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HAMILTONIAN PATHS IN INFINITE STRONG TRIANGULATIONS

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1. Introduction

We shall, for the most part, use the terminology of [2]. Graphs will be finite or infinite, but have no loops or multiple edges. For a vertex v of G, denote by $N_G(v)$ the set of vertices adjacent to v in G, and by $d_G(v)$ the cardinal number of $N_G(v)$. An *x,y-path* is a path joining vertices x and y in G, and in this case x and y are called the *endvertices* of the path. A path P is *one-side infinite* if it contains infinitely many vertices and $d_P(x) = 1$, for only one vertex x in P. In this case the vertex x is said to be the *endvertex* of P.

Let G be a plane graph and let C be a cycle in G. We denote by $\bar{C}$ the subgraph of G consisted of the vertices and the edges lying on C and lying in the interior of C. A plane graph H is a *circuit graph*, following D.Barnette, if there exists a cycle C in a 3-connected plane graph such that $H = \bar{C}$. A circuit graph H is *triangulated* if all facial cycles of H, up to the outer cycle, are triangles.

A *triangulation* G is a countable locally finite plane graph, of which edges are contained in two non-separating triangles. If a representation of the graph G contains no vertex- or edge-accumulation points, then G is called a *strong triangulation.*

Whitney [8] proved every finite 4-connected maximal planar graph has a Hamiltonian cycle, and Tutte [7] and Thomassen [6] extended his result to all 4-connected planar graphs. In particular, Thomassen [6] showed that every 4-connected planar graph is Hamiltonian-connected, i.e., it has a Hamiltonian path connecting any two prescribed vertices. On the other hand, Dillencourt [3] observed the condition for internally maximal planar graphs to have a Hamiltonian cycle, and so he proved that every triangulated circuit graph without separating triangles, which contains at most three chordal edges, is Hamiltonian.

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A simplified proof of Whitney's theorem and a linear algorithm for finding a Hamiltonian cycle in such a graph, can also be found in [1].

Nash-Williams ([4], see also in [5]) conjectured that this theorem is also true for all infinite 4-connected planar graphs, i.e., every infinite 4-connected planar graph has a one-side infinite Hamiltonian path.

In this paper Whitney's theorem will be extended to the infinite strong triangulations under the corresponding hypothesis, which is a part of Nash-williams' conjecture.

Namely, we prove the following theorem.

THEOREM. *Let G be a 4-connected infinite strong triangulation. Then there exists a one-side infinite Hamiltonian path in G originating from any prescribed vertex.*

For the proof, important tools are the structure theorem (in section 2), Whitney's theorem (in section 3) and the so-called *Kőnig's Unendlichkeitslemma*, which is stated below:

LEMMA (KÖNIG). *Let $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \dots\}$ be an infinite sequence of disjoint non-empty finite sets and $\mathcal{R}$ be a relation in $\mathcal{P} := \bigcup_{j=0}^{\infty} \mathcal{P}_j$, such that*

$$\forall j \in \mathrm{N}, \forall P' \in \mathcal{P}_{j+1}, \exists P \in \mathcal{P}_j \text{ such that } (P, P') \in \mathcal{R}.$$

Then there exists an infinite sequence of paths $\{P_1, P_2, P_3, \dots\}$ such that $P_j \in \mathcal{P}_j$ and $(P_j, P_{j+1}) \in \mathcal{R}$.

To investigate the structure of an infinite strong triangulation, we in addition have to define several important conditions.

Let C and C' be two disjoint cycles in an infinite strong triangulation G, where C lies in the interior of C'. A *(C, C')-ring* is a subgraph of G, which consists of not only C and C' but also the vertices and edges lying between C and C'. For a (C, C')-ring R, a *bridge of R* is either an edge of R joining C and C' (such a bridge is called a *chordal edge*, following Dillencourt [3]), or it is a connected component of $R-(C \cup C')$ together with all edges of R joining this component to $C \cup C'$. A (C, C')-ring R is *normal* if it satisfies the following properties:

(1) C and C' are induced cycles.
(2) $|V(B) \cap V(C')| \leq 2$, for any bridge B of R.

(3) If $V(B) \cap V(C') = \{z, z'\}$, $z \neq z'$, for a bridge B, it must hold $zz' \in E(G)$.

2. Structure of infinite strong triangulations

LEMMA 1. *Let C be an induced cycle of an infinite strong triangulation G. Then there exists a cycle C' such that the (C, C')-ring is normal.*

Proof. First, we construct a cycle C' in G satisfying the hypothesis of this lemma.

