J. Korean Math. Soc. 30 (1993), No. 2, pp. 315–335

ON THE MINIMIZERS OF CERTAIN SINGULAR CONVEX FUNCTIONALS

HI JUN CHOE

1.Introduction

In this paper the minimizers of singular functionals are considered. Suppose that O is a bounded, open, convex subset of R^n and $f : O \to R$ is smooth and uniformly strictly convex. Suppose further that $f \geq 0$ and

$$\lim_{P \to \partial O} f(P) = \infty.$$

Set $f(P) = \infty$ for all $P \in R^n \setminus O$.

For example, f can be one of the following:

$$f(P) = \frac{1}{1- \mid P \mid^2},\ O = \{P : \mid P \mid < 1,\ \ P \in R^n\}$$

or

$$f(P) = \frac{1}{1 - P_1^2} + \frac{1}{1 - P_2^2},\ O = (-1, 1) \times (-1, 1).$$

Consider the functional

$$I(u) = \int_\Omega f(Du)dx$$

defined for appropriate $u : \Omega \to R$, where Ω is a bounded, open subset of R^n with smooth boundary.

Suppose that $u_0 \in W^{1,2}(\Omega)$ and $I(u_0) < \infty$. Then u_0 is Lipschitz on the closure of Ω and it is relatively easy to see that there exists a unique $u \in u_0 + W_0^{1,2}(\Omega)$ such that

$$I(u) \leq I(v)$$

Received May 9, 1992.

The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1991. Project No. BSRI-91-115.

for all $v \in u_0 + W_0^{1,2}(\Omega)$. Of course, u is Lipschitz on the closure of Ω with $u = u_0$ on $\partial\Omega$ and minimizes I among all such functions. The first question addressed is that of the regularity of Du.

Suppose that u_0 satisfies the following "*bounded slope condition*": there exists a constant M such that for each point $x_0 \in \partial\Omega$, there exist linear functions $\pi^{\pm}$ such that

$$f(D\pi^{\pm}) \leq M$$

and

$$\pi^{-}(x - x_0) \leq u_0(x) - u_0(x_0) \leq \pi^{+}(x - x_0)$$

for all $x \in \partial\Omega$. Then it is shown that $u \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$. In fact, $f(Du)$ is bounded, and since f is smooth, it follows that $u \in C^{\infty}(\Omega)$.

This seems to be the first regularity result of this type, i.e., where the function f exhibits this type singular behavior. The study of this question is motivated by models for hyperelastic materials (see Ball [1]) in which one is lead to consider minimizers of functionals over vector-valued mappings where the integrand exhibits a certain type of singular behavior.

In case u_0 does not satisfy the bounded slope condition given above, then the corresponding minimizer need not be in C^1. An example is given, in case $n = 2$,

$$f(P_1, P_2) = (1 - P_1^2)^{-\alpha} + (1 - P_2^2)^{-\alpha}, \ (0 < \alpha < 1)$$

with Ω an open ball in R^2. The minimizer fails to be in C^1 exactly on a line joining two points of $\partial\Omega$.

For systems, it is known that minimizers need not be C^1 even if f is uniformly strictly convex on all of R^n with bounded second derivatives. In case $n = 1$ and f depends on u and Du, J. M. Ball and V. Mizel[2] have given examples showing that singularities can occur in the interior of Ω. As far as we know the examples given here are the first in the higher-dimensional scalar case showing that singularities can occur in the interior even if f depends only on Du.

ACKNOWLEDGEMENT.. The problem was asked by Professor L. C. Evans and R. Gariepy. Especially the author would like to thank Professor R. Gariepy. The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1991 Project No. BSRI-91-115

2. Existence and uniqueness

Suppose that O is a bounded open convex subset of R^n and that $f : O \to R$ is $C^2(O)$ and for some constant $\lambda > 0$,

$$\frac{\partial^2}{\partial P_i \partial P_j} f(P)\xi_i\xi_j \geq \lambda \mid \xi \mid^2 \tag{1}$$

for all $P \in O$, $\xi \in R^n$. Suppose further that

$$\lim_{P \to \partial O} f(P) = \infty \tag{2}$$

and $f(P) = \infty$ for all $P \in R^n \setminus O$.

Suppose Ω is an open connected subset of R^n. Suppose $u_0 \in W^{1,1}(\Omega; R)$ and

$$I[u_0] = \int_\Omega f(Du_0)dx < \infty.$$

LEMMA 1. *Let $K = \{v \in u_0 + W^{1,1}(\Omega) : I[v] < \infty\}$. Then K is convex.*

proof. Let $v_1, v_2 \in K$, then for each $0 \leq t \leq 1$, $tv_1 + (1-t)v_2 \in u_0 + W_0^{1,1}$ and $I[v_1]$, $I[v_2] < \infty$. Since f is convex, we have

$$\begin{aligned} I[tv_1 + (1-t)v_2] &= \int_\Omega f(tDv_1 + (1-t)Dv_2)dx \\ &\leq t\int_\Omega f(Dv_1)dx + (1-t)\int_\Omega f(Dv_2)dx < \infty. \end{aligned}$$

So $tv_1 + (1-t)v_2 \in K$ and K is convex.

