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SUBMANIFOLDS WITH NONVANISHING PARALLEL MEAN CURVATURE VECTOR FIELD OF A SASAKIAN SPACE FORM

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0. Introduction

The theory of a submanifold of a Sasakian manifold was investigated from two different points of view, namely, one is the case where submanifolds are tangent to the structure vector field, and other is the case where those are normal to the structure vector field (cf. [1], [8], [9]).

The purpose of the present paper is to study submanifolds tangent to the structure vector field with nonvanishing parallel mean curvature vector field immersed in a Sasakian space form.

In §1 we state general formulas on submanifolds of a Sasakian manifold, especially those of a Sasakian space form. §2 is devoted to the study submanifolds with nontrivial parallel mean curvature vector field. Moreover, we suppose that the shape operator in the direction of unit normals is parallel along the structure vector field of the submanifold. We compute the restricted Laplacian for the shape opeator in the direction of the mean curvature vector field in §3. As applications of this, in the last §4 we prove our main theorems.

1. Preliminaries

In this section, the basic properties of submanifolds of a Sasakian manifold are recalled [2], [3], [9].

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Let $\tilde{M}$ be a Sasakian manifold of dimension $2m+1$ with almost contact metric structure (ϕ, G, V). Then for any vector fields X and Y on $\tilde{M}$, we have

$$\phi^2 X = -X + v(X)V,\ G(\phi Y, \phi X) = G(Y, X) - v(Y)v(X),$$
$$v(\phi X) = 0,\ \phi V = 0,\ v(V) = 1,\ G(X, V) = v(X).$$

Since $\tilde{M}$ is a Sasakian manifold, we obtain

$$\tilde{\nabla}_X V = \phi X,\ (\tilde{\nabla}_X \phi)Y = -G(X, Y)V + v(Y)X, \tag{1.1}$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be an $(n+1)$-dimensional Riemannian manifold covered by a system of local coordinate neighborhoods $\{U; x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i : M \longrightarrow \tilde{M}$. When the argument is local, we may identify M with $i(M)$. We represent the immersion i locally by

$$y^A = y^A(x^1, \cdots, x^{n+1}), \quad (A = 1, \cdots, n+1, \cdots, 2m+1)$$

and put $B_j{}^A = \partial_j y^A$, $(\partial_j = \partial/\partial x^j)$ then $B_j = (B_j{}^A)$ are $(n+1)$-linearly independent local tangent vector fields of M. We choose $2m-n$ mutually orthogonal unit normals $C_x = (C_x{}^A)$ to M. Throughout this paper, the indices $h, i, j, \cdots$ run over the range $\{1, \cdots, n+1\}$ and $u, v, w, \cdots$ the range $\{n+2, \cdots, 2m+1\}$ and the summation convention will be used with respect to those indices.

The immersion being isometric, the induced Riemannian metric tensor g on M and the metric tensor δ of the normal bundle are then respectively obtained :

$$g_{ji} = G(B_j, B_i), \quad \delta_{yx} = G(C_y, C_x).$$

In the sequel, we assume that *the submanifold M of $\tilde{M}$ is tangent to the structure vector field V.* Then we have

$$V = \xi^i B_i, \quad \xi_i = G(B_i, V). \tag{1.2}$$

The transforms of B_i and C_x by ϕ are respectively represented in each coordinate neighborhood as follows :

$$\phi B_j = f_j{}^i B_i + J_j{}^x C_x, \tag{1.3}$$

$$\phi C_x = -J_x{}^i B_i + Q_x{}^y C_y, \tag{1.4}$$

where we have put

$$f_{ji} = G(\phi B_j, B_i),\ J_{jx} = (\phi B_j, C_x),\ J_{xj} = -G(\phi C_x, B_j),$$

$$Q_{xy} = G(\phi C_x, C_y),\ f_j{}^h = f_{ji} g^{ih},\ J_j{}^x = J_{jy}\delta^{yx},\ Q_x{}^y = Q_{xz}\delta^{zy},$$

δ^{yz} being the contravariant components of δ_{yz} and $(g^{ji}) = (g_{ji})^{-1}$. From these definitions, we verify that $f_{ji} + f_{ij} = 0$, $J_{jx} = J_{xj}$ and $Q_{xy} + Q_{yx} = 0$.

