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CHARACTERISTIC POLYNOMIALS OF SOME GRAPH BUNDLES

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1. Introduction

Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ be the adjacency matrix of G. Then the characteristic polynomial of G is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$. We denote the characteristic polynomial of G by $\Phi(G;\lambda)$. A root of $\Phi(G;\lambda)$ is called an eigenvalue of G, and the set of eigenvalues of G is called the spectrum of G. If the distinct eigenvalues of G are $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ with their multiplicities $m(\lambda_1), m(\lambda_2), \ldots, m(\lambda_s)$, then we shall write

$$\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_s) \end{pmatrix}.$$

In [7], A.J. Schwenk studied relations between the characteristic polynomials of some related graphs. In particular, he computed the characteristic polynomials of graphs formed by certain binary operations: union, cartesian product, tensor product and strong product of two graphs. In [4], T. Kitamura and M. Nihei studied the characteristic polynomials of regular double coverings of graphs. The aim of this paper is to study a relation between the characteristic polynomial of a graph G and the characteristic polynomials of graph bundles over G. In the case that the fibre graph F is the complete graph K_2 on two vertices or its complement $\overline{K_2}$, we give the complete computations for the characteristic polynomials of F-bundles over a graph G. Finally, we investigate the structures of K_2 (or $\overline{K_2}$)-bundles over graphs by using their eigenvalues and some algebraic characterizations of two isomorphic graph bundles. Generally, we follow N. Biggs [1] for terminology.

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2. Graph bundles

We begin by introducing the notion of a graph bundle. Every edge of a graph G gives rise to a pair of oppositely directed edges. We denote the set of directed edges of G by $D(G)$. By e^{-1} we mean the reverse edge to an edge $e \in D(G)$. We may denote the directed edge e of G by uv if the initial and terminal vertices of e are u and v respectively. For any finite group Γ, a Γ-voltage assignment of G is a function $\phi : D(G) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in D(G)$. We denote the set of all Γ-voltage assignments of G by $C^1(G;\Gamma)$. Let F be also a graph and let $\phi \in C^1(G; \mathrm{Aut}(F))$, where $\mathrm{Aut}(F)$ is the group of all graph automorphisms of F. Now, we construct a graph $G \times^{\phi} F$ as follows: $V(G \times^{\phi} F) = V(G) \times V(F)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times^{\phi} F$ if either $u_1u_2 \in D(G)$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. We call $G \times^{\phi} F$ the *F-bundle over G associated with* ϕ and the natural map $p^{\phi} : G \times^{\phi} F \to G$ the *bundle projection.* We also call G and F the *base* and the *fibre* of $G \times^{\phi} F$, respectively. Note that the map p^{ϕ} maps vertices to vertices but an image of an edge can be either an edge or a vertex. Moreover, if $F = \overline{K_n}$, then every F-bundle of G is just an n-fold covering graph of G, and if $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in D(G)$, then $G \times^{\phi} F$ is just the cartesian product of G and F.

Let H be a subgroup of $\mathrm{Aut}(F)$. Two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are said to be *isomorphic* with respect to H if there exists an isomorphism $\Theta : G \times^{\phi} F \longrightarrow G \times^{\psi} F$ and $h \in H$ such that the diagram

$$\begin{array}{ccc} G \times^{\phi} F & \xrightarrow{\ \Theta\ } & G \times^{\psi} F \\ {\scriptstyle p_{\phi}}\downarrow & & \downarrow{\scriptstyle p_{\psi}} \\ G & \xrightarrow[\ h\ ]{} & G \end{array}$$

commutes.

Let $C^0(G; \mathrm{Aut}\,(F))$ denote the set of functions $f : V(G) \to \mathrm{Aut}\,(F)$. The following two algebraic characterizations of two isomorphic graph bundles may be found in [5].

THEOREM 1. *Two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are isomorphic with respect to H, $H \le \mathrm{Aut}(G)$, if and only if there exist $h \in H$ and*

$f \in C^0(G; Aut(F))$ such that $\psi(h^{-1}(u)h^{-1}(v)) = f(v)\phi(uv)f(u)^{-1}$ for all $uv \in D(G)$.

Let T be a spanning tree of G with root v_0. Define a map $\Im^{\#} : C^1(G; \mathrm{Aut}(F)) \to C^0(G; \mathrm{Aut}(F))$ as follows: for any $v \in V(G)$ there exists a unique path $e_1e_2\cdots e_m$ in the tree T from v_0 to v, and we define

$$\Im^{\#}(\phi)(v) = \phi(e_1)^{-1}\cdots\phi(e_m)^{-1}.$$

We denote $C^1_T(G; \mathrm{Aut}(F))$ by

$$\{\phi \in C^1(G; \mathrm{Aut}(F)) | \phi(e) \text{ is the identity for each } e \in D(T)\},$$

and define $\Im^* : C^1(G; \mathrm{Aut}(F)) \to C^1_T(G; \mathrm{Aut}(F))$ by

$$\Im^*(\phi)(uv) = \Im^{\#}(\phi)(v)\phi(uv)\Im^{\#}(\phi)(u)^{-1}$$

for any $\phi \in C^1(G; \mathrm{Aut}(F))$ and any $uv \in D(G)$. Then, $\Im^*$ is clearly well-defined and the identity on $C^1_T(G; \mathrm{Aut}(F))$. Hence, we have

