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SPINOR GENERIC THETA–SERIES

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§0. Introduction and Notations

Let Q be a positive definite integral quadratic form in m variables and N be a semi-positive definite integral quadratic form in n variables. It is well known [**H**] that the representation of N by the genus of Q is equivalent to that by the spinor genus of Q if $m \geq n+3$.

In this article, we prove that the generic theta-series associated to Q is equal to the spinor generic theta-series associated to Q and thereby recapture the above result as a corollary (under an additional dimensional restriction, which can be removed, however, by a better estimation of the growth of eigenvalues of cusp forms). In the context, we will give arguments only for even m, for convenience. Analogous arguments can easily be established for odd m modulo the so called canonical decomposition of Siegel modular forms of half integral weight, which is not yet given in any literature, while that of Siegel modular forms of integral weight is at hand [**E**].

For $g \in M_{2n}(\mathbf{R})$, let A_g, B_g, C_g, and D_g denote the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices. Let

$$G_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R})\,;\, {}^t g J_n g = rJ_n,\ r > 0\}$$
$$\mathcal{H}_n = \{Z \in M_n(\mathbf{C})\,;\, {}^t Z = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\}$$

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where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ and $r = r(g)$ is a real number determined by g. We set for $g \in G_n$ and $Z \in \mathcal{H}_n$

$$(0.1) \qquad g\langle Z\rangle = (A_g Z + B_g)(C_g Z + D_g)^{-1} \in \mathcal{H}_n,$$

and for a complex valued function F on $\mathcal{H}_n$

$$(0.2) \qquad (F|_k g)(Z) = r(g)^{nk-<n>} \det(C_g Z + D_g)^{-k} F(g\langle Z\rangle),$$

where k is a positive integer and $< n >= n(n+1)/2$.

Let q be positive integer and p be a prime relatively prime to q. Let

$$\begin{gathered} \Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z})\,;\, {}^tMJ_nM = J_n\} \\ L^n = L^n_p = \{g \in M_{2n}(\mathbf{Z}[p^{-1}])\,;\, {}^tgJ_ng = p^\delta J_n, \delta \in \mathbf{Z}\} \\ \Gamma^n_0(q) = \{M \in \Gamma^n\,;\, C_M \equiv 0 (\operatorname{mod} q)\} \\ L^n_0(q) = L^n_{0,p}(q) = \{g \in L^n\,;\, C_g \equiv 0 \,(\operatorname{mod} q)\} \\ \Gamma^n_0 = \{M \in \Gamma^n\,;\, C_M = 0\}, \;\; L^n_0 = L^n_{0,p} = \{g \in L^n\,;\, C_g = 0\} \\ \Lambda^n = SL_n(\mathbf{Z}), \;\; V^n = V^n_p = \{D \in M_n(\mathbf{Z}[p^{-1}])\,;\, \det D = p^d, \; d \in \mathbf{Z}\} \end{gathered}$$

where $\delta = \delta(g)$ is an integer determined by g in L^n above. We also set

$$\begin{gathered} E^n = E^n_p = \{g \in L^n\,;\, \delta \in 2\mathbf{Z}\}, \\ E^n_0(q) = E^n_{0,p}(q) = E^n \cap L^n_0(q) \;\text{ and }\; E^n_0 = E^n_{0,p} = E^n \cap L^n_0. \end{gathered}$$

For $Z \in M_n(\mathbf{C})$, let $e(Z) = \exp(2\pi i\sigma(Z))$ where $\sigma(Z)$ is the trace of Z. For other standard terminologies and basic facts, we refer the readers [**A1**], [**M**], [**O**].

§1. Preliminaries

Let $m \geq n$ be positive integers and let $Q \in \mathcal{N}^+_m$. We define the theta-series of degree n associated to Q by

$$(1.1) \qquad \theta^n(Z,Q) = \sum_{X \in M_{m,n}(\mathbf{Z})} e({}^tXQXZ) = \sum_{N \in \mathcal{N}_n} r(N,Q) e(NZ)$$

for $z \in \mathcal{H}_n$, where $r(N,Q) = \left|\{X \in M_{m,n}(\mathbf{Z}) ;\ {}^tXQX = N\}\right| < \infty$ is the representation number of N by Q.