In what follows we denote the index $n+2$ by the symbol $*$.

By the properties of the Sasakian structure tensor, it follows from (1.2), (1.3) and (1.4) that we have

$$f_j{}^t f_t{}^h = -\delta_j{}^h + \xi_j \xi^h + J_j{}^x J_x{}^h,\ Q_x{}^y Q_y{}^z = -\delta_x{}^z + J_x{}^t J_t{}^z, \tag{1.5}$$

$$f_j{}^t J_t{}^x + J_j{}^y Q_y{}^x = 0, \tag{1.6}$$

$$\xi^j J_j{}^x = 0,\ \xi^j f_j{}^h = 0,\ \xi_j \xi^j = 1. \tag{1.7}$$

By denoting ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\nabla_j B_i = A_{ji}{}^x C_x, \quad \nabla_j C_x = -A_j{}^h{}_x B_h, \tag{1.8}$$

where $A_{ji}{}^x$ are the second fundamental forms in the direction of C_x and related by

$$A_j{}^h{}_x = A_{jix} g^{ih} = A_{ji}{}^y g^{ih} \delta_{yx}.$$

Differentiating (1.3) and (1.4) covariantly along M and making use of (1.1), (1.8) and these equations, we easily find

$$\nabla_j f_i{}^h = \delta_j{}^h \xi_i - g_{ji}\xi^h + A_j{}^h{}_x J_i{}^x - A_{ji}{}^x J_x{}^h, \tag{1.9}$$

$$\nabla_j J_i{}^x = A_{ji}{}^y Q_y{}^x - A_{jt}{}^x f_i{}^t, \tag{1.10}$$

$$\nabla_j Q_y{}^x = A_{jt}{}^x J_y{}^t - A_{jty} J^{tx}. \tag{1.11}$$

We also have from (1.2)

$$\nabla_j \xi_i = f_{ji}, \tag{1.12}$$

$$A_{jt}{}^x \xi^t = J_j{}^x \tag{1.13}$$

because of (1.1), (1.3) and (1.8).

In the rest of this section we suppose that the ambient Sasakian manifold $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature c, which is called a Sasakian sapce form, and is denoted by $\tilde{M}^{2m+1}(c)$. Then we see, using (1.2), (1.3), (1.4) and (1.8), that equations of the Gauss, Codazzi and Ricci for M are respectively given by

(1.14)
$$\begin{aligned} R_{kjih} = &\frac{1}{4}(c+3)(g_{kh}g_{ji} - g_{jh}g_{ki}) + A_{kh}{}^x A_{jix} - A_{jh}{}^x A_{kix} \\ &+ \frac{1}{4}(c-1)(\xi_k\xi_i g_{jh} - \xi_j\xi_i g_{kh} + \xi_j\xi_h g_{ki} - \xi_k\xi_h g_{ji} \\ &+ f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}), \end{aligned}$$

$$\nabla_k A_{ji}{}^x - \nabla_j A_{ki}{}^x = \frac{1}{4}(c-1)(J_k{}^x f_{ji} - J_j{}^x f_{ki} - 2J_i{}^x f_{kj}), \tag{1.15}$$

$$\begin{aligned} R_{jiyx} = &\frac{1}{4}(c-1)(J_{jx}J_{iy} - J_{ix}J_{jy} - 2f_{ji}Q_{yx}) \\ &+ A_{jtx}A_i{}^t{}_y - A_{itx}A_j{}^t{}_y, \end{aligned} \tag{1.16}$$

where R_{kjih} and R_{jiyx} are the Riemannian curvature tensor of M and that with respect to the connection induced in the normal bundle of M respectively. We see from (1.14) that the Ricci tensor S of M can be expressed as follows :

(1.17)
$$R_{ji} = \frac{1}{4}\{n(c+3)+2(c-1)\}g_{ji} - \frac{1}{4}(c-1)(n+2)\xi_j\xi_i - \frac{3}{4}(c-1)J_j{}^zJ_{iz} + h^xA_{jix} - A_j{}^{tx}A_{itx}$$

with the aid of (1.5), where $h^x = g^{ji}A_{jix}$.

