J. Korean Math. Soc. 30 (1993), No. 2, pp. 361–369

SURFACES WITH SIMPLE GEODESICS

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1. Introduction

Geodesics of a submanifold in Euclidean space play an important role in characterizing the submanifold. For example, if all the geodesics of submanifold are plane curves in the ambient Euclidean space, it is isometric to n-plane or a compact rank one symmetric space ([8] and [10]). Such a submanifold is called a submanifold with planar geodesics. A submanifold is called helical if all the geodesics have the same constant curvatures. K. Sakamoto [10] and other geometers studied helical submanifolds.

In 1981, B.-Y. Chen and P. Verheyen [3] introduced the notion of submanifolds with geodesic normal sections and classified surfaces with geodesic normal sections in a Euclidean space. They also proved that helical submanifolds have geodesic normal sections if the ambient manifold is Euclidean. Later, Verheyen [12] proved that submanifold with geodesic normal sections in Euclidean space is helical. Thus, the concept of submanifolds with geodesic normal sections coincides with that of helical submanifold if the ambient space is Euclidean.

In 1982, D. Ferus and S. Schirrmacher [4] studied the surface in Euclidean 4-space E^4 with simple geodesics which are geodesics of the surface as W-curves in E^4. A W- curve in a Euclidean space means all Frenet curvatures of the curve are constant along the curve. They classified surfaces in 4-dimensional Euclidean space E^4 in which all geodesics are W-curves. More precisely, we give the definition of W-curve ([4]): a regular curve $c : I \subset R \longrightarrow E^m$ is called a W-curve of rank d if for all $t \in I, c'(t) \wedge c''(t) \wedge \cdots \wedge c^{(d)}(t) \neq 0, c'(t) \wedge c''(t) \wedge \cdots \wedge c^{(d)}(t) \wedge c^{(d+1)}(t) = 0$ and the Frenet curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{d-1}$:

Received June 1, 1992.
1980 Mathematical Subject Classification (1985 Revision): Primary 53C40.
This research was partially supported by KOSEF, 1990.

$I \longrightarrow R_+$ are constant. By the fundamental theorem of curves, if a W-curve is of even rank, then there are positive constants $a_1, a_2, \cdots, a_k$, unique up to order, corresponding positive constants $r_1, r_2, \cdots, r_k$ and orthonormal vectors $e_1, e_2, \cdots, e_{2k} \in E^m$ such that

$$c(t) = \text{const.} + \sum_{i=1}^{k} r_i\{e_{2i-1} \ \cos a_i t + e_{2i} \sin a_i t\}.$$

The rank of unbounded W-curve is odd and the equation for such a curve contains an additional linear term in t.

We now extend our definition of W-curve in a Riemannian manifold $\tilde{M}$. Let $\tilde{\nabla}$ be the Riemannian connection of $\tilde{M}$. A regular curve $c : I \longrightarrow \tilde{M}$ is called a W-curve of rank d in $\tilde{M}$ if for all $t \in I, c'(t) \wedge \tilde{\nabla}_{c'(t)}c'(t) \wedge \cdots \wedge \tilde{\nabla}^{d-1}_{c'(t)}c'(t) \neq 0$, $c'(t) \wedge \tilde{\nabla}_{c'(t)}c'(t) \wedge \cdots \tilde{\nabla}^{d}_{c'(t)}c'(t) = 0$ and the Frenet curvatures $\kappa_1, \kappa_2, ..., \kappa_{d-1}$ are constant along c. Throughout this paper, a simple geodesic of surface means a W-curve regarded as a curve in the ambient manifold. In the present paper, we prove

THEOREM A. *Let M be a compact connected surface in 5-dimesional Euclidean space E^5. Then M has simple geodesics if and only if M is a standard torus $S^1(a) \times S^1(b) \subset E^4$, a 2-sphere $S^2(r) \subset E^3$ or a Veronese surface which is minimal in $S^4(r) \subset E^5$.*

THEOREM B. *Let M be a complete connected surface in 4 - sphere $S^4(r)$. Then M has simple geodesics if and only if M is a 2-sphere, a Veronese surface or a standard torus $S^1(a) \times S^1(b)$, $a^2 + b^2 = r^2$.*

2. Proof of Theorem A

The sufficiency is clear. We prove the necessity.