Since O is bounded, K is a bounded subset of $W^{1,p}$ for all $1 \leq p \leq \infty$. Since any convex function is bounded below, we assume that f is nonnegative and $f(0) = 0$ is the minimum of f in O. The next theorem proves that I is weakly sequentially lower semicontinuous in $W^{1,1}(\Omega)$.

THEOREM 1. *Suppose that $u, u_n \in K$ for each n and $u_n \rightharpoonup u$ weakly in $W^{1,1}(D)$ for each $D \subset\subset \Omega$. Then*

$$I[u] \leq \liminf_{n\to\infty} I[u_n]. \tag{3}$$

proof. Let $d = \mathrm{dist}(D, \partial\Omega)$ and ϕ be a nonnegative smooth function supported in the unit ball such that

$$\int_{R^n} \phi(x)dx = 1.$$

Define $w_\rho(x)$ by

$$w_\rho(x) = \frac{1}{\rho^n}\int_{R^n} \phi(\frac{x-y}{\rho})w(y)dy \tag{4}$$

for each function $w \in W^{1,1}(\Omega)$ and $\rho > 0$. Since $Du_\rho \to Du$ almost everywhere in D as $\rho \to 0$ and f is continuous, we have

$$f(Du_\rho) \to f(Du)$$

almost everywhere in D as $\rho \to 0$. Since f is nonnegative,

$$I[u:D] = \int_D f(Du)dx \leq \liminf_{\rho\to 0}\int_D f(Du_\rho)dx$$

by Fatou's lemma. From Jensen's inequality, we have

$$f(Du_\rho) \leq f(Du)_\rho$$

and

$$f(Du_{n,\rho}) \leq f(Du_n)_\rho$$

on D, for each n and $\rho < d$.

So we see that

$$I[u_{n,\rho}:D] \leq \int_D f(Du_n)_\rho dx \leq \int_\Omega f(Du_n)dx.$$

Since $Du_n \rightharpoonup Du$ weakly in $L^1(D')$ for each $D \subset\subset D' \subset\subset \Omega$, $Du_{n,\rho} \to Du_\rho$ pointwisely in $D \subset\subset \Omega$ as $n \to \infty$. Thus, combining the previous inequalities, we have

$$\begin{aligned} I[u_\rho : D] &= \int_D f(Du_\rho)dx \\ &= \int_D \lim_{n\to\infty} \inf f(Du_{n,\rho})dx \\ &\leq \lim_{n\to\infty} \inf \int_D f(Du_{n,\rho})dx \\ &\leq \lim_{n\to\infty} \inf \int_\Omega f(Du_n)dx. \end{aligned}$$

So we have

$$\begin{aligned} I[u : D] &= \int_D f(Du)dx \\ &\leq \liminf_{\rho\to 0} \int_D f(Du_\rho)dx \\ &\leq \lim_{n\to\infty} \inf \int_\Omega f(Du_n)dx. \end{aligned}$$

Since D is chosen arbitrarily, we have

$$I[u : \Omega] = I[u] \leq \lim_{n\to\infty} \inf I[u_n]$$

and I is lower semicontinuous.

Now we prove the existence and uniqueness of the minimizer. The theorem follows essentially from the weak compactness of the bounded subset of $W^{1,2}(\Omega)$.

THEOREM 2. *Let $\mu = \inf_{v\in K} I[v]$. Then there is a unique $u \in K$ such that*

$$I[u] = \mu.$$

proof. We note that μ is a finite number, since I is convex and $u_0 \in K$. Let $\{u_n\}$ be a sequence in K such that $I[u_n] \to \mu$ as $n \to \infty$. Since K is a bounded subset of $W^{1,2}(\Omega)$, there is a subsequence $\{u_{n_k}\}$

such that $u_{n_k} \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$ for some $u \in W^{1,2}(\Omega)$ as $k \to \infty$. We see that $u - u_0 \in W_0^{1,2}(\Omega)$. From the lower semicontinuity of I we have

$$I[u] \le \lim_{k\to\infty} \inf I[u_{n_k}] = \mu$$

So $I[u] = \mu$ from the fact that $\mu = \inf_{v\in K} I[v]$.

We prove the uniqueness by using the strict convexity of f and a variational inequality which the minimizer u satisfies. First we show that u satisfies a variational inequality. Suppose that $v \in K$. Then $I[v] < \infty$. Since f is convex,

$$f(Dv(x)) - f(Du(x)) \ge \frac{f(Du(x) + t(Dv(x) - Du(x))) - f(Du(x))}{t}$$

for all $x \in \Omega$ and $0 < t \le 1$. Moreover we see that

$$F_t(x) = \frac{f(Du(x) + t(Dv(x) - Du(x))) - f(Du(x))}{t}$$

is monotone decreasing as $t \to 0$ and $F_t(x)$ converges to $f_{P_i}(Du(x))$ $(D_i v - D_i u)$ for almost all $x \in \Omega$. Since $F_t(x) \le f(Dv(x)) - f(Du(x))$ for all $x \in \Omega$ and F_t converges to $f_{P_i}(Du(x))(D_i v - D_i u)$ monotonically as $t \to 0$, by the monotone convergence theorem, we have

$$\lim_{t\to 0} \int_\Omega F_t(x)dx = \int_\Omega f_{P_i}(Du)(D_i v - D_i u)dx.$$

Since $I[v] - I[u] \ge \int_\Omega F_t(x) \ge 0$ for all $0 < t \le 1$,

$$I[v] - I[u] \ge \int_\Omega f_{P_i}(Du(x))(D_i v - D_i u)dx \ge 0$$

for all $v \in K$.

