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INDICES OF IRREDUCIBLE BOOLEAN MATRICES

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1.Introduction

Let $\beta = \{0, 1\}$ be the Boolean algebra of order two with operations $(+, \cdot)$: $1+0 = 0+1 = 1+1 = 1 \cdot 1 = 1$ & $0+0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$ and an order : $0 < 1$. Then under these Boolean operations, the set B_n of all $n \times n$ matrices over β (Boolean matrices) forms a multiplicative matrix semigroup. There have been many researches on various semigroup properties of B_n. In this paper we study the indices of irreducible Boolean matrices in B_n.

DEFINITION 1.1. Let A be an $n \times n$ Boolean matrix in B_n. The index of A and the period of A are the least positive integers q and p respectively such that $A^q = A^{q+p}$. For $A \in B_n$, $index(A)$ and $period(A)$ denote the index and the period of A respectively, and $\sigma(A)$ denotes the number of one's of A. $A \in B_n$ is called a $J-$matrix if $\sigma(A)$ is n^2, and A is primitive if A^q is a $J-$matrix for some integer q.

For an $n \times n$ Boolean matrix $A = [a_{ij}]$, the associated digraph of A, denoted by $G(A)$, is the digraph with vertices $\{1, 2, ..., n\}$ such that there is an arc from i to j if and only if $a_{ij} > 0$. A path from i to j of length l in $G(A)$ is a sequence of vertices $(v_0 = i, v_1, \ldots, v_l = j)$ such that $a_{v_k v_{k+1}} = 1$ for each $k \in \{0, 1, \ldots, l-1\}$. A path is a simple path if $v_1, \ldots, v_l$ are all distinct, and a simple path is a cycle if $v_0 = v_l$. Then we can interpretes many properties of $A \in B_n$ in terms of its digraph $G(A)$. For example, a Boolean matrix $A \in B_n$ is primitive if and only if its associated digraph $G(A)$ is strongly connected (i.e. for any vertices i and j in $G(A)$ there is a path from i to j).

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DEFINITION 1.2. Let $A \in B_n$ be an $n \times n$ Boolean matrix. Then A is irreucible if A is not permutationally similar to a matrix of the form $\begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix}$, where B_i's are square matrices. A is reducible if A is not irreducible. A is nearly reducible if deleting any positive entry of A results in a reducible matrix. If A is an irreducible matrix of period p, then A is called a p–irreducible matrix. $A \in B_n$ is called a cyclically p–partite matrix if A is permutationally similar to the following matrix

$$\Pi = \begin{pmatrix} D_1 & B_1 & & & 0 \\ & D_2 & B_2 & & \\ 0 & & D_3 & \cdot & \\ & & & \cdot & B_{p-1} \\ B_p & & 0 & & D_p \end{pmatrix},$$

where the block matrices D_k's on the main diagonal of Π are square zero matrices. For each k and m, we let $\pi_k(m) = B_k \cdot B_{k+1} \cdots B_{k+m-1}$ and $\pi_k = \pi_k(p)$, where B_{k+i} represents B_j of Π if $k + i \equiv j (mod\ p)$.

LEMMA 1.3. *Consider the matrix Π in the Definition 1.2. Then,*

(1) *period(π_i) = period(π_j).*
(2) *| index(π_i) - index(π_j) | ≤ 1.*
(3) *If Π is irreducible, then index(Π) is the smallest integer q such that $\pi_k(q)$ is a J–matrix for any k.*

Proof. Refer to Cho [3].

2. Frobenius Numbers and Circum-diameters

For the semigroup R_n of $n \times n$ real matrices, we say $M \in R_n$ is power convergent in R_n if the powers $M, M^2, M^3, \cdots, M^q, \cdots$ form a convergent sequence in R_n. It is well known that the power convergence of M is closely related to the set of eigenvalues of M. For the semigroup B_n of Boolean matrices, any p–irreducible matrix $A \in B_n$ has a finite index in B_n, and the circumferences of A is closely related to the index of A as follows.