Let k, q be positive integers and let χ be a Dirichlet character modulo q. Let $\mathcal{M}_k^n(q,\chi)$ be the space (over $\mathbf{C}$) of Siegel modular forms F of degree n, weight k, level q, with character χ, i.e., the space of holomorphic functions $F : \mathcal{H}_n \to \mathbf{C}$ satisfying (i) $F\big|_k M = \chi(\det D_M)F$ for every $M \in \Gamma_0^n(q)$ and (ii) $F\big|_k M$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$, for every $M \in \Gamma^1 = SL_2(\mathbf{Z})$ when $n = 1$. It is known [**Kö**] that the boundedness condition (ii) when $n \geq 2$ follows from the holomorphicity of F and (i). $\mathcal{M}_k^n(q,\chi)$ is a finite dimensional vector space.

When m is even, the following is known [**A-M**] :

$$\theta^n(Z,Q) \in \mathcal{M}_k^n(q,\chi), \quad Z \in \mathcal{H}_n \tag{1.2}$$

where $k = m/2$, q is the level of Q, and $\chi = \chi_Q$ is a Dirichlet character modulo q defined by

$$\chi(d) = \begin{cases} (d/|d|)^k \left(\dfrac{(-1)^k \det 2Q}{|d|}\right)_{\mathrm{Jac}} & \text{if } q > 1 \\ 1 & \text{if } q = 1 \end{cases} \tag{1.3}$$

for integers d relatively prime to q. (See [**K**] for the analogy when m is odd.)

Let $\mathcal{M}_s^n$ be the space of even($s = 0$) or odd($s = 1$) modular forms of degree n, i.e., the space of holomorphic functions $F : \mathcal{H}_n \to \mathbf{C}$ satisfying (i)$'$ $(\det D_M)^s F(M\langle Z\rangle) = F(Z)$, $Z \in \mathcal{H}_n$ for every $M \in \Gamma_0^n$ and (ii)$'$ $F(z)$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$ when $n = 1$. Observe

$$\mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_s^n \ \text{ if } \ \chi(-1) = (-1)^{k+s}. \tag{1.4}$$

Let $\mathcal{L}_0^n(q) = \mathcal{L}_{0,p}^n(q)$, $\mathcal{L}_0^n = \mathcal{L}_{0,p}^n$, and $\mathcal{D}^n = \mathcal{D}_p^n$, be the Hecke rings of the Hecke pairs $(\Gamma_0^n(q), L_0^n(q))$, (Γ_0^n, L_0^n), and (Λ^n, V^n), respectively. Similary, let $\mathcal{E}_0^n(q) = \mathcal{E}_{0,p}^n(q)$ and $\mathcal{E}_0^n = \mathcal{E}_{0,p}^n$ be the Hecke rings of the Hecke pairs $(\Gamma_0^n(q), E_0^n(q))$ and (Γ_0^n, E_0^n), which are the even subrings of $\mathcal{L}_0^n(q)$ and $\mathcal{L}_0^n$, respectively. There exists an injective homomorphism [**A1**] $\beta^n : \mathcal{L}_0^n(q) \to \mathcal{L}_0^n$ defined by

$$\beta^n\left(\sum a_i(\Gamma_0^n(q)g_i)\right) = \sum a_i(\Gamma_0^n g_i) \tag{1.5}$$

where g_i are chosen to be in L_0^n. We set

$$\mathbf{L}_0^n = \beta^n(\mathcal{L}_0^n(q)) \quad \text{and} \quad \mathbf{E}_0^n = \beta^n(\mathcal{E}_0^n(q)). \tag{1.6}$$

We introduce a homomorphism $\psi_n : \mathcal{L}_0^n \to \mathbf{C}_n[\underline{x}]$, where $\mathbf{C}_n[\underline{x}] = \mathbf{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $X \in \mathcal{L}_0^n$. X can be written in the form $X = \sum a_i(\Gamma_0^n g_i)$, where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$, with $\delta_i = \delta(g_i) \in \mathbf{Z}$, $B_i \in M_n(\mathbf{Z}[p^{-1}])$, $D_i \in V^n$ and $D_i^* = ({}^tD)^{-1}$. We define

$$\omega_n : \mathcal{L}_0^n \to \mathcal{D}^n[t^{\pm 1}] \text{ by } \omega_n(X) = \sum a_i t^{\delta_i}(\Lambda^n D_i).$$

Then ω_n is a surjective ring homomorphism. Let $W = \sum a_i t^{\delta_i}(\Lambda^n D_i) \in \mathcal{D}^n[t^{\pm 1}]$. We may assume that each D_i is an upper triangular matrix with diagonal entries $p^{d_{i1}}, \cdots, p^{d_{in}}$. We define

$$\phi_n : \mathcal{D}^n[t^{\pm 1}] \to \mathbf{C}_n[\underline{x}] \text{ by } \phi_n(W) = \sum a_i x_0^{\delta_i}\left(\prod_{1 \le j \le n} (x_j p^{-j})^{d_{in}}\right).$$