Let $I[u] = I[v]$ for some $v \in K$. Then since f is strictly convex, we have

$$\begin{aligned}
0 =& I[u] - I[v] \\
=& \int_\Omega [f(Du) - f(Dv)]\, dx \\
=& \int_\Omega f_{P_i}(Du)(D_i v - D_i u)\, dx \\
&+ \int_\Omega \int_0^1 (1-t) f_{P_i P_j}(Du + t(Dv - Du))dt \\
&\times (D_i v - D_i u)(D_j v - D_j u)\, dx \\
\geq& \frac{1}{2}\lambda \int_\Omega | Du - Dv |^2\, dx.
\end{aligned}$$

Since $u - v \in W_0^{1,2}(\Omega)$, from Sobolev's inequality,

$$\| u - v \|_2 \leq C \| Du - Dv \|_2 = 0$$

for some C, where $\| u \|_2$ is $L^2(\Omega)$ norm of u.

3. Approximation

We approximate f with functions f^ρ which grow quadratically by using the implicit function theorem.

Let $E_\rho = \{P \in R^n : f(P) \leq \rho\}$. Then E_ρ is a strictly convex, bounded and closed subset of R^n.

Now we construct a uniformly strictly convex function with quadratic growth. First we recall the Implicit Function Theorem.

LEMMA 2. *Let $g \in C^2$ and $D_y g(x_0, y_0) \neq 0$. Then there exists a function $h(y)$ such that $x_0 = h(y_0)$ and $g(h(y), y) = 0$ in some neighborhood U of y_0. Moreover $h \in C^2(U)$.*

The following theorem is fundamental to the approximation.

THEOREM 3. *Suppose $g \in C^2(R^n)$ and*

$$g_{P_i P_j}(P)\xi_i \xi_j \geq \nu \mid \xi \mid^2 \tag{5}$$

for some $\nu > 0$ and all $P, \xi \in R^n$. Suppose further that $g(0) = 0$ is the minimum. Define by $r(P) > 0$

$$g\left(\frac{P}{\sqrt{r(P)}}\right) = c > 0$$

for all $P \in R^n \setminus \{0\}$. Then $r(P) \in C^2(R^n \setminus \{0\})$ and

$$\nu_1 \mid \xi \mid^2 \leq r_{P_i P_j}(P)\xi_i\xi_j \leq \nu_2 \mid \xi \mid^2 \tag{6}$$

for all $P, \xi \in R^n \setminus \{0\}$. Moreover ν_1 and ν_2 depend only on c and ν.

proof. We see that g is radially strictly increasing. Since we are assuming $r > 0$, $r(P)$ is well defined for all $P \in R^n \setminus \{0\}$. By differentiating $g(\frac{P}{\sqrt{r}})$ with respect to r, we have

$$\frac{\partial}{\partial r} g\left(\frac{P}{\sqrt{r}}\right) = g_{P_i}\left(\frac{P}{\sqrt{r}}\right)P_i\left(-\frac{1}{2\sqrt{r}^3}\right) \neq 0$$

for all $P \in R^n \setminus \{0\}$. So, from Lemma 2, $r(P) \in C^2$ and

$$g\left(\frac{P}{\sqrt{r(P)}}\right) = c.$$

Define

$$g_i = g_{P_i}\left(\frac{P}{\sqrt{r}}\right),\ g_{ik} = g_{P_i P_k}\left(\frac{P}{\sqrt{r}}\right),\ r_i = r_{P_i}\left(\frac{P}{\sqrt{r}}\right),\ r_{ik} = r_{P_i P_k}\left(\frac{P}{\sqrt{r}}\right).$$

We see that

$$\frac{\partial}{\partial P_i} g\left(\frac{P}{\sqrt{r}}\right) = g_i \frac{1}{\sqrt{r}} - \frac{1}{2} g_j P_j \frac{r_i}{\sqrt{r}^3} = 0,$$

and hence that

$$r_i = 2\frac{g_i r}{g_j P_j}.$$

Setting $T = g_j P_j$ and differentiating with respect to P_k,

$$\begin{aligned}\frac{1}{2} r_{ik} T^2 =& T g_{ik} \sqrt{r} - \frac{1}{2} T g_{il} P_l \frac{r_k}{\sqrt{r}} + T g_i r_k \\ & - g_i g_{jk} P_j \sqrt{r} + \frac{1}{2} g_{jl} P_l g_i P_j \frac{r_k}{\sqrt{r}} - r g_i g_k.\end{aligned}$$