DEFINITION 2.1. Let $C = \{c_1, c_2, \cdots, c_\lambda\}$ be a finite set of relatively prime positive integers. The Frobenius number $\varphi(C)$ of C is the smallest integer q such that any integer $h(\geq q)$ can be expressed as a nonnegative linear combination of c_i's (i.e. $h = \sum_{i=1}^{\lambda} a_i c_i$, where a_i's are nonnegative integers). In general, for a finite set of positive integers $C = \{c_1, c_2, \cdots, c_\lambda\}$ with the greatest common divisor $\gcd(C) = p$, the Frobenius number $\varphi(C)$ of C is $p \cdot \varphi(d_1, d_2, \cdots, d_\lambda)$, where $d_i = \frac{c_i}{p}$.

Let $A \in B_n$ be an $n \times n$ Boolean matrix, and let $G(A)$ be its associated digraph with vertices $\{1, 2, ..., n\}$. If there is a $c-$cycle (cycle of length c), then such integer c is called a circumference of A (and of $G(A)$). ΓA and $\lambda(A)$ denote respectively the set of all the circumferences of A and the cardinality of ΓA. Now let A be irreducible and C be a subset of ΓA. For any vertices s and t in $G(A)$, $P(s,t) = \{\tau | \tau$ is a path from s to $t\}$ and $Q(C, s, t) = \{\tau | \tau$ is a circumpath from s to t w.r.t. $C\}$ (i.e. τ is a path from s to t such that τ meets with a $p-$cycle for each $p \in C$). Then the distance $d(s,t)$ from s to t is the minimum of $\{$length of $\tau | \tau \in P(s,t)\}$, and the circum-distance $\delta(C, s, t)$ from s to t w.r.t. C is the minimum of the set $\{$length of $\tau | \tau \in Q(C, s, t)\}$. Finally the circum-diameter $\delta_A(C)$ of A w.r.t. C is the maximum of $\{\delta(C, s, t) | s, t \in G(A)\}$, and $\varphi_A(C)$ denotes $\varphi(C)$ if C is a subset of ΓA.

LEMMA 2.2. *Let $A \in B_n$ be a $p-$irreducible matrix. Then index(A) $\leq \varphi_A(C) + \delta_A(C)$ for any subset C of ΓA with* $\gcd(C) = p$.

Proof. Refer to Cho [3].

Consider the 4 by 4 Boolean matrix $W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. Then the characteristic polynomial of W (as a real matrix) is $\lambda^4 - \lambda - 1$, and W is power divergent in R_4 since there exists an eigenvalue whose absolute value is greater than 1. But W (as a Boolean matrix) is power convergent in B_n, and $index(A) = \varphi_W(C) + \delta_W(C)$. Now consider the primitive matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\Gamma A = \{4,5,7\}, \varphi_A(4,5,7) = 7, \delta_A(4,5,7) = 7$. But $index(A) = 12 < 14 = \varphi_A(4,5,7) + \delta_A(4,5,7)$. Thus there exists a Boolean matrix A such that $index(A) < \varphi_A(C) + \delta_A(C)$. Note that if p and q are relatively prime integers, then $\varphi(p,q) = (p-1)(q-1)$.