2. Parallel mean curvature vector fields

Let H be a mean curvature vector field of M in a Sasakian manifold. Namely, it is defined by

$$H = g^{ji}A_{ji}{}^xC_x/(n+1) = h^xC_x/(n+1),$$

which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

In the following we suppose that the mean curvature vector field H of M is nonzero and is parallel in the normal bundle. Then we may choose a local field $\{e_x\}$ in such a way that $H = \sigma C_{n+2} = \sigma C_*$, where $\sigma = |H|$ is nonzero constant. Because of the choice of the local field, the parallelism of H yields

$$\begin{cases} h^x = 0, & x \geq n+3 \\ h^* = (n+1)\sigma. \end{cases} \tag{2.1}$$

Differentiating (1.13) covariantly along M and using (1.10) and (1.12), we find

$$(\nabla_kA_{jrx})\xi^r + A_{jrx}f_k{}^r = A_{kj}{}^yQ_{yx} - A_{krx}f_j{}^r,$$

which together with (1.7) and (1.15) implies that

$$\xi^r\nabla_rA_{jkx} = A_{kj}{}^yQ_{yx} - A_{krx}f_j{}^r - A_{jrx}f_k{}^r. \tag{2.2}$$

Thus, it is seen that $h^y Q_{yx} = 0$ since the mean curvature vector field is parallel. Accordingly, by (2.1) we have

$$Q_x{}^* = 0, \tag{2.3}$$

which join with the second equation of (1.5) imples

$$J_{jx} J^{j*} = \delta_x{}^*. \tag{2.4}$$

H being a normal vector field on M, the curvature tensor R_{jiyx} of the connection in the normal bundle shows that R_{ji*x} vanishes identically for any index x. Thus the Ricci equation (1.16) yields

$$A_{jtx} A_i{}^{t*} - A_{itx} A_j{}^{t*} = \frac{1}{4}(c-1)(J_{j*} J_{ix} - J_{i*} J_{jx}) \tag{2.5}$$

because of (2.3).

From now on we suppose that the shape operator in the direction of C_x is parallel along the structure vector field ξ, namely,

$$\nabla_\xi A^x = 0. \tag{C}$$

Then we have by (2.2)

$$A_{jrx} f_i{}^r + A_{irx} f_j{}^r = A_{ji}{}^y Q_{yx}. \tag{2.6}$$

REMARK 1. A submanifold of a Sasakian manifold $\tilde{M}$ is called a generic submanifold if Q_{yx} vanishies identically [7]. If a submanifold of $\tilde{M}$ is generic and $A^x f = f A^x$ holds for any index x, then the condition (C) is satisfied because of (2.2). But, the converse assertion is not always true.

By (2.6) the equation (1.10) is reduced to

$$\nabla_j J_i{}^x = A_{ir}{}^x J_j{}^r. \tag{2.7}$$

Transvecting (2.6) with ξ^j and using (1.7) and (1.13), we find

$$J_{rx} f_i{}^r + J_{iz} Q_x{}^z = 0,$$

which together with (1.6) gives

$$(2.8)\qquad J_{rx}f_i{}^r = 0, \quad J_{iz}Q_x{}^z = 0.$$

Hence, from (1.5) we have $f^3 + f = 0$ and $Q^3 + Q = 0$, namely, f and Q define the f-structure in M and that in the normal bundle of M respectively [5]. In such a case M is called a contact CR submanifold of a Sasakian manifold [8], [10].