Substituting $r_k = \frac{2 g_k r}{T}$, we have

$$\begin{aligned}\frac{1}{2} T^2 r_{ik} \xi_i \xi_k =& T \sqrt{r} g_{ik} \xi_i \xi_k - \sqrt{r} g_{il} P_l g_k \xi_i \xi_k + 2 r g_i g_k \xi_i \xi_k \\ & - \sqrt{r} g_i g_{jk} P_j \xi_i \xi_k + \frac{\sqrt{r}}{T} g_{jl} P_l P_j g_i g_k \xi_i \xi_k - r g_i g_k \xi_i \xi_k.\end{aligned}$$

Let $S = g_i \xi_i$. Then, by using $g_{ik} = g_{ki}$, we have

$$r_{ik} \xi_i \xi_k = 2 \frac{\sqrt{r}}{T} g_{ik} (\xi_i - \frac{S}{T} P_i)(\xi_k - \frac{S}{T} P_k) + 2r \frac{S^2}{T^2}.$$

We have for some M_1, M_2 and M_3, which depend only on c,

$$| g_i |, | g_{ik} | \leq M_3$$

and

$$0 < M_1 \leq \frac{T}{\sqrt{r}} \leq M_2$$

for all $P \in R^n \setminus \{0\}$. Thus we have

$$r_{ik} \xi_i \xi_k \leq \nu_2 | \xi |^2$$

for all $P, \xi \in R^n$ where ν_2 depends on c.

Now we define $h(\xi)$ and $\overline{h}(\xi)$ by

$$h(\xi) = r_{ik} \xi_i \xi_k = | \xi |^2 \, \overline{h}(\xi).$$

Since g is strictly convex,

$$\overline{h} \geq \frac{2}{M_2} \nu \left| \frac{\xi}{|\xi|} - \frac{S}{T} \frac{P}{|\xi|} \right|^2 + \frac{2}{M_2^2} (\frac{S}{|\xi|})^2.$$

Let $m = \max_{g(P)=c} \mid P \mid$. Then

$$\frac{\mid SP \mid}{\mid \xi \mid T} \leq \frac{Sm}{\mid \xi \mid M_1}$$

for all $P \in R^n \setminus \{0\}$. If $\frac{S}{|\xi|} \geq \frac{M_1}{2m}$, then

$$\overline{h} \geq \frac{M_1^2}{2m^2 M_2^2}.$$

On the other hand, if $\frac{S}{|\xi|} \leq \frac{M_1}{2m}$, then

$$\overline{h} \geq \frac{\nu}{M_2}.$$

Thus

$$h(\xi) \geq \nu_1 \mid \xi \mid^2,$$

where $\nu_1 = \min(\frac{M_1^2}{2m^2}, \frac{\nu}{M_2})$, which depends only on ν and c.

Now we approximate f. Let $\psi : R \to R$ be C^∞ such that

$$\psi(t) = 1$$

for $t \in (-\infty, 0]$ and

$$\psi(t) = 0$$

for $t \in [1, \infty)$.

Define $r(P)$ by

$$f(\frac{P}{\sqrt{r(P)}}) = \rho + \frac{\delta}{2}.$$

From the definition of $E_{\rho+\frac{\delta}{2}}$ and $r(P)$, we see that

$$\partial E_{\rho+\frac{\delta}{2}} = \{P : r(P) = 1\}$$

and $r(P) \in C^2$. Let $b(P) = (r(P) - 1)^+$. Then

$$b(P) = 0$$

if $P \in E_{\rho+\frac{\delta}{2}}$ and

$$b(P) = r(P) - 1$$

if $P \in R^n \setminus E_{\rho+\frac{\delta}{2}}$.

Now we regularize $b(P)$ with ϵ' as (4) and get $b_{\epsilon'}(P) \in C^\infty$. Then we see that $b_{\epsilon'}(P)$ satisfies the same growth condition as $r(P)$ if $|P|$ is large enough. So $b_{\epsilon'}(P)$ grows quadratically. Moreover $b_{\epsilon'}(P) = 0$ for $P \in E_\rho$ and

$$\nu_1 \mid \xi \mid^2 \leq b_{\epsilon', P_i P_j}(P)\xi_i\xi_j \leq \nu_2 \mid \xi \mid^2$$

for all $P \in R^n \setminus E_{\rho+\delta}$ for some ν_1 and ν_2 if ϵ' is small enough.

THEOREM 4. *Suppose that f^ρ is defined by*

$$f^\rho(P) = \psi(\frac{f(P) - \rho - \delta}{\delta})f(P) + \mu b_{\epsilon'}(P) \tag{7}$$

for all $P \in R^n$ and for some $\mu > 0$. If μ is sufficiently large, then f^ρ satisfies the following ellipticity condition

$$\lambda_1 \mid \xi \mid^2 \leq f^\rho_{P_i P_j}(P)\xi_i\xi_j \leq \lambda_2 \mid \xi \mid^2 \tag{8}$$

for all $P, \xi \in R^n$ and for some $\lambda_1, \lambda_2 > 0$ and $f^\rho(P) = f(P)$ for $P \in E_\rho$.

