J. Korean Math. Soc. 30 (1993), No. 2, pp. 337–344

UNIQUENESS THEOREMS IN THE AMBROSETTI TYPE SEMILINEAR WAVE EQUATION

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1. Introduction

The uniqueness theorem is important in the study of differential equations. In this paper we deal the uniqueness theorem in the Ambrosetti type semilinear wave equation,

$$\begin{aligned} u_{tt} - u_{xx} + au^+ - bu^- &= f(x,t) \text{ in } (c,d) \times \mathbb{R} \\ u(c,t) = u(u,t) &= 0 \\ u(x,t+T) &= u(x,t), \end{aligned} \tag{1.1}$$

where the period T is given.

For simplicity, we consider only the case $T = \pi$. By obvious changes of variavles, Problem (1.1) can be reduced to

$$\begin{aligned} u_{tt} - u_{xx} + au^+ - bu^- &= f(x,t) \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R \\ u\left(\pm\frac{\pi}{2}, t\right) &= 0 \\ u(x,t+\pi) &= u(x,t) \end{aligned} \tag{1.2}$$

Here u^+ is a upward restoring force and u^- a downward restoring force. We shall assume that f is even in x and periodic in t with period π, and we shall look for π-periodic solutions of (1.2).

Received May 21, 1992. Revised January 18, 1993.
* Partially supported by Inha University Research Foundation 1992 and GARC-KOSEF. ** Partially supported by KOSEF and GARC-KOSEF.

2. The Banach space spanned by eigenfunctions and uniqueness theorems

In this section we investigate the properties of the Banach space spanned by the eigenfunctions of the wave operator and we prove the uniqueness theorems in a semilinear wave equation.

Let L be the wave operator, in $\mathbb{R}^2$,

$$Lu = u_{tt} - u_{xx}.$$

When u is even in x and periodic in t with period π, the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \tag{2.1}$$

$$u\left(\pm\frac{\pi}{2}, t\right) = 0$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions ϕ_{mn}, ψ_{mn} $(m, n \leq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x & \text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x & \text{for } m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos(2n+1)x & \text{for } m > 0, n \geq 0. \end{aligned}$$

Let n be fixed and we define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n + 1, \tag{2.2}$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n - 3. \tag{2.3}$$

Letting $n \longrightarrow \infty$, $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$. Hence we can know that the only eigenvalues in the interval (-15, 9) are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal base in H_0. Let us denote an element u, in H_0, as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as follows

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

(i) $|||u||| \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u,
(ii) $\|u\| = 0$ if and only if $|||u||| = 0$,
(iii) $Lu \in H$ implies $u \in H$.

We note that 1 belongs to H_0, but does not to H. Hence we can see that the space H is a proper subspace of H_0. The following lemma is very important in this paper.

LEMMA 2.1. *Let c be not an eigenvalue of L. Let $u \in H_0$. Then we have $(L + c)^{-1}u \in H$.*

Proof. Suppose that c is not an eigenvalue of L and finite. When n is fixed, λ_n^+ and λ_n^- were defined in (2.2) and (2.3)

$$\lambda_n^+ = 4n + 1,$$

$$\lambda_n^- = -4n - 3.$$

We see that $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$ as $n \longrightarrow \infty$. Hence we know that the number of $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}).$$

Then

$$(L+c)^{-1}u = \sum \left(\frac{1}{\lambda_{mn}+c} h_{mn}\psi_{mn} + \frac{1}{\lambda_{mn}+c} k_{mn}\psi_{mn} \right).$$

Hence we have the inequality

$$\begin{aligned} |||(L+c)^{-1}u||| &= \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}+c)^2}(h_{mn}^2 + k_{mn}^2) \\ &\le C \sum (h_{mn}^2 + k_{mn}^2) \end{aligned}$$

for some C, which means that

$$|||(L+c)^{-1}u||| \le C_1 \|u\|, \quad C_1 = \sqrt{C}. \qquad \square$$

With the above Lemma 2.1, we can obtain the folloing lemma.

LEMMA 2.2. *Let $f(x,t) \in H_0$. Let a and b be not eigenvalues of L. Then all the solutions in H_0 of*

$$Lu + au^+ - bu^- = f(x,t) \quad \text{in} \quad H_0$$

belong to H.

Let μ_1 and μ_2 be eigenvalues of L such that there is no eigenvalue in between μ_1 and μ_2. Then we have the uniqueness theorem.

THEOREM 2.1. *Let $f(x,t) \in H_0$ and $-\mu_2 < a, b < -\mu_1$. Then the equation*

$$Lu + au^+ - bu^- = f(x,t) \tag{2.4}$$

has a unique solution in H_0. Furthermore this equation (2.4) has a unique solution in H.