Let $F := \{J | J$ is a facial cycle in G such that $V(J) \cap V(C) \neq \emptyset\}$ and let E be the set of all vertices of the cycles in F. Then we can see that $|E| < \infty$, since E contains only finite cycles and F is also finite. Furthermore, set $H := G[E]$, i.e. H is the induced subgraph of G containing all elements of E, and let C' be its outer cycle of H. We will now show the (C, C')-ring R is normal.

As an induced subgraph H of G, C' is an induced cycle. The assertion (3) is also obvious from the assumption. To show that C and C' are disjoint, we assume: $\exists x \in V(C) \cap V(C')$. Let y be a vertex on C' adjacent to x. Then, from the fact that all facial cycles in G are triangles, we can find a facial cycle $J = \{x, y, z\}$ such that $yz \notin E(C')$. But since the cycle must be contained in F (since $V(J) \cap V(C) \neq \emptyset$), it follows that $y, z \in E$. Hence we have $yz \in E(H)$, which contradicts our construction of C'.

It remains to be shown that $|V(B) \cap V(C')| \leq 2$ for every bridge B of R. Suppose there exists a bridge B such that $V(B) \cap V(C') = \{y_1, \ldots, y_r\}$, $r \geq 3$. Since B is not a chordal edge and $V(B) \backslash V(C \cup C') \neq \emptyset$, it follows that there exists a y_1, y_r-path P in $B - (C \cup \{y_2, \ldots, y_{r-1}\})$. Thus the facial cycle in R containing the edge $y_k y_{k+1}$ $(k = 1, \ldots, r-1)$ is not contained in F, and therefore it holds that $y_k \notin E$, $k = 2, \ldots, r-1$, which also contradicts our construction of C'. □

REMARK. We can prove that such a cycle C' is unique for a given induced cycle C.

LEMMA 2. *For any cycle C of an infinite strong triangulation, the induced subgraph $\bar{C}$ is a triangulated circuit graph.*

Proof. As every strong triangulation has a vertex-accumulation po int free representation, $\bar{C}$ is a finite subgraph, and hence it is a circuit graph. It is also obvious that $\bar{C}$ is triangulated. □

PROPOSITION 3. *Let G be a 4-connected strong triangulation. Let x_0 be a vertex of G and let C_0 be the cycle of G consisting of the vertices adjacent to x_0. Then there exists a sequence of induced cycles $\{C_0, C_1, C_2, \dots, \}$ which holds the following properties:*

(1) *The (C_{j-1}, C_j)-ring is normal for all $j \in \mathrm{N}$.*
(2) $V(G) = V(\bigcup_{j=0}^{\infty} \bar{C}_j)$.

Proof. It is clear that C_0 is an induced cycle by the fact of 4-connectedness of G. For $j \in \mathrm{N}$ the existence of C_j, related to C_{j-1}, satisfying the condition (1) follows from lemma 1. It remains only to show that the resulting cycles $\{C_0, C_1, C_2, \dots\}$ hold the condition (2).

Let $x \in V(G)$ be an arbitrary vertex. Since C_{j-1} lies in the interior of C_j ($j \in \mathrm{N}$), it follows that $x \in V(\bar{C}_{n_x})$, where n_x is a metric distance between x and x_0. Because of $V(\bar{C}_{n_x}) \subset V(\bigcup_{j=0}^{\infty} \bar{C}_j)$, we have $V(G) \subseteq V(\bigcup_{j=0}^{\infty} \bar{C}_j)$. Since it holds clearly that $V(G) \supseteq V(\bigcup_{j=0}^{\infty} \bar{C}_j)$, we can conclude $V(G) = V(\bigcup_{j=0}^{\infty} \bar{C}_j)$. □

REMARK. We can also prove that, for an arbitrary given vertex x_0, such a sequence of induced cycles with the condition (1)–(2) is unique.

Let C be an induced cycle in an infinite strong triangulation G. According to lemma 2 we can construct a cycle C' in G such that (C, C')-ring R is normal. We let F be the set of all chordal edges of R and let $BG(R) := (C \cup C') \cup F$. Then we have exactly $|F|$ facial cycles in $BG(R)$, up to the interior of C and the exterior of C'. For a facial cycle J of $BG(R)$ the induced subgraph $\bar{J}$ of G is called a *chamber of* R. If $J = \bar{J}$, then the chamber J is *empty*. Clearly in the interior of a chamber lies at most one bridge of R since G is maximal planar.