Case 1. There exists a non-periodic geodesic γ of M.

The exact same proof for the case 1 of Theorem 2 of [4] is applied to prove that M is sometric to a standard torus $S^1(a) \times S^1(b)$. We now give the sketch the proof. Let $\gamma : R \longrightarrow M$ be a non-periodic geodesic. Then, $x \circ \gamma$ is also non-periodic, where $x : M \longrightarrow E^5$ is an isometric immersion. And $x \circ \gamma$ must be of rank 4. Since $x \circ \gamma$ is not periodic, $\overline{(x \circ \gamma)(R)}$ is a torus $S^1(a) \times S^1(b)$ in $x(M)$. Since x is an immersion, there exists an non-empty open subset U of M such that

$x(U) \subset S^1(a) \times S^1(b)$. Then images of geodesics in U are geodesics in $S^1(a) \times S^1(b)$. Since M is compact, any point of M and a point of U can be joined with a geodesic. On the other hand, every geodesic of $S^1(a) \times S^1(b)$ is a W-curve. By the uniqueness of geodesics with initial Frenet data, $x(M) \subset S^1(a) \times S^1(b)$. Thus, M is sometric to a standard torus $S^1(a) \times S^1(b)$.

Case 2. All the geodesics of M are periodic.

Case 2-1. Every geodesic through p is of rank 2 for every point p of M. In this case, M has planar geodesics. By classification Theorem ([8],[9]), M is a sphere $S^2(r)$ or a Veronose surface.

Case 2-2. There exists a geodesic γ of rank 4 through p for some point p of M. We identify γ with $x \circ \gamma$. Then, $\gamma' \wedge \gamma'' \wedge \gamma''' \wedge \gamma^{(4)} \neq 0$ along γ. Let $\gamma(0) = p$. By continuity, a geodesic γ_1 through p which is sufficiently close to γ is also of rank 4. Let (s, θ) be the geodesic polar coordinates around p associated with an orthonormal basis e_1 and e_2 for T_PM. Then the immersion x can be expressed by

$$(*) \qquad \begin{aligned} x(s,\theta) = C(\theta) &+ r_1(\theta)\cos\alpha(\theta)s\mathbf{f}_1(\theta) + r_1(\theta)\sin\alpha(\theta)s\mathbf{f}_2(\theta) \\ &+ r_2(\theta)\cos\beta(\theta)s\mathbf{f}_3(\theta) + r_2(\theta)\sin\beta(\theta)s\mathbf{f}_4(\theta), \end{aligned}$$

where $x(0,\theta) = p, C(\theta)$ is a vector valued function, $r_1(\theta)$ and $r_2(\theta)$ are nonnegative valued functions, $\alpha(\theta)$ and $\beta(\theta)$ are positive valued functions and $\mathbf{f}_1(\theta), \mathbf{f}_2(\theta), \mathbf{f}_3(\theta)$ and $\mathbf{f}_4(\theta)$ are orthonormal vectors in E^5. For some $\theta_o \in (0, 2\pi), x(s, \theta_o) = \gamma(s)$. Then, $r_1(\theta) > 0$ and $r_2(\theta) > 0$ for $\theta \in I$, where I is an open interval containing θ_o. Let κ_1, κ_2 and κ_3 be the first, second and third Frenet curvatures of $x(s,\theta)$ respectively for $\theta \in I$. Then we have the following relations :

$$(1) \qquad \kappa_1(\theta)^2 + \kappa_2(\theta)^2 + \kappa_3(\theta)^2 = \alpha(\theta)^2 + \beta(\theta)^2,$$

$$(2) \qquad \kappa_1(\theta)^2\kappa_3(\theta)^2 = \alpha(\theta)^2\beta(\theta)^2$$

for every $\theta \in I$. Since all the geodesics are periodic, they have a common period L (See Besse [1], p. 182). Since $\alpha(\theta)$ and $\beta(\theta)$ are integer multiples of $2\pi/L$, they are constant on I. Set $\alpha = \alpha(\theta)$ and $\beta =$