Transforming (2.6) by $f_k{}^i$ and making use of (1.5) and (1.13), we get

$$(A_{jrx}J_z{}^r)J_k{}^z + \xi_k J_{jx} - A_{jix} + A_{srx}f_j{}^r f_k{}^s = -A_{jrz}f_k{}^r Q_x{}^z,$$

from which, taking the skew-symmetric part with respect to indices j and k and using (2.6),

$$\begin{aligned}(A_{jrx}J_z{}^r)J_k{}^z - (A_{krx}J_z{}^r)J_j{}^z + \xi_k J_{jx} - \xi_j J_{kx}\\ = -2A_{jrz}f_k{}^r Q_x{}^z - A_{jkw}Q_z{}^w Q_x{}^z.\end{aligned}$$

If we transvect $J_y{}^k$ to above equation and make use of (1.5) and (2.8), we obtain

$$(2.9)\qquad A_{jrx}J_y{}^r = P_{yzx}J_j{}^z + \xi_j(\delta_{yx} + Q_{zx}Q_y{}^z) - (A_{jrw}J_y{}^r)Q_z{}^w Q_x{}^z,$$

where we have put $P_{yzx} = A_{jix}J_y{}^i J_z{}^i$ and hence $P_{yzx}Q_{wy} = 0$. Thus, by transvecting (2.9) with $Q_{wx}J_j{}^y$ and taking account of (2.8), we get

$$(2.10)\qquad P_{yzx}Q_{wx} = 0.$$

From (2.9) we also have

$$(2.11)\qquad A_{jr}{}^*J_y{}^r = P_{yz*}J_j{}^z + \delta_y{}^*\xi_j$$

because of (2.3).

Multiplying $J_z{}^j J_y{}^i$ to (2.5) and summing for j and i, we find

$$P_{uzx}P_y{}^{u*} - P_{uyx}P_z{}^{u*} = \frac{1}{4}(c+3)\{\delta_z{}^* J_y{}^i J_{xi} - \delta_y{}^* J_z{}^i J_{xi}\},$$

where we have used (1.7), (2.4), (2.8), (2.10) and (2.11). Consequently we have

$$P_{yzx}P^{yz*} - P^{z}P_{zx*} = \frac{1}{4}(c+3)(p-1)\delta_{x*}, \tag{2.12}$$

$$P_{zx}{}^{*}P_{y}{}^{z*} - P_{zyx}P^{z**} = \frac{1}{4}(c+3)(J_{y}{}^{i}J_{ix} - \delta_{y}{}^{*}\delta_{x}{}^{*}), \tag{2.13}$$

where we denote $P_{z}{}^{zx} = P^{x}$ and $J_{jx}J^{jx} = p$.

Differentiating (2.11) covariantly and substituting (1.12) and (2.7), we find

$$(\nabla_k A_{jr}{}^{*})J_{y}{}^{r} + A_{j}{}^{r*}A_{rsy}f_{k}{}^{s} = (\nabla_k P_{yz*})J_{j}{}^{z} + P_{yz*}A_{jr}{}^{z}f_{k}{}^{r} + \delta_{y}{}^{*}f_{kj}$$

and hence, taking the skew-symmetric part with respect to k and j and taking account of (1.15), (2.4), (2.6) and (2.8),

$$\begin{aligned} &2A_{j}{}^{r*}A_{rsy}f_{k}{}^{s} + A_{k}^{r*}A_{jrz}Q_{y}{}^{z} - 2P_{yz*}A_{jr}^{z}f_{k}{}^{r} \\ =&(\nabla_k P_{yz*})J_{j}{}^{z} - (\nabla_j P_{yz*})J_{k}{}^{z} + \frac{1}{2}(c+3)\delta_{y*}f_{kj}. \end{aligned} \tag{2.14}$$