Now we let G be 4-connected and let L be a nonempty chamber of R. Because of the conditions (2) and (3) in definition of normality, L must be one of following two types:

(*i*) $|V(L) \cap V(C')| = 1$,
(*ii*) $|V(L) \cap V(C')| = 2$.

In the former case we say that L *is of type 1* and in the latter case that L *is of type 2*.

3. Whitney's theorem and its extensions

The following notations are useful for the concept and proof of Whitney's lemma and its corollaries.

A path P on C is *wh-induced* if there exists no edge $xy \in E(H)\setminus E(C)$, $x, y \in V(P)$. For distinct vertices u, v (resp. u, v, w) on C, we say that *(H,u,v) (resp. (H,u,v,w)) satisfies condition W1 (resp. W2)* if the two u, v-paths (resp. the $u, v-$, $v, w-$ and w, u-path) on C are wh-induced. Note that according to our definition (H, u, v) satisfies $W1$ if and only if (H, u, v, w) satisfies $W2$ for every vertex w on C.

LEMMA 4 (H.WHITNEY). *Let H be a triangulated circuit graph without separating triangles and let C be its outer cycle. Finally let u and v be two distinct vertices on C. If (H, u, v) satisfies the condition $W1$ or if (H, u, v, w) satisfies $W2$ for some vertex w on C, then H contains a Hamiltonian u, v-path.*

Proof. See in [8]. □

LEMMA 5. *Let H be a 3-connected triangulated circuit graph without separating triangles and let C be its outer cycle with $|V(C)| \geq 4$.*

(1) *Let $y \in V(C)$, and let u, v be the vertices adjacent to y on C. Then there exists a Hamiltonian u, v-path in $H - y$.*
(2) *Let $yy' \in E(C)$ and let u (resp. v) be the vertex adjacent to y (resp. y') on C such that $u \neq y'$ and $v \neq y$. Then there exists a Hamiltonian u, v-path in $H - \{y, y'\}$.*

Proof. (1) Set $H' := H - y$. Then H' clearly is a triangulated circuit graph, because it is 2-connected. Let J be the outer cycle of H' and let J_1 and J_2 be the u, v-paths on J with $J_1 = C - y$. Then the vertices of J_2 are identical to the vertices adjacent to y in H since H is triangulated. Note that J_1 is wh-induced. We will show J_2 also is wh-induced.

Suppose that there is an edge xx' in $E(H)\setminus E(J_2)$ contained in the interior of J. Then the vertices $\{x, x', y\}$ separate H in two components, and hence H contains a separating triangle since $xx', x'y \in E(H)$. So we have a contradiction to the hypothesis of this lemma.

Therefore (H', u, v) satisfies $W1$, and so $H' = H - y$ contains a Hamiltonian u, v-path, by the Whitney's lemma.

(2) From $yy' \in V(C)$ and $|V(C)| \geq 4$, $H' := H - \{y, y'\}$ is 2-connected, and from this it is a triangulated circuit graph. Let J be the outer cycle of H', and let J_1 and J_2 be the u,v-paths on J such that $J_1 = C - \{y, y'\}$. Then we have $V(J_2) = N_G(\{y, y'\})$. We first note that J_1 is wh-induced. Let us consider the path J_2.

Since H' is triangulated we can easily verify that there exists a u,v-path J_2' such that

(i) $V(J_2') \subseteq V(J_2)$,

(ii) J_2' is induced path if $|V(C)| \geq 5$,
and $J_2' \cup \{uv\}$ is induced cycle if $|V(C)| = 4$.

If $V(J_2') = V(J_2)$, then J_2 is wh-induced, and so (H', u, v) satisfies $W1$.

Now assume that $V(J_2') \subset V(J_2)$. From the fact H contains no separating triangles, it is easy to see that there exists only one edge $e \in E(H)$ such that $e \in E(J_2')\backslash E(J_2)$. Let w be the vertex of J_2 such that $\{y, y', w\}$ constitutes a facial cycle of H. Then, as in the proof of (1), it can be verified that the $u,w-$ and v,w-path on J_2 are wh-induced. Therefore (H', u, v, w) satisfies $W2$, and hence, in both cases, we can find a Hamiltonian u,v-path in $H' = H - \{y, y'\}$ by Whitney's lemma. □