$\beta(\theta)$ on I. Since $x(0,\theta) = p, C(\theta) = x(0,\theta) - r_1(\theta)\mathbf{f}_1(\theta) - r_3(\theta)\mathbf{f}_3(\theta)$. Therefore, (*) can be written as

$$\begin{aligned}x(s,\theta) = x(0,\theta) &+ r_1(\theta)(\cos\alpha s - 1)\mathbf{f}_1(\theta) + r_1(\theta)\sin\alpha s\mathbf{f}_2(\theta)\\ &+ r_2(\theta)(\cos\beta s - 1)\mathbf{f}_3(\theta) + r_2(\theta)\sin\beta s\mathbf{f}_4(\theta)\end{aligned}$$

for $\theta \in I$. Then we have

$$\begin{aligned}x_*(\partial/\partial s) =& -\alpha r_1(\theta)\sin\alpha s\mathbf{f}_1(\theta) + \alpha r_1(\theta)\cos\alpha s\mathbf{f}_2(\theta)\\ &- \beta r_2(\theta)\sin\beta s\mathbf{f}_3(\theta) + \beta r_2(\theta)\cos\beta s\mathbf{f}_4(\theta),\end{aligned}\tag{3}$$

$$\begin{aligned}x_*(\partial/\partial\theta) =& r_1'(\theta)(\cos\alpha s - 1)\mathbf{f}_1(\theta) + r_1(\theta)(\cos\alpha s - 1)\mathbf{f}_1'(\theta)\\ &+ r_1'(\theta)\sin\alpha s\mathbf{f}_2(\theta) + r_1(\theta)\sin\alpha s\mathbf{f}_2'(\theta)\\ &+ r_2'(\theta)(\cos\beta s - 1)\mathbf{f}_3(\theta) + r_2(\theta)(\cos\beta s - 1)\mathbf{f}_3'(\theta)\\ &+ r_2'(\theta)\sin\beta s\mathbf{f}_4(\theta) + r_2(\theta)\sin\beta s\mathbf{f}_4'(\theta),\end{aligned}\tag{4}$$

$$\begin{aligned}&\tilde{\nabla}_{x_*(\partial/\partial s)}x_*(\partial/\partial s)\\ &= -\alpha^2 r_1(\theta)\cos\alpha s\mathbf{f}_1(\theta) - \alpha^2 r_1(\theta)\sin\alpha s\mathbf{f}_2(\theta)\\ &\quad - \beta^2 r_2(\theta)\cos\beta s\mathbf{f}_3(\theta) - \beta^2 r_2(\theta)\sin\beta s f_4(\theta),\end{aligned}\tag{5}$$

$$\begin{aligned}&\tilde{\nabla}_{x_*(\partial/\partial s)}x_*(\partial/\partial\theta)\\ &= -\alpha r_1'(\theta)\sin\alpha s\mathbf{f}_1(\theta) - \alpha r_1(\theta)\sin\alpha s\mathbf{f}_1'(\theta)\\ &\quad + \alpha r_1'(\theta)\cos\alpha s\mathbf{f}_2(\theta) + \alpha r_1(\theta)\cos\alpha s\mathbf{f}_2'(\theta)\\ &\quad - \beta r_2'(\theta)\sin\beta s\mathbf{f}_3(\theta) - \beta r_2(\theta)\sin\beta s\mathbf{f}_3'(\theta)\\ &\quad + \beta r_2'(\theta)\cos\beta s\mathbf{f}_4(\theta) + \beta r_2(\theta)\cos\beta s\mathbf{f}_4'(\theta),\end{aligned}\tag{6}$$

where $\tilde{\nabla}$ is the Riemann connection in E^5. From the Gauss lemma, we get $\langle x_*(\partial/\partial s), x_*(\partial/\partial\theta)\rangle = 0$, where $\langle , \rangle$ is the scalar product in E^5.