If we transvect f^{kj} to the last equation and make use of (1.5), (2.5) and (2.8), we can get

$$\begin{aligned} &A_{j}^{r*}A_{rsy}(g^{js} - \xi^{j}\xi^{s} - J^{jz}J_{z}{}^{s}) \\ &= P_{yz}{}^{*}A_{jr}{}^{z}(g^{jr} - \xi^{j}\xi^{r} - J^{jw}J_{w}^{r}) + \frac{1}{4}(c+3)\delta_{y}{}^{*}(n-p), \end{aligned}$$

which together with (1.13), (2.1), (2.4) and (2.11) yields

$$A_{ji}{}^{*}A^{ji}{}_{y} = h^{*}P_{y**} + P_{zwy}P^{zw*} - P^{z}P_{zy*} + 2\delta_{y}{}^{*} + \frac{1}{4}(c+3)(n-p)\delta_{y}{}^{*}.$$

Thus, it follows that we obtain

$$A_{ji}{}^{*}A^{ji}{}_{y} = h^{*}P_{y**} + \frac{1}{4}(n-1)(c+3)\delta_{y}{}^{*} + 2\delta_{y}{}^{*} \tag{2.15}$$

because of (2.12).

For the shape operator A^* in the direction of the mean curvature vector field, a tensor field $(A^*)^a$ and a function $h_{(a)}$ for any integer $a \geq 2$ are introduced as follows :

$$(A_{ji}{}^*)^a = A_{ji_1}{}^* A_{i_2}{}^{i_1 *} \cdots A_i{}^{i_{a-1}*}, \quad h_{(a)} = \sum_i (A_{ii}{}^*)^a.$$

Thus, (2.15) implies that

$$h_{(2)} = h^* P_{***} + \frac{1}{4}(n-1)(c+3) + 2. \tag{2.16}$$

3. Lemmas

In this section we prepare some lemmas for later use.

LEMMA 1. *Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field of a $(2m+1)$-dimensional Sasakian space form. Suppose that the mean curvature vector field is nonzero and parallel in the normal bundle. If $\nabla_\xi A^x = 0$ on M, then we have*

$$R_{js} A_i{}^{s*} A^{ji*} - R_{kjih} A^{kh*} A^{ji*} = \frac{1}{16}(c-1)^2(n-p). \tag{3.1}$$

Proof. When $y = *$ in (2.14), we have

$$\begin{aligned} &2A_j{}^{r*} A_{rs*} f_k{}^s \\ &= (\nabla_k P_{z**}) J_j{}^z - (\nabla_j P_{z**}) J_k{}^z + 2P_{z**} A_{jrz} f_k{}^r + \frac{1}{2}(c+3) f_{kj} \end{aligned}$$

because of (2.3). Transforming by $A_t{}^{j*} f^{kt}$ and making use of (1.5), (2.8) and (2.11), we find

$$\begin{aligned} A_j{}^{r*} A_{rs*} A_t{}^{j*} (g^{st} - \xi^s \xi^t - J^{sw} J_w{}^t) &= P_{z**} A_{jr}^{\ z} A_t{}^{j*} (g^{rt} - \xi^r \xi^t - J^{rw} J_w{}^t) \\ &\quad + \frac{1}{4}(c+3) A_t{}^{j*} (\delta_j{}^t - \xi_j \xi^t - J_j{}^w J_w{}^t), \end{aligned}$$

which combine with (1.13), (2.4), (2.9), (2.10) and (2.11) gives forth

$$h_{(3)} = P_{wz*}P^{zx*}P_x{}^{w*} - P_{xwz}P^{xw*}P^{z**} + P_{***} + P_{z**}A_{ji}{}^{z}A^{ji*} + \frac{1}{4}(c+3)(h^* - P^*).$$

If we take account of (2.13) and (2.15), then the last equation is reduced to

$$(3.2)\quad h_{(3)} = h^*|P_{z**}|^2 + \frac{1}{4}(c+3)(n-2)P_{***} + \frac{1}{4}(c+3)h^* + 3P_{***}.$$