Then ϕ_n is an injective ring homomorphism. Finally, we set

$$\psi_n = \phi_n \circ \omega_n : \mathcal{L}_0^n \to \mathbf{C}_n[\underline{x}]. \tag{1.7}$$

§2. Hecke Operators

Let n, k, q, χ be as above. For $F \in \mathcal{M}_k^n(q, \chi)$ and $X = \sum a_i(\Gamma_0^n(q) g_i) \in \mathcal{L}_0^n(q)$, we set

$$F\big|_{k,\chi} X = \sum a_i \chi(\det A_{g_i}) F\big|_k g_i \in \mathcal{M}_k^n(q, \chi). \tag{2.1}$$

As for $F \in \mathcal{M}_s^n$ and $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$, we set

$$F\big|_{k,\chi} X = \sum a_i \chi(\det A_{g_i}) F\big|_k g_i \in \mathcal{M}_s^n \tag{2.2}$$

where $\chi(-1) = (-1)^{k+s}$. X acting on modular spaces as above are called Hecke operators.

Let $F \in \mathcal{M}_k^n(q,\chi)$ with $\chi(-1) = (-1)^{k+s}$ and let $X = \sum a_i(\Gamma_0^n(q)g_i) \in \mathcal{L}_0^n(q)$ with g_i chosen to be in L_0^n. Then from (1.4), (1.5) and the above definitions follows that

$$(2.3) \qquad F\big|_{k,\chi}X = F\big|_{k,\chi}\beta^n(X).$$

Let $F \in \mathcal{M}_s^n$. We define the Siegel operator $\Phi : \mathcal{M}_s^n \to \mathcal{M}_s^{n-1}$ by

$$(2.4) \qquad (\Phi F)(Z') = \lim_{\lambda \to +\infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}\right), \ Z' \in \mathcal{H}_{n-1} \text{ and } \lambda > 0$$

($\mathcal{M}_s^0 = \mathbf{C}$, $\mathcal{H}_0 = \{0\}$). Every $F \in \mathcal{M}_s^n$, hence every $F \in \mathcal{M}_k^n(q,\chi)$ with $\chi(-1) = (-1)^{k+s}$, has a Fourier expansion of the form

$$(2.5) \qquad F(Z) = \sum_{N \in \mathcal{N}_n} f(N)e(NZ), \ \ Z \in \mathcal{H}_n.$$

So from (2.4) and (2.5) follows that

$$(2.6) \qquad (\Phi F)(Z') = \sum_{N' \in \mathcal{N}_{n-1}} f\left(\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix}\right) e(N'Z'), \ \ Z' \in \mathcal{H}_{n-1}$$

($\mathcal{N}_0 = \{0\}$) and that $\Phi F \in \mathcal{M}_k^{n-1}(q,\chi)$ if $F \in \mathcal{M}_k^n(q,\chi)$.

Let $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$ with $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$. By multiplying $\begin{pmatrix} U_i^* & 0 \\ 0 & U_i \end{pmatrix} \in \Gamma_0^n$ for a suitable $U_i \in GL_n(\mathbf{Z})$ from the left of g_i, we may assume that all the D_i are of the form $D_i = \begin{pmatrix} D_i' & * \\ 0 & p^{d_i} \end{pmatrix}$, $d_i \in \mathbf{Z}$, where $D_i' \in V^{n-1}$ is upper triangular. We set

$$(2.7) \qquad \Psi(X,u) = \sum a_i u^{-\delta_i}(up^{-n})^{d_i}(\Gamma_0^{n-1}g_i') \in \mathcal{L}_0^{n-1}[u^{\pm 1}]$$

where $g_i' = \begin{pmatrix} p^{\delta_i}(D_i')^* & B_i' \\ 0 & D_i' \end{pmatrix} \in L_0^{n-1}$. Here B_i' and D_i' denote the blocks of size $(n-1) \times (n-1)$ in the upper left corner of B_i and D_i, respectively. If $n = 1$, we set $\Psi(X,u) = \sum a_i u^{-\delta_i}(up^{-1})^{d_i}$. $\Psi(-,u)$ is a well defined ring homomorphism : $\mathcal{L}_0^n \to \mathcal{L}_0^{n-1}[u^{\pm 1}]$ (see [Z]).