So, we get

$$
\begin{aligned}
(7)\qquad & \alpha r_1^2\langle \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta) > +\beta r_2^2(\theta)\langle \mathbf{f}_3'(\theta), \mathbf{f}_4(\theta)\rangle \\
& + \alpha r_1(\theta)(r_1'(\theta) + r_2(\theta)\langle \mathbf{f}_1(\theta), \mathbf{f}_3'(\theta)\rangle) \sin \alpha s \\
& + \alpha r_1(\theta)(r_1(\theta)\langle \mathbf{f}_1(\theta), \mathbf{f}_2'(\theta)\rangle - r_2(\theta)\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle) \cos \alpha s \\
& + \beta r_2(\theta)(r_2'(\theta) + r_1(\theta)\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle) \sin \beta s \\
& + \beta r_2(\theta)(r_1(\theta)\langle \mathbf{f}_1(\theta), \mathbf{f}_4'(\theta)\rangle - r_2(\theta)\langle \mathbf{f}_4(\theta), \mathbf{f}_3'(\theta)\rangle) \cos \beta s
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{(\alpha-\beta)}{2} r_1(\theta) r_2(\theta)\{\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle + \langle \mathbf{f}_2(\theta), \mathbf{f}_4'(\theta)\rangle\} \sin(\alpha+\beta)s \\
& + \frac{(\alpha+\beta)}{2} r_1(\theta) r2(\theta)\{< \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle - \langle \mathbf{f}_2(\theta), \mathbf{f}_4'(\theta)\rangle\} \sin(\alpha-\beta)s \\
& + \frac{(\alpha-\beta)}{2} r_1(\theta) r_2(\theta)\{\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle - \langle \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta)\rangle\} \cos(\alpha+\beta)s \\
& + \frac{(\alpha+\beta)}{2} r_1(\theta) r_2(\theta)\{\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle + \langle \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta)\rangle\} \cos(\alpha-\beta)s \\
& = 0 \quad \text{for all} \quad \theta \in I.
\end{aligned}
$$

We now prove $\alpha \neq \beta, \alpha \neq 2\beta$ and $\beta \neq 2\alpha$.
Suppose that $\alpha = \beta$. Since $\langle x_*(\partial/\partial s), x_*(\partial/\partial s)\rangle = 1$, we have

$$1 = \alpha^2 r_1^2(\theta) + \beta^2 r_2^2(\theta) = \alpha^2(r_1^2(\theta) + r_2^2(\theta)).$$

On the other hand, we have

$$\kappa_1^2(\theta) = \|\tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s)\|^2 = \alpha^4(r_1^2(\theta) + r_2^2(\theta)).$$

Therefore, we get $\kappa_1(\theta) = \alpha$. By means of (1) and (2), we see that $\kappa_2(\theta) = 0$. It contradicts that $x(s,\theta)$ is of rank 4 for every $\theta \in I$. Thus, $\alpha \neq \beta$ on I. We now suppose that $\beta = 2\alpha$. We obtain the following from (7) : For every $\theta \in I$,

$$(8)\qquad r_1^2(\theta)\langle \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta)\rangle + 2r_2^2(\theta)\langle \mathbf{f}_3'(\theta), \mathbf{f}_4(\theta)\rangle = 0,$$

$$
\begin{aligned}
(9)\qquad & (r_1'(\theta) + r_2(\theta)\langle \mathbf{f}_1(\theta), \mathbf{f}_3'(\theta)\rangle) \\
& - \frac{3}{2} r_2(\theta)(\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle - \langle \mathbf{f}_2(\theta), \mathbf{f}_4'(\theta)\rangle = 0,
\end{aligned}
$$

$$(r_1(\theta)((\langle \mathbf{f}_1(\theta), \mathbf{f}_2'(\theta)\rangle - r_2(\theta)\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle) \\ + \frac{3}{2}r_2(\theta)((\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle - \langle \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta)\rangle = 0, \tag{10}$$