By using (1.13), (2.1) and (2.5), the equation (1.17) implies

$$\begin{aligned} R_{js}A_i{}^{s*}A^{ji*} = {} & \frac{1}{4}\{n(c+3)+2(c-1)\}h_{(2)} - \frac{1}{4}(c-1)(n+2) \\ & - \frac{3}{4}(c-1)A_s{}^{j*}A^{si*}J_{jz}J_i{}^{z} + h^*h_{(3)} - A_{jr}^{\ x}A_{isx}A^{rs*}A^{ji*} \\ & - \frac{1}{4}(c-1)A_j{}^{rx}A^{ji*}(J_{r*}J_{ix} - J_{i*}J_{rx}), \end{aligned}$$

which together with (2.4), (2.10), (2.11), (2.12) and (3.2) leads to

$$(3.3)\quad \begin{aligned} R_{js}A_i{}^{s*}A^{ji*} = {} & \frac{1}{4}\{n(c+3)+2(c-1)\}h_{(2)} - \frac{1}{4}(c-1)(n+2) \\ & + \frac{1}{4}(c-1)^2 - \frac{3}{4}(c-1)P^xP_{x**} - \frac{1}{4}(c-1)(c+2)p \\ & + |h^*P_{x**}|^2 + \frac{1}{4}(c+3)(h^*)^2 + 3h^*P_{***} \\ & + \frac{1}{4}(c+3)(n-2)h^*P_{***} - A_{jr}{}^{x}A_{isx}A^{rs*}A^{ji*}. \end{aligned}$$

On the other hand, by means of (1.13), (2.4) and (2.15), the equation (1.14) gives

$$(3.4)\quad \begin{aligned} R_{kjih}A^{kh*}A^{ji*} = {} & \frac{1}{4}(c+3)\{(h^*)^2 - h_{(2)}\} + \frac{1}{2}(c-1) \\ & + |h^*P_{z**}|^2 + \frac{3}{4}(c-1)A^{kh*}A^{ji*}f_{kj}f_{hi} \\ & - A_{jrx}A_{isx}A^{rs*}A^{ji*} + \frac{1}{2}(c+3)(n-1)h^*P_{***} \\ & + 4h^*P_{***} + \{\frac{1}{4}(c+3)(n-1)+2\}^2. \end{aligned}$$

By the way, making use of (1.5), (1.13), (2.4), (2.11) and (2.12) we obtain

$$A^{kh*}A^{ji*}f_{kj}f_{hi} = h_{(2)} - P^x P_{x**} - \frac{1}{4}(c-1)(p-1) - p - 1. \tag{3.5}$$

From the last three equations we easily see that

$$\begin{aligned}
&R_{js}A_i{}^{s*}A^{ji*} - R_{kjih}A^{kh*}A^{ji*} \\
&=\{\frac{1}{4}(c+3)n+1\}(h_{(2)} - h^* P_{***}) - \frac{1}{4}(c-1)(n+2) \\
&\quad + \frac{1}{4}(c-1)^2 - \{\frac{1}{4}(c+3)(n-1)+2\}^2 - \frac{1}{2}(c-1) \\
&\quad + \frac{3}{16}(c-1)^2(p-1) + \frac{3}{4}(c-1)(p+1) - \frac{1}{4}(c-1)(c+2)p,
\end{aligned}$$

which joined with (2.16) implies (3.1). This completes the proof.

LEMMA 2. *Under the same assumptions as that in Lemma 1, the function $h_{(2)}$ is harmonic.*

Proof. By definition we have $P_{***} = A_{ji*}J_*{}^i J_*{}^j$. Differentiation covariantly we have

$$\nabla_k P_{***} = (\nabla_k A_{ji*})J_*{}^j J_*{}^i,$$

where we have used (2.7), (2.8) and (2.11). Thus, the Laplacian of the function P_{***} is given by

$$\Delta P_{***} = (\Delta A_{ji*})J_*{}^j J_*{}^i + 2(\nabla_k A_{ji*})J_*{}^i \nabla^k J^{j*},$$

which together with (1.5), (1.13), (1.15) and (2.7) gives

$$\Delta P_{***} = (\Delta A_{ji*})J_*{}^j J_*{}^i - \frac{1}{2}(c-1)(h^* - P^*). \tag{3.6}$$