THEOREM 2.1. *Let $F \in \mathcal{M}_k^n(q,\chi)$ with $\chi(-1) = (-1)^{k+s}$, and let $X \in \mathcal{L}_0^n(q)$. Then*

$$\Phi(F\big|_{k,\chi}X) = (\Phi F)\big|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1})$$

where $Y = \beta^n(X) \in \mathcal{L}_0^n$. (If $n = 1$, then the action on the right hand side is nothing but a multiplication of complex numbers.)

Proof. [**A2**]. (See [**KKO**] for the analogy when k is a half integer.)

The following result is also given by Andrianov [**A2**].

THEOREM 2.2. *$\Psi(-,u) : \mathbf{E}_0^n \to \mathbf{E}_0^{n-1}$ is a surjective ring homomorphism for any $u \in \mathbf{C}$, $u \neq 0$.*

§3. Theta Operators

Let Θ_m^n be the space (over $\mathbf{C}$) spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$ and let $\Theta_m^n(q,d)$ be its subspace spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$ with $d = \det 2Q$ and $q =$ the level of Q for given positive integers d and q. Then

$$\Theta_m^n(q,d) \subset \mathcal{M}_k^n(q,\chi) \quad \text{and} \quad \Theta_m^n \subset \mathcal{M}_0^n \tag{3.1}$$

for even m where χ is a character (1.3).

Let $Q \in \mathcal{N}_m^+$. We denote the genus, the spinor genus, and the class of Q by $[Q]$, $\{Q\}$, and (Q), respectively. So $(Q) \subseteq \{Q\} \subseteq [Q]$ and $[Q]$ contains a finite number of classes (see, for instance, [**O**]). Note that $\theta^n(Z,Q_1) = \theta^n(Z,Q)$ for any $Q_1 \in (Q)$. Since $d = \det 2Q$ and the level q of Q are invariants of $[Q]$,

$$\Theta_m^n[Q] \subset \Theta_m^n(q,d) \subset \Theta_m^n \tag{3.2}$$

where $\Theta_m^n[Q]$ is the subspace of Θ_m^n spanned by $\theta^n(Z,Q_i)$, $Q_i \in [Q]$.

It is well known that

(3.3)
$$\Phi(\theta^n(Z,Q)) = \theta^{n-1}(Z',Q), \;\; Z = \begin{pmatrix} Z' & * \\ * & * \end{pmatrix} \in \mathcal{H}_n, \;\; Z' \in \mathcal{H}_{n-1}.$$

In particular, $\Phi : \Theta_m^n[Q] \to \Theta_m^{n-1}[Q]$, $\Phi : \Theta_m^n(q,d) \to \Theta_m^{n-1}(q,d)$ are epimorphisms for all $n \geq 1$ and isomorphisms [**F**] if $n > m$.

We now introduce theta operators. Let $m, n \geq 1$ and let p be a prime relatively prime to q. Let $\alpha : L_0^m \to \mathbf{C}^\times$ be a character such that $\alpha(\Gamma_0^m) = 1$. For $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ O & D_0 \end{pmatrix} \in L_0^m$ and $\theta^n(Z,Q) \in \Theta_m^n$ with $Q \in \mathcal{N}_m^+$, we set

$$\theta^n(Z,Q) \circ_\alpha X = \alpha(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda / \Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D) \theta^n(Z, p^\delta Q[D^*]) \tag{3.4}$$

where $\Lambda = \Lambda^m$ and

$$l_X(Q,D) = \sum_{B \in B_X(D)/\mathrm{mod}\, D} e(QBD^{-1}). \tag{3.5}$$

Here $B_X(D) = \left\{ B \in M_m(\mathbf{Z}[p^{-1}]) ; \begin{pmatrix} p^\delta D^* & B \\ O & D \end{pmatrix} \in \Gamma_0^m g_0 \Gamma_0^m \right\}$ and $B_1, B_2 \in B_X(D)$ are said to be congruent modulo D on the right if $(B_1 - B_2)D^{-1} \in M_m(\mathbf{Z})$. This congruence is obviously an equivalent relation and the summation in (3.5) is over the equivalent classes in $B_X(D)$ modulo D on the right. We extend (3.4) by linearity to the whole space Θ_m^n and the whole ring $\mathcal{L}_0^m$. We set

$$\mathcal{L}_{00}^m = \left\{ \sum a_i (\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^m \, ; \, \delta_i m - 2b_i = 0, b_i = \log_p |\det D_{g_i}| \right\}$$

and $\mathcal{E}_{00}^m = \mathcal{E}_0^m \cap \mathcal{L}_{00}^m$, where $g_i \in L_0^m$ with $\delta_i = \delta(g_i)$.