$$r_2'(\theta) + r_1'(\theta)\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle = 0, \tag{11}$$

$$r_1(\theta)\langle \mathbf{f}_1(\theta), \mathbf{f}_4'(\theta)\rangle - r_2(\theta)\langle \mathbf{f}_4(\theta), \mathbf{f}_3'(\theta)\rangle = 0, \tag{12}$$

$$\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle + \langle \mathbf{f}_2(\theta), \mathbf{f}_4'(\theta)\rangle = 0, \tag{13}$$

$$\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle - \langle \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta)\rangle = 0. \tag{14}$$

We now compute $\langle \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s), \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial\theta)\rangle$ on I. From (6) and (7), we get after a long computation

$$\begin{aligned}
&\langle \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s), \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial\theta)\rangle \\
&= \alpha^3 r_1^2(\theta)\langle \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta)\rangle - 8\alpha^3 r_2^2(\theta)\langle \mathbf{f}_3(\theta), \mathbf{f}_4'(\theta)\rangle \\
&+ 2\alpha^3 r_1(\theta) r_2(\theta)(-\cos\alpha s + \sin\alpha s\cos 2\alpha s)\langle \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta)\rangle \\
&- 2\alpha^3 r_1(\theta) r_2(\theta)(\cos\alpha s + \sin\alpha s\sin 2\alpha s)\langle \mathbf{f}_1(\theta), \mathbf{f}_4'(\theta)\rangle \\
&+ 2\alpha^3 r_1(\theta) r_2(\theta)(\cos\alpha s + \cos\alpha s\cos 2\alpha s)\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle \\
&+ 2\alpha^3 r_1(\theta) r_2(\theta)(\sin as + \sin 2\alpha s\cos\alpha s)\langle \mathbf{f}_2(\theta), \mathbf{f}_4'(\theta)\rangle.
\end{aligned}$$

Making use of (13) and (14), it is reduced to

$$\begin{aligned}
&\langle \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s), \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial\theta)\rangle \\
&= \alpha^3 r_1^2(\theta)\langle \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta)\rangle - 8\alpha^3 r_2^2(\theta)\langle \mathbf{f}_3(\theta), \mathbf{f}_4'(\theta)\rangle \\
&- 6\alpha^3 r_1(\theta) r_2(\theta)\sin\alpha s\langle \mathbf{f}_3(\theta), \mathbf{f}_1'(\theta)\rangle \\
&+ 6\alpha^3 r_1(\theta) r_2(\theta)\cos\alpha s\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle
\end{aligned}$$

Differentiating the last equation covariantly along $x(s,\theta)$ for all $\theta \in I$, we obtain

$$\begin{aligned}
&x_*(\partial/\partial s)\langle \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s), \tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial\theta)\rangle \\
&= -6\alpha^4 r_1(\theta) r_2(\theta)\cos\alpha s\langle \mathbf{f}_3(\theta), \mathbf{f}_1'(\theta)\rangle \\
&- 6\alpha^4 r_1(\theta) r_2(\theta)\sin\alpha s\langle \mathbf{f}_2(\theta), \mathbf{f}_3'(\theta)\rangle.
\end{aligned} \tag{15}$$

We now identify $x_*(\partial/\partial s)$ with $\partial/\partial s$ and $x_*(\partial/\partial\theta)$ with $\partial/\partial\theta$ and denote them by T and Q respectively. The second fundamental form h on M is defined by $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$, where X and Y are vector fields on M, ∇ the Riemannian connection on M. Let $\bar{\nabla}$ be the operator of covariant differentiation of h defined on $T(M)\oplus T^{\perp}(M)$ as follows :

