$ for some positive integer n, define

$$A_i = \begin{cases} (C_{2i-1}, & a_{2i-1}) & \text{if } \quad 1 \leq i \leq n \\ (\frac{1}{4}(C_{2n-1} + 3C_{2n+1}), & a_{2n+1}) & \text{if } \quad i = n+1 \\ (C_{4n+4-2i}, & a_{4n+4-2i}) & \text{if } \; n+2 \leq i \leq 2n+1 \\ (C_{4n+4-2i}, & -a_{2i-4n-3}) & \text{if } 2n+2 \leq i \leq 3n+2 \\ (\frac{1}{2}(C_{-2n} + C_{-2-2n}), & a_{2n+3}) & \text{if } \quad i = 3n+3 \\ ((0,0), & a_{2n+1}) & \text{if } \quad i = 3n+4 \\ (\frac{1}{4}(C_{-1-2n} + 2C_{-2-2n} + C_{-3-2n}), & a_{2n+1}) & \text{if } \quad i = 3n+5 \\ (C_{2i-8n-13}, & -a_{8n+14-2i}) & \text{if } 3n+5 \leq i \leq 4n+6. \end{cases}$$

Then the projection of the $(s + 3)$-gon $k = A_1A_2 \ldots A_{s+3}$ into the xy-plane is, if $s = 11$, as in Figure 5. An argument as in the proof of Theorem 6 shows that the projection is an alternating diagram so that k represents a copy of W_s. □

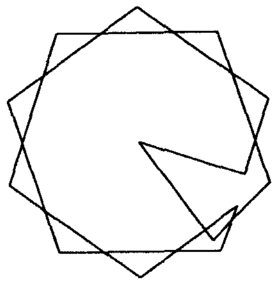

FIGURE 5.

THEOREM 11. *Let $\Sigma(a_1, a_2, \ldots, a_n)$ denote the pretzel knot (or link) of type $(a_1, a_2, \ldots, a_n)$. Then*

$$P(\Sigma(a_1, a_2, \ldots, a_n)) \leq n + \sum_{i=1}^{n} |a_i|.$$

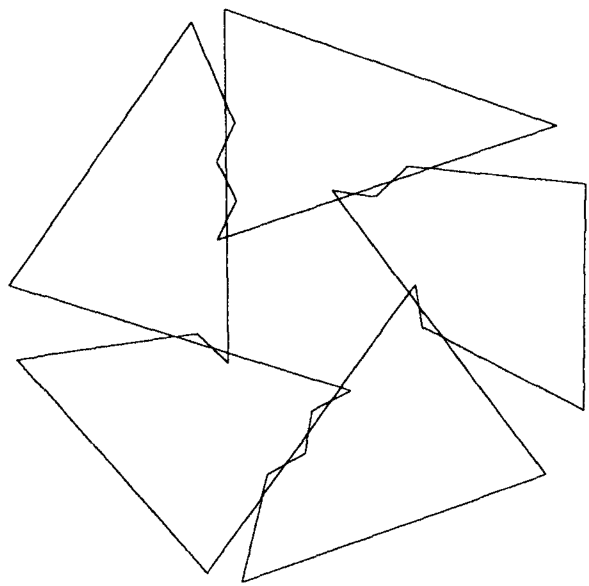

FIGURE 6.

Proof. We construct an $(n + \sum_{i=1}^{n} |a_i|)$-gon whose projection into the xy-plane is as in Figure 6. Figure 6 shows the case $(a_1, a_2, \ldots, a_n) = (3,4,5,3,5)$ so that $(n + \sum_{i=1}^{n} 6n\,|a_i|) = 25$. It is not hard to choose the z-coordinates of the vertices so that the polygon is actually a copy of $\Sigma(a_1, a_2, \ldots, a_n)$. See [4] for detail. □

4.The polygon index of composite knots

Let $k\sharp l$ denote the connected sum of k and l. A vertex of a polygonal knot willl be callled an *external vertex* if it is contained in the boundary of the convex hull of the knot.

THEOREM 12. *Let k and l be knots. Then $P(k\sharp l) \leq P(k)+P(l)-2$.*

Proof. We may assume that k and l are polygonal knots with vertices $P_0, \ldots, P_{P(k)-1}$ and $Q_0, \ldots, Q_{P(l)-1}$, respectively, such that Q_0 is external. By affine transformations of $\mathbf{R}^3$, we may assume that $\angle P_{P(k)-1}P_0P_1$ and $\angle Q_{P(l)-1}Q_0Q_1$ are right angles. By resizing, if necessary, we may put k and l together so that

(i) $k \cap \bar{l} = P_0 = Q_0$, where $\bar{l}$ is the convex hull of l,
(ii) $P_{P(k)-1}, P_0$ and Q_1 are on one line,
(iii) $Q_{P(l)-1}, Q_0$ and P_1 are on one line.

Now we can choose a point R near P_1 so that the curve obtained by joining the points $R, P_2, \ldots, P_{P(k)-1}, Q_1, \ldots, Q_{P(l)-1}, R$ by straight edges in that order is the connected sum $k\sharp l$. Therefore $p(k\sharp l) \leq P(k)+P(l)-2$. □

An edge of a polygonal knot will be called an *external edge* if it is contained in the boundary of the convex hull of the knot. An edge of a polygonal knot $k = V_1V_2\ldots V_n$, say $\overline{V_2V_3}$,is of *positive type* if

$$(\overrightarrow{V_1V_2} \times \overrightarrow{V_2V_3}) \cdot \overrightarrow{V_3V_4} > 0,$$

and of *negative type* if

$$(\overrightarrow{V_1V_2} \times \overrightarrow{V_2V_3}) \cdot \overrightarrow{V_3V_4} < 0.$$

THEOREM 13. *Let k and l be knots. If k admits a minimal polygonal embedding which has an external edge, then $P(k\sharp l) \leq P(k)+P(l)-3$. Furthermore, if l admits a minimal polygonal embedding such that one of the edge is of the same type with the external edge for k, then $P(k\sharp l) \leq P(k)+P(l)-4$.*

Proof. Suppose that k has a minimal polygonal embedding $k_1 = P_1P_2\dots P_n$ such that the edge $\overline{P_2P_3}$ is external, say, of positive type.