proof. We note that $b_{\epsilon'}(P) = 0$ and $\psi(\frac{f(P)-\rho-\delta}{\delta}) = 1$ for $P \in E_\rho$. So $f^\rho(P) = f(P)$ for $P \in E_\rho$. By differentiating f^ρ with respect to P_i, we have

$$f^\rho_{P_i} = \frac{1}{\delta}\psi_t f f_{P_i} + \psi f_{P_i}(P) + \mu b_{\epsilon', P_i}(P)$$

and

$$f^\rho_{P_i P_j} = \frac{1}{\delta^2}\psi_{tt} f f_{P_i} f_{P_j} + \frac{1}{\delta} f\psi_t f_{P_i P_j} + \frac{2}{\delta}\psi_t f_{P_i} f_{P_j} + \psi f_{P_i P_j} + \mu b_{\epsilon', P_i P_j}(P).$$

Since $b_{\epsilon'}$ is convex,

$$b_{\epsilon', P_i P_j}\xi_i\xi_j \geq 0$$

for all $P \in R^n$. Let $P \in E_{\rho+\delta}$. Then $\psi = 1, \psi_t = 0$ and $\psi_{tt} = 0$. Hence we have

$$f^\rho_{P_i P_j}(P)\xi_i\xi_j = f_{P_i P_j}\xi_i\xi_j + \mu b_{\epsilon', P_i P_j}\xi_i\xi_j \geq \lambda \mid \xi \mid^2$$

for all $\xi \in R^n$. Since $| f_{P_iP_j}(P) | \leq M$ for some M and for all $P \in E_{\rho+\delta}$, we have

$$f_{P_iP_j}\xi_i\xi_j \leq C \mid \xi \mid^2$$

for all $\xi \in R^n$, where C depends on μ and M.

Let $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$. Then we have

$$\nu_1 \mid \xi \mid^2 \leq b_{\epsilon',P_iP_j}\xi_i\xi_j \leq \nu_2 \mid \xi \mid^2$$

for all $\xi \in R^n$. If $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$, then

$$\mid (\psi f)_{P_iP_j} \mid \leq M$$

for some M. Hence if μ is large enough, we have that for all $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$

$$\mu b_{\epsilon',P_iP_j}\xi_i\xi_j + (\psi f)_{P_iP_j}\xi_i\xi_j \geq C_2 \mid \xi \mid^2$$

for some C_2 which depends on M and μ.

Since $\mu b_{\epsilon',P_iP_j}\xi_i\xi_j \leq \mu\nu_2 \mid \xi \mid^2$ and $\mid (\psi f)_{P_iP_j} \mid \leq M$ for all $P \in E_{\rho+\delta} \setminus E_\rho$, we have

$$\mu b_{\epsilon',P_iP_j}\xi_i\xi_j + (\psi f)_{P_iP_j}\xi_i\xi_j \leq C_3 \mid \xi \mid^2$$

where C_3 depends on M and μ.

If $P \in R^n \setminus E_{\rho+2\delta}$, then $\psi = 0$ and $f^\rho(P) = \mu b_{\epsilon'}(P)$. Hence by Theorem 3 we have

$$f^\rho_{P_iP_j}\xi_i\xi_j \leq \mu\nu_2 \mid \xi \mid^2$$

for all $P, \xi \in R^n$.

4. Regularity

By using a maximum principle and existence theorem for quasilinear elliptic equations due to P. Hartman and G. Stampaccia [3] we obtain $C^{1,\alpha}(\overline{\Omega})$ regularity for a minimizer if $(u_0, \partial\Omega)$ satisfies a certain *bounded slope condition.*

THEOREM 5. *Suppose that Ω is a bounded open connected subset of R^n with $\partial\Omega \in C^{1,1}$. Moreover suppose that u_0 satifies the following "bounded slope condition" : there exists a constant M such that for each point $x_0 \in \partial\Omega$, there exist linear functions $\pi^{\pm}_{x_0}$ such that*

$$f(D\pi^{\pm}_{x_0}) \leq M$$

and

$$\pi^{-}_{x_0}(x - x_0) \leq u_0(x) - u_0(x_0) \leq \pi^{+}_{x_0}(x - x_0) \tag{9}$$

for all $x \in \partial\Omega$. Then the minimizer u with respect K is $C^{1,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha < 1$.

J. Moser observed in [4] that if v is a solution of a linear elliptic equation

$$D_i(a_{ij}(x)D_j u) = 0 \tag{10}$$

with a_{ij} measurable and

$$c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq c_1|\xi|^2$$

for some positive constants c_0 and c_1, then for any convex function h, $h(u)$ is a subsolution of (10). We prove a similar theorem for the derivatives of solutions of quasilinear elliptic equations.

THEOREM 6. *Suppose that $A_i \in C^1(R^n)$ satisfies the following ellipticity condition :*

$$c_0|\xi|^2 \leq A_{i,P_j}(P)\xi_i\xi_j \leq c_1|\xi|^2 \tag{11}$$

for all $P, \xi \in R^n$ and for some positive constants c_0 and c_1. Moreover suppose that $g : R \to R$ is nonincreasing and in C^1.

Let $v \in W^{1,2}(\Omega)$ be a solution to the quasilinear equation

$$D_i(A_i(Dv)) + g(v) = 0. \tag{12}$$

Suppose that $G \in C^2(R^n; R)$ is a convex function with $G(0) = 0$ as minimum. Then $G(Dv)$ is a subsolution to

$$D_i(A_{i,P_j}(Dv)D_j w) = 0. \tag{13}$$

proof. First we prove that $v \in W^{2,2}_{loc}(\Omega)$ by difference quotient argument.