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where $T(M)$ denotes the tangent bundle, $T^{\perp}(M)$ the normal bundle of M, X, Y and Z vector fields on M and $\nabla^{\perp}$ the normal connection defined on the normal bundle $T^{\perp}(M)$. We denote $(\bar{\nabla}_X h)(Y,Z)$ by $(\bar{\nabla}h)(X,Y,Z)$. Then we have

$$\begin{aligned}
(16)\qquad & T\langle \tilde{\nabla}_T T, \tilde{\nabla}_T Q\rangle \\
&= T\langle h(T,T), h(T,Q)\rangle \\
&= \langle(\bar{\nabla}h(T,T,T), h(T,Q)\rangle + \langle h(T,T), (\bar{\nabla}h)(T,T,Q) \\
&\quad - h(T, \nabla_T Q)\rangle \\
&= \langle(\bar{\nabla}h)(T,T,T), h(T,Q)\rangle + \langle h(T,T), (\bar{\nabla}h)(T,T,Q)\rangle \\
&\quad - \langle h(T,T), h(T, \nabla_Q T)\rangle.
\end{aligned}$$

Together with (15) and (16), we get $T\langle \tilde{\nabla}_T T, \tilde{\nabla}_T Q\rangle \longrightarrow 0$ as $s \longrightarrow 0$ and hence $\langle \mathbf{f}_3(\theta), \mathbf{f}_1'(\theta)\rangle = 0$ on I. Combining (9),(11) and (12), we can conclude that $r_1(\theta)$ and $r_2(\theta)$ are constant on I. Therefore, all the curvatures κ_1, κ_2 and κ_3 are constant on for all $\theta \in I$. Let $\Theta_p = \{\theta \in I | x(s,\theta) \text{ is of rank } 4\}$. Then Θ_p is a non-empty open subset of $(0, 2\pi)$. By the continuity of $\kappa_i's$, Θ_p is closed. Thus $\Theta_p = (0, 2\pi)$. Hence all the geodesics through p is of rank 4 and have the same constant Frenet curvatures. Let q be any point of M. Since M is compact, there exists a geodesic joining p and q. By the same argument as above, we see that $\Theta_q = (0, 2\pi)$.

Therefore we can conclude that M is helical in E^5. By [10], M is isometric to a standard sphere or a Veronese surface. It would be a contradiction. Thus $\beta \neq 2\alpha$. Similarly, we can prove that $\alpha \neq 2\beta$. Therefore, the functions $1, \sin\alpha s, \cos\alpha s, \sin\beta s, \cos\beta s, \sin(\alpha \pm \beta)s$ and $\cos(\alpha \pm \beta)s$ in (7) are linearly independent. By solving the system of equations formed by the coefficients of them, we have $r_1'(\theta) = r_2'(\theta) = 0$ on I, i.e., $r_1(\theta)$ and $r_2(\theta)$ are constant on I. Therefore, the curvatures κ_1, κ_2 and κ_3 are constant for all $\theta \in I$. The same argument as we did above implies that M is helical in E^5. Thereby Case 2-2 cannot occur. This completes the proof of Theorem A.

3. Proof of Theorem B

Suppose that M has simple geodesics in $S^4(r)$. Let $x : M \longrightarrow S^4(r)$ be an isometric immersion. Without loss of generality we may assume that $r = 1$. Let $i : S^4(1) \longrightarrow E^5$ be the canonical inclusion of $S^4(1)$ into E^5 whose origin coincides with the center of $S^4(1)$. Let γ be a geodesic of M in $S^4(1)$ and let $\lambda_1, \lambda_2, \lambda_3$ and $\kappa_1, \kappa_2, \kappa_3$ be the curvatures of $x \circ \gamma$ and $i \circ x \circ \gamma$ respectively. Then we obtain (cf. [11]) :

$$\kappa_1^2 = \lambda_1^2 + 1, \kappa_1^2\kappa_2^2 = \lambda_1^2\lambda_2^2, \quad \kappa_2^2 + \kappa_3^2 = \lambda_2^2 + \lambda_3^2.$$

Therefore, M has simple geodesics in E^5. According to Theorem A, M is a 2-sphere or a Vereonese surface or a standard torus $S^1(a) \times S^1(b), a^2 + b^2 = 1$ since M is a surface of $S^4(1)$. The converse is obvious. This completes the proof of Theorem B.

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