3. Gaps in the Index Set E_n^p

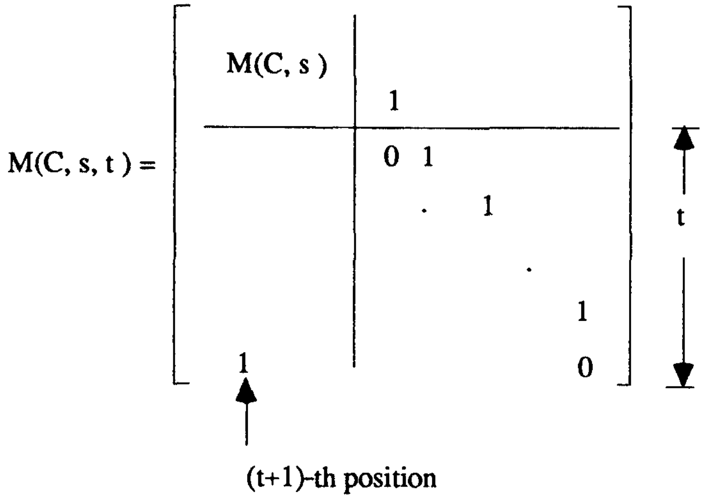
For the given positive integers n and $p(p \le n)$, let E_n^p be the index set $\{index(A) | A \in B_n \text{ is } p\text{–irreducible}\}$ and mE_n^p be the maximum element of E_n^p. Also let G_n^p be the set $\{g | g \text{ is a positive integer less than } mE_n^p \text{ and } g \notin E_n^p\}$. Then any integer in G_n^p is called a gap of E_n^p. Shao [11] and Min [9] proved that there is no gap less than $\omega_n + 1$ in the index set $E_n^1 (n \neq 11)$, where $\omega_n = \frac{n^2-2n+2}{2}$. Moreover Lewin and Vitek [7] specified the gaps greater than $\omega_n + 1$ in E_n^1. In this section we investigate the gaps in the index set E_n^p of $n \times n$ p–irreducible matrices using their results and the index properties of cyclically p–partite matrices.

DEFINITION 3.1. $E_n(i,j)$ denotes an $n \times n$ Boolean matrix whose (i,j)–entry is the only nonzero entry. Each $n \times n$ permutation matrix P can be expressed as a Boolean sum $\sum_{i=1}^{n} E_n(i, \sigma(i))$, where σ is an element of the symmetric group S_n representing P. Let $n = p\alpha + \beta$ with $\alpha = [\frac{n}{p}]$ for some positive integers $p(\le n)$, and let M_n^p denote the set $\{A \in B_n | A \text{ is } p\text{–irreducible and } index(A) > p(\omega_\alpha + 1) + \beta\}$.

LEMMA 3.2. *Let $C = \{c_1, c_2, \cdots, c_\lambda\}$ be a finite set of positive integers with $2 \le c_1 < c_2 < \cdots < c_\lambda \le n$ and $gcd(C) = p$. If $c_\lambda + c_{\lambda-1} \ge n$, then $E_n^p(C) \subseteq E_n^p$, where $E_n^p(C) = \{\varphi(C) + c_\lambda - p, \cdots, \varphi(C) + c_\lambda - p + (n - c_{\lambda-1})\}$.*

Proof. Let $u = c_{\lambda-1}$, $v = c_\lambda$, and $M(C) = \pi(v) + \sum_{i=1}^{\lambda} E_v(c_i)$. Here, $+$ is the Boolean sum, $\pi(v) = E_v(v,1) + \sum_{i=1}^{v-1} E_v(i,i+1)$, and

$E_v(c_i) = \sum_{j \in S_i} E_v(c_i, j)$, where $S_i = \{j | j > 0 \text{ and } c_i - j + 1 \in C\}$. For integers s and t with $0 \leq s \leq v - u$ and $0 \leq t < u$, consider the $(v + t) \times (v + t)$ matrix



where $M(C, s) = M(C) + \sum_{i=1}^{s} E_v(i + u)$. For n with $v \leq n < u + v$

$$M_n(C, s, t) = \begin{pmatrix} M(C, s, t) & Q \\ R & 0 \end{pmatrix}$$

is an $n \times n$ matrix such that the i-th row (respectively column) of $R(Q)$ is the first row (column) of $M(C, s, t)$ for any i. If $u + v > n$, then the index of the above $M_n(C, s, t)$ is $\varphi(C) + (v - p) + (v - u - s) + t$. Now let $u + v = n$, and consider the $n \times n$ matrix $S = \begin{pmatrix} M_{n-1}(C, 0, u - 1) & 0 \\ 0 \cdots 0 & 0 \end{pmatrix} + E_n(u - 1, n) + E_n(n, u + 1)$. Then the index of S is $\varphi(C) + (v - p) + (n - u)$. Therefore we have the lemma.