THEOREM 3.1. (1) *The action (3.4) is a well defined action.*

(2) $\Theta_m^n(q,d)$ *is invariant under the theta operators of* $\mathcal{L}_{00}^m$ *if* p *is relatively prime to* q.

(3) $\Theta_m^n[Q]$ *is invariant under the theta operators of* $\mathcal{E}_{00}^m$ *if* p *is relatively prime to* $2q$, *where* q *is the level of* Q.

Proof. See [**A2**] for even m and [**K**] for odd m.

Let $Q \in \mathcal{N}_m^+$ with m even. We define $\Psi = \Psi_Q : \mathcal{E}_0^n(q) \to \mathcal{E}_0^{n-1}(q)$ by requiring the following diagram commutes :

$$\begin{tikzcd} \mathcal{E}_0^n(q) \arrow[r, "\sim", "\beta^n"'] \arrow[d, "\Psi"'] & \mathbf{E}_0^n \arrow[d, "\Psi(-, p^{n-k}\chi^{-1}(p))"] \\ \mathcal{E}_0^{n-1}(q) \arrow[r, "\sim", "\beta^{n-1}"'] & \mathbf{E}_0^{n-1} \end{tikzcd} \tag{3.6}$$

where $k = m/2$ and $\chi = \chi_Q$ is the character (1.3). Ψ is surjective since the right vertical arrow is surjective by Theorem 2.2 . We let Ψ^r be the r-th iteration of Ψ for $r > 0$ and $\Psi^0 =$ the identity map. For $X \in \mathcal{E}_0^{n-r}(q)$, $0 \le r \le n$, let $\Psi^{-r}(X)$ denote any element in $\mathcal{E}_0^n(q)$ whose image under Ψ^r is X.

Let $X = (\Gamma_0^m g \Gamma_0^m) \in \mathcal{L}_0^m$ We define the signature $s(X)$ of X by $s(X) = 2b - m\delta$ where $\delta = \delta(g)$ and $b = \log_p |\det D_g|$. A linear combination of double cosets with the same signature $s \in \mathbf{Z}$ in $\mathcal{L}_0^m$ is said to be s-homogeneous of signature s. For general $X = \sum a_i(\Gamma_0^m g_i) \in \mathcal{L}_0^m$, we denote the s-homogeneous part of signature s in X by $X_{(s)}$, i.e.,

$$X_{(s)} = \sum_{i, 2b_i - m\delta_i = s} a_i(\Gamma_0^m g_i)$$

where $\delta_i = \delta(g_i)$ and $b_i = \log_p |\det D_{g_i}|$. Let $X \in \mathcal{E}_0^m(q)$ and $Y = \beta^m(X) \in \mathbf{L}_0^m$. We define a homomorphism $\xi^m = \xi_Q^m : \mathcal{E}_0^m(q) \to \mathcal{E}_{00}^m$ by

$$\xi^m(X) = \sum_{s \ge 0} (\chi(p)p^{m/2})^s Y_{(-2s)} X_m^{+s} \tag{3.7}$$

where $\chi = \chi_Q$ and

$$X_m^{+s} = p^{-sm} \sum_{\substack{D \in \Lambda^m \backslash M_m(\mathbf{Z})/\Lambda^m \\ \det D = p^s}} (\Gamma_0^m \begin{pmatrix} D^* & 0 \\ 0 & D \end{pmatrix} \Gamma_0^m) \in \mathcal{E}_0^m.$$

THEOREM 3.2. *Let $m \ge n \ge 1$ be integers, m even. Let $Q \in \mathcal{N}_m^+$ with level q and let p be a prime relatively prime to $2q$. Then for $X \in \mathcal{E}_0^n(q)$, we have*

$$\theta^n(Z, Q)|_{k,\chi} X = \theta^n(Z, Q) \circ_\alpha \xi^m(\Psi^{n-m}(X)) \tag{3.8}$$

where $k = m/2$, $\chi = \chi_Q$, and $\alpha = \alpha_{k,\chi} : L_0^m \to \mathbf{C}^\times$ is a character defined by

$$\alpha_{k,\chi}(g) = \chi(p^{\delta m - b}) p^{\delta(mk - <m>) - bk} \tag{3.9}$$

for any $g \in L_0^m$ with $\delta = \delta(g)$ and $b = \log_p |\det D_g|$.

Proof. [**A2**]. (See [**K**] for the analogy when m is odd.)