Proof. Let $f(x,t) \in H_0$ and $-\mu_2 < a, b < -\mu_1$. Let $\delta = -\frac{1}{2}(\mu_1 + \mu_2)$. The equation (2.4) is equivalent to

$$u = (L+\delta)^{-1}[(\delta - a)u^+ - (\delta - b)u^- + f(x,t)],$$

where $(L+\delta)^{-1}$ is a compact, self-adjoint, linear map from H_0 into H_0 with norm $\frac{2}{\mu_2 - \mu_1}$. We note that

$$\begin{aligned}\|(\delta - a)(u_2^+ - u_1^+) - (\delta - b)(u_2^- - u_1^-)\| &\le \max\{|\delta - a|, |\delta - b|\}\|u_2 - u_1\| \\ &< \frac{1}{2}(\mu_2 - \mu_1)\|u_2 - u_1\|.\end{aligned}$$

It follows that the right hand side of (2.4) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$ of (2.4).

On the other hand, by Lemma 2.2, if $f(x,t) \in H_0$ then we know that the solution of (2.4) belongs to H. □

We now state a symmetry theorem which was proved in [4].

THEOREM A. *Assume that $L : \mathcal{D}(L) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ is a linear self-adjoint operator which posseses two closed invariant subspace H_1 and $H_2 = H_1^{\perp}$. Let σ denote the spectrum of L and σ_i the spectrum of $L|H_i$ ($i = 1,2$; $\sigma = \sigma_1 \cup \sigma_2$). Let $\frac{\partial f}{\partial u}(u,x) \equiv f_u$ be piecewise smooth and assume that $f_u \in [a,b]$ for all $u \in \mathbb{R}$ and $x \in \Omega$.*

If $[a,b] \cap \sigma_2 = \phi$ and if the Nemytzki operator $u \mapsto Fu = f(u(x),x)$ maps H_1 into itself, then every solution of

$$Lu = f(u,x) \quad \text{in} \quad L^2(\Omega)$$

is in H_1.

With the Theorem A, we have the follwing theorem, which is important in the study of nonlinear oscillations in the wave of a string

$$Lu + au^+ - bu^- = f(x,t) \quad \text{in} \quad H \tag{2.5}$$

THEOREM 2.2. *Let $-1 < a, b < 7$. We assume that*

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} \neq 1. \tag{2.6}$$

Then the equation

$$Lu + au^+ - bu^- = 0 \quad \text{in} \quad H_0 \tag{2.7}$$

has only the trivial solution $u \equiv 0$.

Proof. The space $H_1 = \text{span}\{\cos x \cos 2mt : m \geq 0\}$ is invariant under L and under the map $u \mapsto au^+ - bu^-$. The spectrum σ_1 of L restricted to H_1 contains $\lambda_{10} = -3$ and does not contain any other point in the interval $(-7, 1)$. The sectrum σ_2 of L restricted to $H_2 = H_1^\perp$ does not intersect the interval $(-7, 1)$. From Theorem A, we conclude that any solution of (2.7) belongs to H_1, i.e., it is of the form $y(t)\cos x$, where y satisfies

$$y'' + y + ay^+ - by^- = 0, \tag{2.8}$$

since $\cos x$ is positive in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Any nontrivial periodic solution of this equation is periodic with period

$$\frac{\pi}{\sqrt{a+1}} + \frac{\pi}{\sqrt{b+1}} \neq \pi.$$

In fact, if y_1 is a solution (with period $\frac{2\pi}{\sqrt{a+1}}$) of $y'' + (a+1)y = 0$ and y_2 is a solution (with period $\frac{2\pi}{\sqrt{b+1}}$) of $y'' + (b+1)y = 0$, then the only nonzero candidate $y = {y_1}^+ - {y_2}^-$ is a nonzero solution of (2.8) when y and y' has no discontinuity. This candidate y must be periodic with period $\frac{\pi}{\sqrt{a+1}} + \frac{\pi}{\sqrt{b+1}}$. This shows that there is no nontrivial solution of (2.7). □

The condition (2.6) is essential. When

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} = 1,$$

we can construct a nontrvial solution u_0 of (2.7) and any $ku_0 (k > 0)$ becomes a nontrivial solution of (2.7).

THEOREM 2.3. *Let μ_1 and μ_2 be the nearest eigenvalues to the left and right of $\lambda_{m0}(m \geq 2)$, respectively. Let $-\mu_2 < a < \lambda_{m0} < b < -\mu_1$. Then the equation (2.7) has only the trivial solution.*

Proof. We can know that

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} < 1,$$

since $m \geq 2$. Hence, if we follow the proof of Theorem 2.2, we can obtain Theorem 2.3. □

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