LEMMA 6. *Let H be a triangulated circuit graph without separating triangles and let C be its outer cycle. Let $u, v \in V(C), u \neq v$, and $e \in E(C)$ arbitrary (but $e \neq uv$ if $uv \in E(C)$). If (H, u, v) satisfies the condition $W1$, then H has a Hamiltonian u,v-path which contains the edge e.*

Proof. Let $e := xy \in E(C)$ and let w be a further vertex not in H. We construct a graph $\tilde{H}$ as follows:

$$V(\tilde{H}) := V(H) \cup \{w\} \quad E(\tilde{H}) := E(H) \cup \{xw, yw\}.$$

Then $\tilde{H}$ again is triangulated and (H, u, v, w) further satisfies $W2$, and hence there exists a Hamiltonian u,v-path $\tilde{P}$ in $\tilde{H}$ by Whitney's lemma. Let $V(P) := V(\tilde{P})\backslash\{w\}$ and $E(P) := E(\tilde{P}) \cup \{xy\}\backslash\{xw, yw\}$. Since $\tilde{P}$ must contain the edge xw and yw, the u,v-path P is Hamiltonian in H containing the edge $e = xy$. □

4. Proof of the main theorem

Let R be a normal (C, C')-ring in a 4-connected infinite strong triangulation G. We choose an arbitrary vertex y_0 in $V(C')$. Let $\bar{y}$ be the first vertex adjacent to y_0 on C', counterclockwise, and set $M := N_G(\bar{y}) \cap V(C)$. We note that M is non-empty since G is maximal planar. Let x_1 be the first vertex in M, also counterclockwise, and $\{x_1, \ldots, x_k\} \subseteq V(C) \cap N_G(C')$. (i.e. for every $i \in \{1, \ldots, m\}$, there exists a vertex $y \in V(C')$ with $x_i y \in E(G)$, and conversely). Then for every $i \in \{1, \ldots, m\}$ and for each pair $\{x_i, x_{i+1}\}$ we can find exactly one chamber L_i such that $x_i, x_{i+1} \in V(L_i)$. Let x_0 be the vertex adjacent to x_m on C lying between x_m and x_0. (If $x_m x_1 \in E(C)$ we let $x_0 = x_1$). We will prove there exists a Hamiltonian x_0, y_0-path in $R - (V(C') \setminus \{y_0\})$.

(1) The chamber $L_i (i = 1, \ldots, m-1)$.

Case 1: L_i is of type 1:

Let $y \in V(L_i) \cap V(C')$. If L_i is empty we let $P_i := \{x_i, x_{i+1}\}$. Otherwise L_i clearly is a 3-connected triangulated circuit graph without separating triangles. Since x_i and x_{i+1} are adjacent to y on the outer cycle of L_i we can find a Hamiltonian x_i, x_{i+1}-path P_i in $L_i - y$ by lemma 5 (1).

Case 2: L_i is of type 2:

Let $y, y' \in V(L_i) \cap V(C')$. Since R is normal, yy' must be an edge of L_i. Analogously it can be verified that L_i satisfies the hypothesis of (2) in lemma 5. Therefore we can also find a Hamiltonian x_i, x_{i+1}-path P_i in $L_i - \{y, y'\}$.

(2) The chamber L_m.

Let J be the outer cycle of L_m. Because of the choice of x_1, it is clear that $y_0 x_m, y x_1 \in E(G)$. We will construct a Hamiltonian x_1, y_0-path $\bar{P}$ in L_m (resp. $L_m - \bar{y}$) containing the edge $x_m x_0$ if L_m is of type 1 (resp. type 2), where the vertex x_0 is defined at the beginning of this section.

Case 1: L_m is of type 1:

Let $V(L_m) \cap V(C') =: \{y_0\}$. If L_m is empty, then we let $\bar{P} = L_m - \{x_1 y_0\}$. Otherwise L_m is 3-connected and $|V(J)| \geq 4$. Because $x_1 y_0 \neq x_m x_0$, L_m satisfies the hypothesis of lemma 5 (corresponding to the vertices x_1, y_0 and the edge $x_m x_0$). Therefore there exists a Hamiltonian x_1, y_0-path $\bar{P}$ in L_m containing the edge $x_m x_0$.