It is well known that any p-irreducible matrix is a cyclically p-parti te matrix. Also note that permutationally similar matrices have the same index. Thus without loss of generality we will assume that the matrix A of Lemma 3.3 and 3.4 is of the form Π (in Definition 1.2).

LEMMA 3.3. *If* $A \in M_n^p(\alpha \geq 3)$, *then* $\Gamma A = \{pg, ph\}$ *with* $g + h > \alpha$.

Proof. First let $p = 1$. It is known that for a primitive matrix $A \in B_\alpha$, $\lambda(A) \geq 3$ means $index(A) \leq \omega_\alpha + 1$ [7]. If $\lambda(A)$ is 1, then the girth of A is 1 and $index(A) \leq \alpha \leq \omega_\alpha + 1$. Also if $\Gamma A = \{g, h\}$ with $g + h \leq \alpha$, then $index(A) \leq (g-1)(\alpha - g - 1) + 2\alpha - g - 1 = -g^2 + g(\alpha - 1) + g \leq \omega_\alpha + 1$. Now let $p \geq 2$. If $index(\pi_1)$ is the minimum of the set $\{index(\pi_i)\}$ and if there are β many π_i's such that $index(\pi_i) > index(\pi_1)$, then $index(A) \leq p(index(\pi_1)) + \beta$. Thus if $\lambda(A) = 1$, then the girth of A is p and $index(A) \leq p\alpha + \beta \leq p(\omega_\alpha + 1) + \beta$. Next if $\lambda(A) \geq 3$, then $index(A) \leq p(\omega_\alpha + 1) + \beta$ since there are at most β many π_i's whose order is greater than α. Now let $\Gamma A = \{pg, ph\}$ with $g < h$, and let γ be the minimum of $\{order\ of\ \pi_i\}$ and the order of π_k be γ. Note that if $g + h \leq \alpha$, then $index(\pi_k) \leq (g-1) \cdot (\alpha - g - 1) + 2\gamma - g - 1 \leq \omega_\alpha + 1$. Since at least $p - \beta$ many π_i's are of order less than $\alpha + 1$, $index(A) \leq p(\omega_\alpha + 1) + \beta$. Thus if $index(A) > p(\omega_\alpha + 1) + \beta$, then $\Gamma A = \{pg, ph\}$ with $g + h > \alpha$.

LEMMA 3.4. *Let $A \in B_n(\alpha \geq 3)$ be p–irreducible, $\Gamma A = \{pg, ph\}$ $(g < h)$, and $q = g(h-1)$. If every π_i is nonsymmetric and $g + h > \alpha$, then $pq \leq index(A) \leq pq + p(\alpha - g) + \beta$ is a sharp inequality.*

Proof. If $p = 1$, then A is primitive and the lemma holds by the results of Lewin and Vitek [7]. From their results, if A is symmetric, then $index(A) \leq \omega_\alpha + 1$. For each h–cycle τ of $G(A)$, there is an arc (s, t) in τ with $d(t, s) = h - 1$, and $g(h-1) \leq index(A) \leq g(h-1) + (\alpha - g)$. Thus in the following proof, we will assume that $p \geq 2$. We also let $u = pg$ and $v = ph$.

(Upper bound) Since A is a p–irreducible matrix, without loss of generality, let A be of the form Π (in Definition 1.2). If the order of π_i is m for some i, then $index(\pi_i) \leq q + (m - g)$. If $m < \alpha$, then $index(\pi_j) \leq q + (\alpha - g)$ for any j, and $index(A) \leq pq + p(\alpha - g)$ in this case. Now let the minimum of $\{order\ of\ \pi_i\}$ be α. Then there are at most β many π_i's such that the order of each π_i is greater than α. Therefore by the structure of $A (= \Pi)$ and Lemma 1.3, $index(A) \leq pq + p(\alpha - g) + \beta$.