Since the submanifold M has parallel mean curvature vector field, the Laplacian ΔA_{ji*} of A^* is given , using the Ricci formula for A^* and (1.15), by

$$\begin{aligned}
\Delta A_{ji*} = {} & R_{jr}A_i{}^{r*} - R_{kjih}A^{kh*} \\
& + \frac{1}{4}(c-1)\nabla_k(J_*{}^k f_{ji} + J_{j*}f_i{}^k + 2J_{i*}f_j{}^k).
\end{aligned} \tag{3.7}$$

Thus, it follows that we get

$$(\Delta A_{ji*})J_*{}^jJ_*{}^i = R_{jr}A_i{}^{r*}J_*{}^jJ_*{}^i - R_{kjih}A^{kh*}J_*{}^jJ_*{}^i + \frac{3}{4}(c-1)(h^* - P^*)$$

because of (1.5), (1.7), (1.9), (2.7), (2.8) and (2.11). Therefore (3.6) turns out to be

$$\Delta P_{***} = R_{jr}A_i{}^{r*}J_*{}^jJ_*{}^i - R_{kjih}A^{kh*}J_*{}^jJ_*{}^i + \frac{1}{4}(c-1)(h^* - P^*). \tag{3.8}$$

From (1.17) we obtain

$$\begin{aligned} R_{ji}J_*{}^jA_r{}^{i*}J_*{}^r &= h^* - P^* + \frac{1}{4}(n-1)(c+3)P_{***} + h^*|P_{z**}|^2 \\ &\quad - P_{z**}P_{ux*}P^{zux} \end{aligned} \tag{3.9}$$

because of (1.7), (1.13), (2.4) and (2.11). We also have by (1.14)

$$\begin{aligned} R_{kjih}J_*{}^jJ_*{}^iA^{kh*} &= \frac{1}{4}(c+3)(h^* - P_{***}) + P_{z**}A_{jiz}A^{ji*} \\ &\quad - A^{kh*}(P_{wx*}J_k{}^w + \delta_{x*}\xi_k)(P_z{}^{x*}J_h{}^z + \delta^{x*}\xi_h) \\ &= \frac{1}{4}(c+3)(h^* - P_{***}) + h^*|P_{z**}|^2 \\ &\quad + \frac{1}{4}(n-1)(c+3)P_{***} - P_{ux*}P_z{}^{u*}P^{zx*}, \end{aligned} \tag{3.10}$$

where we have used (1.7), (1.13), (2.11) and (2.15).

Substituting (3.9) and (3.10) into (3.8) and taking account of (2.13), we get $\Delta P_{***} = 0$. Thus, the equation (2.16) implies that $\Delta h_{(2)} = 0$ because the mean curvature vector field is parallel. Therefore we arrive at the conclusion.

LEMMA 3. *Under the same assumptions as that in Lemma 1, we have*

$$\nabla_k A_{ji}{}^* = -\frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{i*}f_{kj}). \tag{3.11}$$

Proof. By (1.9) we have

$$\nabla_k f_i{}^k = n\xi_i + h^* J_{i*} - A_{irx} J_x{}^r$$

because of (2.1). Hence we have

$$A^{ji*}\nabla_k(J_{j*}f_i{}^k) = A^{ji*}A_{jr*}f_k{}^r f_i{}^k + (P_{z**}J^{iz}+\xi^i)(n\xi_i + h^* J_{i*} - A_i{}^{rx} J_{xr})$$

by virtue of (2.7) and (2.11), or using (1.5), (1.7), (1.13) and (2.4) we obtain

(3.12)
$$\begin{aligned} &A^{ji*}\nabla_k(J_{j*}f_i{}^k) \\ &= -h_{(2)} + P_{xy*}P^{xy*} - P^x P_{x**} + h^* P_{***} + n - p + 2 \\ &= -\frac{1}{4}(c-1)(n-p) \end{aligned}$$

by means of (2.13) and (2.16).