Case 2: L_m is of type 2:

Let $V(L_m) \cap V(C') =: \{y_0, \bar{y}\}$. In this case L_m is not empty, so it is 3-connected. As in the proof of lemma 5, $(L_m - \bar{y}, x_1, y_0)$ satisfies the condition $W1$, so, by Whitney's lemma, there exists a Hamiltonian x_1, y_0-path $\bar{P}$ in $L_m - \bar{y}$ containing the edge $x_m x_0$

In each case we let P_0 be the x_0, x_1-path of $\bar{P}$ and P_m be the x_1, x_m-path of $\bar{P}$. Then $V(P_0) \cup V(P_m) = V(\bar{P})$ and $E(P_0) \cup E(P_m) = E(\bar{P}) \backslash \{x_m x_0\}$ since $\bar{P}$ contains the edge $x_m x_0$.

Now we summerize all chambers $L_1, \ldots, L_{m-1}, L_m$. For a given normal (C, C')-ring R in G and for an arbitrary given vertex y_0 on C', the chambers $L_1, \ldots, L_m$ are fixed. From (1) and (2) we have $m+1$ paths $P_0, P_1, \ldots, P_m$ in R, such that:

(1) for $i = 0, \ldots, m-1$ the endvertices of P_i are x_i and x_{i+1}, and those of P_m are x_m and y_0.
(2) $V(P_i) = V(L_i - C')$ for $i = 0, \ldots, m-1$, and $V(P_0) \cup V(P_m) = V(L_m)$.

Let $P := \bigcup_{i=0}^{m} P_i$. Then P is a x_0, y_0-path in R which covers all vertices of $(R - C') \cup \{y_0\}$. Thus we have:

PROPOSITION 7. *Let R be a normal (C, C')-ring in a 4-connected infinite strong triangulation and let x_0, y_0 be the vertices defined at the beginning of this section. Then there exists a x_0, y_0-path in R which covers all vertices of $(R - C') \cup \{y_0\}$.* □

We can now prove the main theorem of this paper with the aid of König's lemma.

Let G be a 4-connected infinite strong triangulation and x_0 an arbitrary given vertex of G. We let C_0 be the induced cycle of G consisting of the vertices adjacent to x_0. Then, by proposition 3, we have a sequence of induced cycles $\{C_0, C_1, C_2, \ldots\}$ satisfying the same conditions (1)–(2) in the proposition.

For $j \in \mathrm{N}$, let R_j be the (C_{j-1}, C_j)-ring and let $\mathcal{P}_j$ be the set of all paths in R_j such that:

$$P \in \mathcal{P}_j \qquad \text{if and only if} \quad \begin{cases} i)\ P \text{ is a } x, y\text{-path in } R_j \\ \quad \text{with } x \in V(C_{j-1}) \text{ and } y \in V(C_j), \\ ii)\ E(P) = E(R_j - C_j) \cup \{y\}. \end{cases}$$

We will further define a relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$, where $\mathcal{P} := \bigcup_{j=1}^{\infty} \mathcal{P}_j$:

$$(P, P') \in \mathcal{R} \quad \text{if and only if} \quad \begin{cases} i)\ \exists j \in \mathrm{N};\ P \in \mathcal{P}_j \text{ and } P' \in \mathcal{P}_{j+1}, \\ ii)\ \text{P and P' have a common endvertex.} \end{cases}$$

We will show that the relation $\mathcal{R}$ holds the hypothesis of König's lemma. Clearly we have $\mathcal{P}_j \neq \emptyset$ and $|\mathcal{P}_j| < \infty$ for all $j \in \mathrm{N}$. For any $j \in \mathrm{N}$, let $P' \in \mathcal{P}_{j+1}$ be an arbitrary element. Then, by the definition of P', one of its endvertices of P', say x', is contained in C_j and the another in C_{j+1}. By proposition 7, we can find a x, x'-path P in R_j with $x \in V(C_{j-1})$ and $V(P) = V(R_j - C_j) \cup \{x'\}$, and from this we have $P \in \mathcal{P}_j$ and $(P, P') \in \mathcal{R}$. Thus, by König's lemma, there exists an infinite sequence of paths $\{P_1, P_2, \dots\}$ such that $P_j \in \mathcal{P}_j$ and $(P_j, P_{j+1}) \in \mathcal{R}$ for all $j \in \mathrm{N}$. We now let x_1 be the endvertex of P_1 lying on C_0 and set $P_0 := \{x_0 x_1\}$. Then $P := \bigcup_{j=0}^{\infty} P_j$ clearly is a one-side infinite path in G. Because of $V(G) = \{x_0\} \cup V(\bigcup_{j=1}^{\infty} P_j) = V(\bigcup_{j=0}^{\infty} \bar{C}_j)$ by proposition 3, P is a Hamiltonian path in G originating from the endvertex x_0, and this completes the proof of the main theorem.

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