Let $\Omega' \subset\subset \Omega$ and $d < \operatorname{dist}(\Omega', \partial\Omega)$. Let $h \leq \frac{1}{4}d$ and e_k be k-th direction unit coordinate vector for $k = 1, ..., n$. Let $\psi \in C^\infty(\Omega)$, $|D\psi| \leq \frac{c}{d}$ for some c and $\operatorname{supp}(\psi) \pm \frac{1}{4}de_k \subset \Omega$. We apply $(v(x+he_k) - v(x))\psi^2(x)$ as a test function to (12). Hence we have that

$$\int_\Omega [A_i(Dv(x+he_k)) - A_i(Dv(x))]D_i[(v(x+he_k) - v(x))\psi^2(x)]dx$$

$$-\int_\Omega [g(v(x+he_k)) - g(v(x))][(v(x+he_k) - v(x))\psi^2(x)]dx = 0$$

for all $k = 1, ..., n$. Since g is nonincreasing, we have that

$$[g(v(x+he_k)) - g(v(x))][v(x+he_k) - v(x)]\psi^2(x) \leq 0$$

for all $x \in \Omega$ and $k = 1, ..., n$. By using the ellipticity of A_i and the equation we have

$$\begin{aligned}
&\frac{c_0}{h^2}\int_\Omega |Dv(x+he_k) - Dv(x)|^2\psi^2 dx \\
\leq &\frac{1}{h^2}\int_\Omega [A_i(Dv(x+he_k)) - A_i(Dv(x))][D_iv(x+he_k) - D_iv(x)]\psi^2 dx \\
\leq &-\frac{2}{h^2}\int_\Omega [A_i(Dv(x+he_k)) - A_i(Dv(x))] \\
&\times [v(x+he_k) - v(x)]D_i\psi(x)\psi(x)dx \\
\leq &2c_1\int_\Omega \left|\frac{Dv(x+he_k) - Dv(x)}{h}\right| \left|\frac{v(x+he_k) - v(x)}{h}\right| |D\psi(x)|\,\psi(x)dx.
\end{aligned}$$

Now by using Holder's inequality on the right hand side of the last inequality, we have

$$\int_{\Omega'} \left|\frac{Dv(x+he_k) - Dv(x)}{h}\right|^2 dx \leq \frac{c}{d^2}\int_{\Omega''} \left|\frac{v(x+he_k) - v(x)}{h}\right|^2 dx$$

for some $\Omega' \subset \Omega'' \subset \Omega$, for all $0 < h < \frac{1}{4}d$ and k. So $v \in W^{2,2}_{loc}(\Omega)$ and we can differentiate formally with respect to x_k to obtain

$$D_i(A_{i,P_j}(Dv)D_jD_kv) + g'(v)D_kv = 0$$

for each k. Let η be a nonnegative $C_0^\infty(\Omega)$ function. Then

$$\int_\Omega A_{i,P_j}(Dv)D_jG(Dv)D_i\eta dx = \int_\Omega A_{i,P_j}(Dv)G_{P_k}(Dv)D_jD_kvD_i\eta dx.$$

Since

$$D_i(G_{P_k}(Dv)\eta) = G_{P_kP_l}D_iD_lv\eta + G_{P_k}(Dv)D_i\eta$$

and $G_{P_k}(Dv)\eta \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega A_{i,P_j}G_{P_k}(Dv)D_jD_kvD_i\eta dx \\
&= \int_\Omega A_{i,P_j}D_jD_kvD_i(G_{P_k}(Dv)\eta)dx \\
&\quad - \int_\Omega A_{i,P_j}G_{P_kP_l}(Dv)D_jD_kvD_iD_lv\eta dx \\
&= \int_\Omega g'(v)D_kvG_{P_k}(Dv)\eta dx - \int_\Omega A_{i,P_j}G_{P_kP_l}(Dv)D_jD_kvD_iD_lv\eta dx.
\end{aligned}$$

Since G is radiallly increasing and $g' \le 0$,

$$\int_\Omega g'(v)D_kvG_{P_k}(Dv)\eta dx \le 0.$$

Since A_{i,P_j} and $G_{P_kP_l}$ are positive definite matrices,

$$A_{i,P_j}G_{P_kP_l}(Dv)D_jD_kvD_iD_lv \ge 0.$$

Therefore we have

$$\int_\Omega A_{i,P_j}(Dv)D_jG(Dv)D_i\eta dx \le 0$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. Hence $G(Dv)$ is a subsolution to

$$D_i(A_{i,P_j}D_jw) = 0$$

and this completes the proof.

We have the following lemma for the solutions of homogeneous equations.

LEMMA 3. *Let $v \in W^{1,2}(\Omega)$ be a solution to*

$$D_i(A_i(Dv)) = 0$$

where $A_i : R^n \to R$ satisfies the ellipticity condition (12). Let $G : R^n \to R$ be convex and in C^1. Then $G(Dv)$ is a subsolution to

$$D_i(A_{i,P_j}(Dv)D_j w) = 0.$$

Since $G(Dv)$ is a subsolution of a linear elliptic equation, we have a maximum principle.