§4. Generic and Spinor Generic Theta-series

Let $Q \in \mathcal{N}_m^+$. Let $Q_1, \cdots, Q_g$ be the full set of representatives of the classes in the genus $[Q]$ of Q. We define the generic theta-series of degree n associated to Q by

$$\theta^n(Z,[Q]) = \left(\sum_{i=1}^{g} \frac{\theta^n(Z,Q_i)}{e_i}\right)\left(\sum_{i=1}^{g} \frac{1}{e_i}\right)^{-1}, \quad Z \in \mathcal{H}_n \tag{4.1}$$

where e_i is the order of the orthogonal group $O(Q_i)$.

THEOREM 4.1. *Let $m \geq n \geq 1$ be integers with m even. Let $Q \in \mathcal{N}_m^+$. Let q and $\chi = \chi_Q$ be the level and the character of Q, respectively. Let p be a prime relatively prime to $2q$. Then for any $X \in \mathcal{E}_0^n(q)$,*

$$\theta^n(Z,[Q])\big|_{k,\chi} X = \lambda(X,\chi)\theta^n(Z,[Q]) \tag{4.2}$$

where $k = m/2$ and the eigenvalue $\lambda(X,\chi) = \lambda_p(X,\chi)$ is determined as follows : Let $f(x_0, x_1, \cdots, x_n) = (\psi_n \circ \beta^n)(X) \in W_n[\underline{x}]$. Then

$$\lambda(X,\chi) = f(p^{nk-<n>}\chi(p), p^{1-k}\chi(p), \cdots, p^{n-k}\chi(p)). \tag{4.3}$$

Proof. [**A2**]. Note that $\chi(p) = \chi(p)^{-1}$ in this case. (See [**K**] for the analogy when m is odd.)

Let $Q_1, \ldots, Q_h$ be the full set of representatives of the classes in the spinor genus $\{Q\}$ of Q. We define the spinor generic theta-series of degree n associated to Q by

$$\theta(Z,\{Q\}) = \left(\sum_{i=1}^{h} \frac{\theta^n(Z,Q_i)}{e_i}\right)\left(\sum_{i=1}^{h} \frac{1}{e_i}\right)^{-1}, \quad Z \in \mathcal{H}_n. \tag{4.4}$$

Cleary $\theta^n(Z,\{Q\}) \in \Theta_m^n[Q]$. We now prove the following:

THEOREM 4.2. *Let m be even ≥ 4 such that $m \geq n \geq 1$. Let Q, q, χ, p and k be as in the above theorem. Then for any $X \in \mathcal{E}_0^n(q)$,*

$$\theta^n(Z,\{Q\})\big|_{k,\chi} X = \lambda(X,\chi)\theta^n(Z,\{Q\}), \quad Z \in \mathcal{H}_n \tag{4.5}$$

where the eigenvalue $\lambda(X, \chi)$ is (4.3).

Proof. It is well known[S] that $\theta^1(Z, [Q]) = \theta^1(Z, \{Q\})$ if $m \geq 4$. By Theorem 3.2,

$$\theta^n(Z, \{Q\})\big|_{k,\chi} X = \theta^n(Z, \{Q\}) \circ_\alpha \xi^m(\Psi^{n-m}(X))$$

with $\alpha = \alpha_{k,\chi}$ is the character (3.9). Applying Φ^{n-1} both sides, we get

$$\theta^1(Z, \{Q\})\big|_{k,\chi} \Psi^{n-1}(X) = \theta^1(Z, \{Q\}) \circ_\alpha \xi^m(\Psi^{n-m}(X)).$$

So, $\theta^1(Z, \{Q\}) \circ_\alpha \xi^m(\Psi^{n-m}(X)) = \lambda(\Psi^{n-1}(X), \chi)\theta^1(Z, \{Q\})$. Since $\lambda(\Psi^{n-1}(X), \chi) = \lambda(X, \chi)$ and theta operators do not depend on the degree n, the theorem follows.

Theorem 4.2 says that the generic theta-series and the spinor generic theta-series of Q are both simultaneous eigenforms with respect to the Hecke operators in $\mathcal{E}_0^n(q)$ with the same eigenvalues. Furthermore, since Schulze-Pillot's result that $\theta^1(Z, [Q]) = \theta^1(Z, \{Q\})$ does not depend on the parity of m as long as $m \geq 4$, one can use analogous arguments to show that Theorem 4.2 holds for odd m.