Let l have a minimal polygonal embedding $l_1 = Q_1Q_2\dots Q_m$. Choose an external vertex, say Q_2. By pulling Q_2 outward from the convex hull without making crossings of edges, we may assume that there is a plane $\mathcal{B}$ which separates Q_2 from all other vertices of l_1. We choose a point $Q_{2'}$ near Q_2 so that $l_2 = Q_1Q_{2'}Q_2\dots Q_m$ still represents l and the edge $\overline{Q_{2'}Q_2}$ is of positive type.

Since the edge $\overline{P_2P_3}$ is of positive type, there is an orientation preserving nonsingular affine transformation T of $\mathbb{R}^3$ such that the lines $T(\overleftrightarrow{P_1P_2})$ and $T(\overleftrightarrow{P_3P_4})$ are equal to the lines $\overleftrightarrow{Q_2Q_3}$ and $\overleftrightarrow{Q_1Q_{2'}}$, respectively. Furthermore, if we make det T small enough, we may place all $T(P_i)$'s on one side of the plane $\mathcal{B}$ where Q_2 is. Then it is clear that the $(m+n-3)$-gon $T(P_1)Q_3Q_4\dots Q_mT(P_4)T(P_5)\dots T(P_n)$ represents the knot $k\sharp l$. This proves the first part of the theorem.

Suppose that l_1 has an edge of positive type, say $\overline{Q_2Q_3}$. By applying the Gram-Schmidt orthonormalization processes if necessary, we may assume that $\{\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_3P_4}\}$ and $\{\overrightarrow{Q_1Q_2}, \overrightarrow{Q_2Q_3}, \overrightarrow{Q_3Q_4}\}$ are orthonormal sets. By moving vertices $Q_5,\dots,Q_m$ if necessary we may further assume that a thin cylindrical neighborhood of radius ϵ of the line $\overleftrightarrow{Q_2Q_3}$ does not intersect the edges $\overline{Q_4Q_5},\dots,\overline{Q_{m-1}Q_m}, \overline{Q_mQ_1}$. By an affine transformation which radially shrinks the planes perpendicular to $\overline{P_2P_3}$, we may put k_1 inside a cylinder of diameter ϵ so that the edge $\overline{P_2P_3}$ is a part of longitudal line of the cylinder. Then there is a rigid motion of $\mathbb{R}^3$ which moves k_1 so that $P_2 = Q_3$, $P_3 = Q_2$, $\overleftrightarrow{P_1P_2} = \overleftrightarrow{Q_3Q_4}$ and $\overleftrightarrow{Q_1Q_2} = \overleftrightarrow{P_3P_4}$. Then the $(m+n-4)$-gon $P_1Q_4Q_5\dots Q_mQ_1P_4\dots P_n$ represents $k\sharp l$. This proves the second part. $\square$

Figure 7 shows projections of the octagonal knots with vertices at $(-2,1,1),(2,-1,1),(1,1,-1.9),(1,-1,1.9),(2,1,-1),(-2,-1,-1),$

$(-1,1,1.9),(-1,-1,-1.9)$, and $(-2,1,1),(2,-1,1),(1,1,-1.9),(1,-1,1.9),(2,1,-1),(-2,-1,-1),(3,3,3.1),(3,-3,-3.1)$, into the xy-plane. It can be easily checked that the knot on the left is a square knot and the one on the right a granny knot. As mentioned in [2,Theorem 6], the unknot, the trefoil knots and the figure eight knot are the only knots with polygon indices less than 8. Therefore the square knot and the granny knot have polygon index 8. These are the cases that $P(k\sharp l) = P(k) + P(l) - 4$.

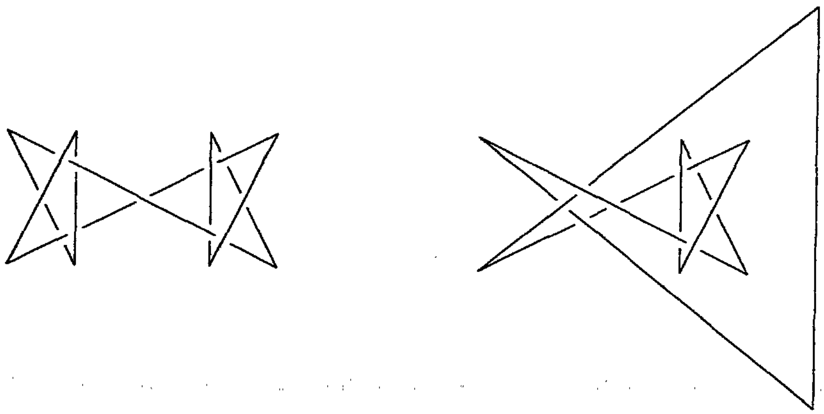

FIGURE 7

It is not hard to see that the polygonal torus knots $T_{r,s}$ as constructed in the proof of Theorem 8 have no external edges for $r \geq 3$. It is not known whether there actually exist knots all of whose minimal polygonal embeddings have no external edges. We guess that some torus knots k and l satisfy

$$P(k\sharp l) = P(k) + P(l) - 3.$$

We also guess that

$$3 \leq P(k) + P(l) - P(k\sharp l) \leq 4$$

for all knots k and l.

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