LEMMA 4. *Let G and $v \in C^1(\overline{\Omega})$ satisfy the same conditions as in Theorem 6. Then we have the following maximum principle*

$$\max_{\Omega} G(Dv) \leq \max_{\partial\Omega} G(Dv). \tag{14}$$

proof. Let $M = \max_{\partial\Omega} G(Dv)$ and $w = (G(Dv) - M - \epsilon)^+$ for some $\epsilon > 0$. Then we see $w \in W_0^{1,2}$. So by using w as a test function to (13), we have

$$\int_{\{x \,:\, M+\epsilon \leq G(Dv(x))\}} A_{i,P_j} D_i(G(Dv))D_j(G(Dv))dx \leq 0$$

and

$$\int_{\{x\in\Omega \,:\, G(Dv(x))\geq M+\epsilon\}} |D(G(Dv))|^2 \, dx = 0.$$

By using Sobolev inequality we have $\text{meas}\{x \in \Omega : M+\epsilon \leq G(Dv(x))\} = 0$ for all $\epsilon > 0$.

Now we prove Theorem 5 by using monotone operator theory as in [3].

proof of Theorem 5. Let f^ρ be the approximation of f in the theorem 4 such that $f^\rho(P) = f(P)$ for all $P \in \{P : f(P) \leq \rho\} \cup \{P : f^\rho(P) \leq \rho\}$ and let f^ρ satisfy the quadratic growth condition. Let u^ρ be the minimizer of

$$I^\rho[u^\rho] = \int_\Omega f^\rho(Du^\rho)dx$$

with respect to $K^\rho = \{v \in W^{1,2} : v - u_0 \in W_0^{1,2}\}$.

From section 2.1, we know that there exists a unique minimizer u^ρ for each ρ. Fix $L \geq 2M$, where M is the constant defined in the bounded slope condition of Theorem 5. We know that u^L satisfies the Euler-Lagrange equation

$$D_i(f^L_{P_i}(Du^L)) = 0 \tag{15}$$

with $u^L - u_0 \in W_0^{1,2}$. We see that $(u_0, \partial\Omega)$ has the ordinary bounded slope condition and

$$\left|D\pi^\pm_{x_0}\right| \leq C$$

for all $x \in \partial\Omega$, where C is independent of x_0.

Since u_0 is Lipschitz and $(u_0, \partial\Omega)$ satisfies the bounded slope condition, there exists a $w^L \in C^{1,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha < 1$ which satisfies the Euler-Lagrange equation (15) by the Theorem 13.1 and 14.1 in [3]. By the uniqueness, $u^L = w^L$. Since f^L is convex, from the maximum principle (Lemma 4), we see that

$$\max_{\Omega} f^L(Du^L) \leq \max_{\partial\Omega} f^L(Du^L).$$

Since $u^L = u_0$ on $\partial\Omega$ and $\pi^-_{x_0}(x) \leq u^L(x) \leq \pi^+_{x_0}(x)$ for all $x \in \Omega$,

$$\frac{\partial}{\partial\eta}u^L(x_0) = \frac{\partial}{\partial\eta}u_0(x_0)$$

for all tangent vector η to $\partial\Omega$ at x_0 and

$$\frac{\partial}{\partial\tau}\pi^+_{x_0} \leq \frac{\partial}{\partial\tau}u^L(x_0) \leq \frac{\partial}{\partial\tau}\pi^-_{x_0}$$

for all outward normal vector τ to $\partial\Omega$ at x_0. So we see that

$$Du^L(x_0) = tD\pi^+_{x_0} + (1-t)D\pi^-_{x_0}$$

for some $0 \leq t \leq 1$ and

$$f^L(Du^L(x_0)) \leq tf^L(D\pi^+_{x_0}) + (1-t)f^L(D\pi^-_{x_0}) \leq M$$

for all $x_0 \in \partial\Omega$. So

$$\max_{\Omega} f^L(Du^L) \leq M.$$

Since $f^L(P) = f(P)$ if $f^L(P) \leq L$, we conclude that $f(Du^L) = f^L(Du^L)$ for all $x \in \Omega$ and hence $u^L \in K$. From the uniqueness of the minimizer, $u^L = u$ and u is $C^{1,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha < 1$.

5. Counterexamples

In this section we construct some counterexamples which exhibit that if the boundary data do not satisfy the bounded slope condition. Then a minimizer may not have a continuous derivative.

Let $0 < \theta < 1$ and $O = (-1,1) \times (-1,1)$.

Let $f : R^2 \to R$ be a function such that

$$f(P) = (1 - P_1^2)^{-\theta} + (1 - P_2^2)^{-\theta}$$

for all $P \in O$ and

$$f(P) = \infty$$

for $P \in R^2 \setminus O$.