We now introduce so called the canonical decomposition of $\mathcal{M}_k^n(q, \chi)$ due to Evdokimov [E]. For a positive integer k, we have

$$\mathcal{M}_k^n(q, \chi) = \bigoplus_{r=0}^{n} \mathcal{M}_k^{n,r}(q, \chi) \tag{4.6}$$

where $\mathcal{M}_k^{n,r}(q, \chi)$ are subspaces defined inductively for $r = n, n-1, \ldots, 0$ as follows: $\mathcal{M}_k^{n,n}(q, \chi)$ is the subspace of cusp forms, i.e., the subspace consisting of $F \in \mathcal{M}_k^n(q, \chi)$ satisfying $\Phi(F\big|_k M) = 0$ for any $M \in \Gamma^n$; for each r, $0 \leq r \leq n-1$, $\mathcal{M}_k^{n,r}(q, \chi)$ is the subspace consisting of F satisfying (i) $\Phi^{n-r}(F\big|_k M)$ is a cusp form for any $M \in \Gamma^n$, (ii) F is orthogonal to $\oplus_{s=r+1}^n \mathcal{M}_k^{n,s}(q, \chi)$ with respect to the canonical inner product induced from Maass-Petersson inner product. $M_k^{n,r}(q, \chi)$ are mutually orthogonal and are invariant under the Hecke operators in $\mathcal{L}_0^n(q)$.

§5. The Main Theorem

THEOREM 5.1. *Let $Q \in \mathcal{N}_m^+$, m even $\geq 2n+3$. Let n, k, q, χ be as above. Then*

$$\theta^n(Z,[Q]),\ \theta^n(Z,\{Q\}) \in \mathcal{M}_k^{n,0}(q,\chi) \tag{5.1}$$

Proof. Let $T^n = \left(\Gamma_0^n(q)\begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix}\Gamma_0^n(q)\right)^2 \in \mathcal{E}_0^n(q)$, $\gcd(p,2q) = 1$. Then $(\psi_n \circ \beta^n)(T^n) = x_0^2(1+x_1)^2\cdots(1+x_n)^2$. Let $F(Z)$ be either $\theta^n(Z,[Q])$ or $\theta^n(Z,\{Q\})$. From (4.2) and (4.5) follows

$$F(Z)\big|_{k,\chi}T^n = \lambda(T^n,\chi)F(Z),\quad Z \in \mathcal{H}_n$$

with the eigenvalue

$$\lambda(T^n,\chi) = \prod_{s=1}^{n}\left(p^{k-s} + \chi(p)\right)^2. \tag{5.2}$$

We decompose $F(Z) = F_0(Z) + F_1(Z) + \cdots + F_n(Z)$ with $F_r(Z) \in \mathcal{M}_k^{n,r}(q,\chi)$, $r = 0, 1, \ldots, n$. Then

$$F_r(Z)\big|_{k,\chi}T^n = \lambda(T^n,\chi)F_r(Z) \tag{5.3}$$

because of the invariance of $\mathcal{M}_k^{n,r}(q,\chi)$ under Hecke operators in $\mathcal{L}_0^n(q)$. On the other hand,

$$\Phi^{n-r}(F_r(Z)\big|_{k,\chi}T^n) = \Phi^{n-r}(F_r(Z))\big|_{k,\chi}\Psi^{n-r}(T^n)$$

for each $r = 0, 1, \ldots, n$ (see (2.3), (3.6), and Theorem 2.1). Since $\Psi^{n-r}(T^n) = \lambda_{n-r}T^r$ where

$$\lambda_{n-r} = \begin{cases} \prod_{s=r+1}^{n}\left(p^{k-s} + \chi(p)\right)^2 & \text{if } r < n \\ 1 & \text{if } r = n, \end{cases} \tag{5.4}$$

we have

$$\lambda(T^n,\chi)\Phi^{n-r}(F_r(Z)) = \lambda_{n-r}(\Phi^{n-r}(F_r(Z))\big|_{k,\chi}T^r). \tag{5.5}$$