Multiplying A^{ji*} to (3.7) and summing for j and i, and making use of (3.1) and (3.12), we have

$$A^{ji*}\Delta A_{ji}{}^* = -\frac{1}{8}(c-1)^2(n-p). \tag{3.13}$$

By the way, generaly we have

$$\frac{1}{2}\Delta h_{(2)} = A^{ji*}\Delta A_{ji}{}^* + |\nabla_k A_{ji}{}^*|^2.$$

Thus because of Lemma 2 and (3.13), we obtain

$$|\nabla_k A_{ji*}|^2 = \frac{1}{8}(c-1)^2(n-p). \tag{3.14}$$

On the other hand, we easily verify that

$$|\nabla_k A_{ji}{}^* + \frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{i*}f_{kj})|^2 = |\nabla_k A_{ji}{}^*|^2 - \frac{1}{8}(c-1)^2(n-p),$$

where we have used (1.5), (1.15), (2.4) and (2.8). Consequently (3.11) is valid by virtue of (3.14). This completes the proof.

REMARK 2. If M is a generic submanifold of a Sasakian space form satisfying $A^x f = fA^x$, then (3.11) holds on M provided that M has nonzero and parallel mean curvature vector field.

4. Main theorems

Let M be a submanifold tangent to the structure vector field of a Sasakian space form $\tilde{M}^{2m+1}(c)$ satisfying (3.11).

Transvecting (3.11) with $(A^{ji*})^{a-1}$ for any integer $a \geq 2$ and taking account of (1.7), (2.8) and (2.11), we then obtain $\nabla_k h_{(a)} = 0$.

For any point q in M we can choose a local orthonormal frame field $\{E_i\}$ so that the shape operator A^* in the direction of the mean curvature vector field is diagonalizable at that point q, say $A_{ji}{}^* = \lambda_j \delta_{ji}$. Then $h_{(a)}$ can be written as

$$h_{(a)} = \sum_i \lambda_i{}^a, \quad (a = 1, 2, \cdots).$$

Since we have $h_{(a)} =$ constant for any integer $a \geq 1$, it is seen that λ_i is constant, namely all eigenvalues of A^* are constant.

We denote by σ_{ji} the sectional curvature of M spanned by E_j and E_i. Then by Lemma 1 we have

$$\sum_{j,i} (\lambda_i - \lambda_j)^2 \sigma_{ji} = \frac{1}{8}(c-1)^2(n-p) \geq 0$$

because of (3.14). Thus, if $\sigma_{ji} \leq 0$, then $(c-1)^2(n-p) = 0$, and consequently $\nabla A^* = 0$ by Lemma 3. Moreover, we have $c = 1$ or $n = p$. If $n = p$, then $f = 0$ and M is a totally real submanifold of $\tilde{M}^{2m+1}(c)$ tangent to the structure vector field V (cf. [6]). Thus we have

THEOREM 4. *Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector field. If $\nabla_\xi A^x = 0$, and if the sectional curvature of M is nonpositive, then the shape operator A^* in the direction of the mean curvature vector field is parallel. Moreover, $c = 1$, or M is totally real in $\tilde{M}^{2m+1}(c)$ with respect to ϕ.*

According to Theorem 3.5 of [3] and Theorem 3.2 of [4] and Theorem 4, we have

THEOREM 5. *Under the same hypothesis as that in Theorem 4, if M is complete and simply connected, then M is a product of Riemannian manifolds $M_1 \times \cdots \times M_s$, where s is the number of the distinct eigenvalues of A^*, and the mean curvature vector field of M is an umbilical section of $M_t (t = 1, \cdots, s)$.*

REMARK 3. Let M be a complete and simply connected generic submanifold of a Sasakian space form with nonvanishing parallel mean curvature vector field. Then it is, taking account of Remark 2, seen that if $A^x f = f A^x$ for any index x, and if the sectional curvature of M is nonpositive, then M is the same type as that in Theorem 5 (See also Theorem 3.8 of [3]).

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