By direct computation, we have

$$f_{P_1}(P) = 2\theta P_1 (1 - P_1^2)^{-\theta - 1},$$

$$f_{P_2}(P) = 2\theta P_2 (1 - P_2^2)^{-\theta - 1},$$

$$f_{P_1 P_1}(P) = 2\theta (1 - P_1^2)^{-\theta - 2}(1 + (2\theta + 1)P_1^2),$$

$$f_{P_1 P_2}(P) = 0,$$

$$f_{P_2 P_2}(P) = 2\theta (1 - P_2^2)^{-\theta - 2}(1 + (2\theta + 1)P_2^2)$$

and we can see

$$f_{P_i P_j}(P)\xi_i \xi_j \geq 2\theta |\xi|^2$$

for all $P \in O$ and $\xi \in R^2$.

Suppose $\Omega = (0, \frac{1}{4}) \times (-1,1)$, $\Omega_1 = (0, \frac{1}{4}) \times (0,1)$ and $\Omega_2 = (0, \frac{1}{4}) \times (-1,0)$.

We define $I[v]$ by

$$I[v] = \int_{-1}^{1} \int_{0}^{\frac{1}{4}} f(Dv)dxdy$$

for all $v \in W^{1,\infty}$. Let $u_1(x,y) = x(1-y)$ in $\overline{\Omega}_1 = [0, \frac{1}{4}] \times [0,1]$. Now reflect u_1 with respect to x axis and set $u_2 = x(1+y)$ in $\overline{\Omega}_2 = [0, \frac{1}{4}] \times [-1,0]$. Define $u = u_1$ in $\overline{\Omega}_1$ and $u = u_2$ in $\overline{\Omega}_2$.

LEMMA 5. *Let w be any admissible function for I(i.e., $w = u$ on $\partial\Omega$) and $I[w] < \infty$). Then*

$$w(x, 0) = x$$

for all $0 \le x \le \frac{1}{4}$.

proof. We prove by contradiction. First we note that w is a Lipschitz function. Suppose that the lemma is false and we assume that $w(x_0, 0) > x_0$ for some x_0, where $0 < x_0 < \frac{1}{4}$. Define $\delta = w(x_0, 0) - x_0$. We regularize w with ϵ as (4). Then $w_\epsilon \to w$ uniformly for all $x \in \Omega' \subset\subset \Omega$ and by Jensen's inequality,

$$f(Dw_\epsilon) \le f(Dw)_\epsilon < \infty$$

for all $x \in \Omega'$ if ϵ is sufficiently small.

Let $\delta_1 > 0$ be so small that

$$w(\delta_1, 0) < \frac{\delta}{5}$$

and let ϵ be so small that

$$| w_\epsilon(\delta_1, 0) - w(\delta_1, 0) | \le \frac{\delta}{5}$$

and

$$| w_\epsilon(x_0, 0) - w(x_0, 0) | \le \frac{\delta}{5}.$$

Then we see that

$$\frac{w_\epsilon(x_0, 0) - w_\epsilon(\delta_1, 0)}{x_0 - \delta_1} \ge 1 + \delta_2$$

for some $\delta_2 > 0$ independently for all small ϵ. So for some $\delta_1 \le x_1 \le x_0$

$$\frac{\partial w_\epsilon}{\partial x}(x_1, 0) > 1$$

and

$$f(Dw_\epsilon(x_1, 0)) = \infty.$$

This contradicts the fact that

$$f(Dw_\epsilon) < \infty$$

for all $x \in \Omega' \subset\subset \Omega$.

THEOREM 7. *u is a minimizer and Du is not continuous.*

proof. By direct computation, we see that $I[u] < \infty$ and u is an admissible function. Moreover for all $\psi \in C_0^\infty(\Omega_1)$,

$$\begin{aligned}&\int_{\Omega_1} f_{P_1}(Du)\frac{\partial\psi}{\partial x} + f_{P_2}(Du)\frac{\partial\psi}{\partial y}dxdy \\ =&-\int_{\Omega_1} f_{P_1P_1}(Du)\frac{\partial^2 u}{\partial x\partial x}\psi + 2f_{P_1P_2}(Du)\frac{\partial^2 u}{\partial x\partial y}\psi dxdy \\ &+ f_{P_2P_2}(Du)\frac{\partial^2 u}{\partial y\partial y}\psi dxdy.\end{aligned}$$

We have, by direct computation,

$$\frac{\partial^2 u}{\partial x\partial x} = \frac{\partial^2 u}{\partial y\partial y} = 0$$

and

$$f_{P_1P_2} = 0.$$

We see that u satisfies the Euler-Lagrange equation in Ω_1. Similarly we see that u satisfies the Euler-Lagrange equation in Ω_2. Since every admissible function must have the same data on the line $y = 0$, we conclude that u is a minimizer.

Since

$$\frac{\partial u}{\partial y} = -x$$

in Ω_1 and

$$\frac{\partial u}{\partial y} = x$$

in Ω_2, Du is not continuous on the line $y = 0$.

REMARK. *We note that the minimizers do not have the unique continuation property.*

References

[1] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **63** (1985), 337-403.

[2] J. M. Ball and V. J. Mizel, *Onedimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*, Arch. Rat. Mech. Anal. **90** (1985), 407-423.

[3] P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential functional equations*, Acta Math. **115** (1971), 271-310.

[4] J. Moser, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, CPAM **13** (1960), 457-468.

Department of Mathematics
POSTECH,
Pohang 790-330, Korea