From the definition of $\mathcal{M}_k^{n,r}(q,\chi)$ follows that $\Phi^{n-r}(F_r(Z))$ is a cusp form in $\mathcal{M}_k^r(q,\chi)$. It is well known [R] that for any cusp form $G(Z') \in \mathcal{M}_k^r(q,\chi)$, there exists a constant C_G independent of $Z' \in \mathcal{H}_r$ such that

$$\left|G(Z')\right| \leq C_G \det(\mathrm{Im} Z')^{-k/2}. \tag{5.6}$$

We now assume in addition that $G(Z')$ is an eigenform with respect to $T^r \in \mathcal{E}_0^r(q)$ with the eigenvalue λ_G. We may write $\beta^r(T^r) = \sum a_i(\Gamma_0^r g_i) \in \mathcal{E}_0^r$ with $a_i > 0$ and $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in E_0^r$. Then from (2.2), (2.3), and (5.6) follows

$$\begin{aligned}
\left|G(Z')\big|_{k,\chi} T^r\right| &\leq \sum a_i \left|\chi(\det p^{\delta_i} D_i^*) G(Z')\big|_k g_i\right| \\
&= \sum a_i p^{\delta_i(rk-<r>)} \left|\det D_i^{-k} G(p^{\delta_i} D_i^* Z' D_i^{-1} + B_i D_i^{-1})\right| \\
&\leq C_G \left(\sum a_i p^{\delta_i(rk/2-<r>)}\right) \det(\mathrm{Im} Z')^{-k/2}.
\end{aligned}$$

But $\sum a_i p^{\delta_i(rk/2-<r>)}$ is precisely the value of $(\psi^r \circ \beta^r)(T^r) \in W_n[\underline{x}]$ evaluated at $x_0 = p^{rk/2-<r>}$, $x_1 = p_1, \ldots, x_r = p^r$. So,

$$\left|\lambda_G G(Z')\right| \leq C_G p^{rk-2<r>} (1+p)^2 \cdots (1+p^r)^2 \det(\mathrm{Im} Z')^{-k/2},$$

and by taking $Z' = iI_r$, we obtain

$$\left|\lambda_G\right| \leq C'_G p^{rk-2<r>} (1+p)^2 \cdots (1+p^r)^2. \tag{5.7}$$

Let $G(Z') = \Phi^{n-r}(F_r(Z))$. Then from (5.5) and (5.7) follows

$$\left|\lambda(T^n,\chi)\right| = O(p^{n(2k-n-1)-r(k-r-1)}). \tag{5.8}$$

Since $m \geq 2n+3$ implies $k-r-1 > 0$ for any $r = 0, 1, \cdots, n$, (5.8) contradicts to (5.2) unless $r = 0$. Therefore $F_r(Z) = 0$ for $r = 1, 2, \ldots, n$ and this proves the theorem.

COROLLARY 5.2. *Let $m \geq 2n+3$, m even. Then for any $Q \in \mathcal{N}_m^+$,*

$$\theta^n(Z,[Q]) = \theta^n(Z,\{Q\}). \tag{5.9}$$

Proof. Let $Q_1, \ldots, Q_h, Q_{h+1}, \ldots, Q_g$ be the full set of representatives of classes in $[Q]$ while $Q_1, \ldots, Q_h$ is that in $\{Q\}$. It is known [Ki] that the constant term of $\theta^n(Z,Q_i) - \theta^n(Z,Q_j)$ vanishes at every cusp for each pair i,j. In other words,

$$\theta^n(Z,Q_i) - \theta^n(Z,Q_j) \in \bigoplus_{r=1}^{n} \mathcal{M}_k^{n,r}(q,\chi).$$

Fix j and take a weighted average over $i = 1, \ldots, g$ as in (4.1) and then take a weighted average over $j = 1, \ldots, h$ as in (4.4). Then we have

$$\theta^n(Z,[Q]) - \theta^n(Z,\{Q\}) \in \bigoplus_{r=1}^{n} \mathcal{M}_k^{n,r}(q,\chi).$$

Hence by Theorem 5.1, $\theta^n(Z,[Q]) - \theta^n(Z,\{Q\}) = 0$.

COROLLARY 5.3. (HSIA). *Let $Q \in \mathcal{N}_m^+$ and $N \in \mathcal{N}_n$ with $2m \geq n+3$. Then $r(N,Q_i) > 0$ for some $Q_i \in [Q]$ if and only if $r(N,Q_j) > 0$ for some $Q_j \in \{Q\}$.*

Proof. Clear from Corollary 5.2. Hsia proves this for $m \geq n+3$ [**H**]. Note that our dimensional restriction $2m \geq n+3$ can be improved with a better estimation of $|\lambda_G|$ (see (5.7)).

REMARK 5.4. The canonical decomposition (4.6) of $\mathcal{M}_k^n(q,\chi)$ for half integer k is not yet given in literature. If we assume this, however, Theorem 5.1 and Corollary 5.2 can be extended for odd m immediately because all the necessary analogies are established in [**K**], [**Sh**], [**Zh1,2**], etc., for odd m or half integral k.

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