(Lower bound) Consider a v–cycle of $G(A)$ labeled as $(v_1, v_2, \cdots, v_v, v_1)$. Without loss of generality let $d(v_{p+1}, v_1)$ be $v - p$ (so there is no path of length $pq - p$ from v_{p+1} to v_1). If $index(A) \leq pq - 1$, then there is a path τ_j of length $pq - 1$ from v_{p+1-j} to v_{p-j} for $0 \leq j < p$. Note that the length of τ_i can be expressd as a sum of the length of a simple path from v_{p+1-j} to v_{p-j} and some circumferences of A. Also note that $p(g-1)(h-1) - p(= pgh - pg - ph)$ cannot be expressed as a nonnegative linear combination of u and v. So there is a path of length $u-1$ from v_{p+1-j} to v_{p-j} for $0 \leq j < p$ since $pq-1 = p(gh-g)-1 = pgh-pg-1 = pgh-pg-ph+(v-1)$. By the similar reason, we can construct a path of length $k(u-1)$ from v_{p+1} to v_{p+1-k} for $0 < k \leq p$. Then there is a path of length $pq-p$ from v_{p+1} to v_1 since $pq-p = u(h-1)-p = u(h-p-1)+p(u-1)$, contradiction. Thus $index(A) \geq pq$.

(Sharpness) By the Lemma 3.2.

THEOREM 3.5. *For each* $\theta(\geq 3)$ *and* $n = p\alpha + \beta$ *with* $\alpha = [\frac{n}{p}](\geq 3), [p(\alpha^2 - \theta\alpha + [(\theta+1)^2/4]) + 1, \cdots, p(\alpha^2 - (\theta-1)\alpha + (\theta-2)) - 1]$ *is a gap interval greater than* $p(\omega_\alpha + 1) + \beta$ *in the index set* E_n^p.

Proof. Let $M_n^p[\varepsilon, \delta]$ denote the set $\{A \in M_n^p | \Gamma A = \{pg, ph\}(g < h)$ such that $\varepsilon = h - g + 1$ and $\delta = \alpha - h\}$. For a positive integer θ, $M_n^p[\theta]$ denotes the set $\bigcup_{\varepsilon+2\delta=\theta} M_n^p[\varepsilon, \delta]$. Now choose any $A \in M_n^p$. Then we may assume that $\Gamma A = \{pg, ph\}(g < h)$ with $g + h > \alpha$ by Lemma 3.3, and $A \in M_n^p[\theta]$ for some θ. From Lemma 3.4 we can obtain an inequality $p(\alpha^2 - \theta\alpha + [-\delta^2 + (\theta-2)\delta + (\theta-1)] \leq index(A) \leq p(\alpha^2 - \theta\alpha + [-\delta^2 + (\theta-3)\delta + 2(\theta-1)])$. Since $\delta = (\theta-\varepsilon)/2$ the minimum possible value of $[-\delta^2 + (\theta-2)\delta + (\theta-1)]$ is $\theta - 1$ when $\delta = 0$ by simple calculation, and the maximum possible value of $[-\delta^2 + (\theta-3)\delta + 2(\theta-1)]$ is $[(\theta+1)^2/4]$ when δ is $(\theta-3)/2$. Therefore there does not exist any matrix in M_n^p whose index lies between $p(\alpha^2 - \theta\alpha + [(\theta+1)^2/4]) + 1$ and $p(\alpha^2 - (\theta-1)\alpha + (\theta-2)) - 1$.

It is our belief that if $\alpha \geq 14$ or $\alpha \leq 8$, then there is no gap less than $p(\omega_\alpha + 1) + \beta$ in the index set E_n^p.

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