THEOREM 2. *Any F-bundle $G \times^{\phi} F$ over G, $\phi \in C^1(G; Aut(F))$, is isomorphic to an F-bundle $G \times^{\psi} F$ with respect to the identity automorphism of G for some $\psi \in C^1_T(G; Aut(F))$.*

Figure 1 shows two K_2-bundles over the cycle C_5 of five vertices, and the values of the base graph C_5 represent the $\mathbf{Z}_2$-voltage assignments, where $\mathbf{Z}_2 = \{1, -1\}$. Note that the two K_2-bundles in Figure 1 are all K_2-bundles over C_5 up to isomorphism (with respect to the identity automorphism), by Theorem 2.

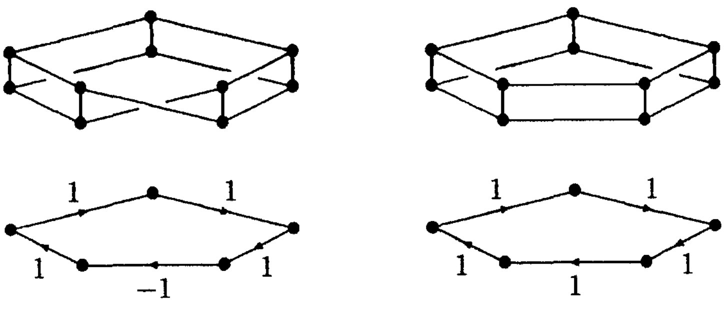


Figure 1. Two K_2-bundles over C_5

3. A relation between $\Phi(G;\lambda)$ and $\Phi(G\times^{\phi} F;\lambda)$

In this section, we show that if the fibre F is k-regular graph, then $\Phi(G;\lambda-k)$ is a divisor of $\Phi(G\times^{\phi} F;\lambda)$ for any Aut(F)-voltage assignment ϕ. For any set S, we denote the cardinality of S by $|S|$. For each $v\in V(G)$, we denote the set of all vertices adjacent to v by $N(v)$ and call it the *neighborhood* of v. We also denote the set of all vertices of $(p^{\phi})^{-1}(v)$ by R^{ϕ}_v for each $v\in V(G)$. A partition $P=(V_1,\dots,V_n)$ of $V(G)$ is said to be *equitable* if for each i and any $u,v\in V_i$, $|N(u)\cap V_j|=|N(v)\cap V_j|$ for all j. We define a matrix $P_G=(p_{ij})$ by $p_{ij}=|N(u)\cap V_j|$ for $u\in V_i$. In [7], A.J. Schwenk showed that the characteristic polynomial of the matrix P_G is a divisor of $\Phi(G;\lambda)$, which is a main tool in proving the next theorem.

THEOREM 3. *Let F be a k-regular graph. Then $\Phi(G;\lambda-k)$ is a divisor of $\Phi(G\times^{\phi} F;\lambda)$ for any $\phi\in C^1(G;Aut(F))$.*

Proof. Let $V(G)=\{v_1,\cdots,v_{|V(G)|}\}$ and $P(\phi)=(R^{\phi}_{v_1},\dots,R^{\phi}_{v_{|V(G)|}})$. Then $P(\phi)$ is an equitable partition of $V(G\times^{\phi} F)$. If $A(G)=(a_{ij})$, for each i and j we have

$$|N(u)\cap R^{\phi}_j|=\begin{cases} k & \text{if } u\in R^{\phi}_{v_i} \text{ and } i=j\\ a_{ij} & \text{if } u\in R^{\phi}_{v_i} \text{ and } i\neq j,\end{cases}$$

because F is k-regular. Thus $P(\phi)_G=A(G)+kI_{|V(G)|}$, where I_n is the identity matrix of order n. Then the characteristic polynomial of $P(\phi)_G$ is

$$\begin{aligned}\det\left(\lambda I_{|V(G)|}-A(G)-kI_{|V(G)|}\right)&=\det\left((\lambda-k)I_{|V(G)|}-A(G)\right)\\&=\Phi(G;\lambda-k).\end{aligned}$$

Since the partition $P(\phi)$ is equitable, $\Phi(G;\lambda-k)$ is a divisor of $\Phi(G\times^{\phi} F;\lambda)$.

Since $\overline{K_n}$ is a 0-regular graph, we have the following corollary.

COROLLARY 1. *If $\tilde{G}$ is a covering graph of G, then $\Phi(G;\lambda)$ is a divisor of $\Phi(\tilde{G};\lambda)$.*

Note that this corollary shows that if $\Phi(G;\lambda)$ is not a divisor of $\Phi(\tilde{G};\lambda)$, then $\tilde{G}$ cannot be a covering graph of G. Also note that the converse of Corollary 1 is false, because there exist two graphs G_1 and G_2 such that $\Phi(G_1;\lambda) = \Phi(G_2;\lambda)$ but G_1 and G_2 are not isomorphic (see [1], p.13).

4. Characteristic polynomials of double coverings

A *signed graph* is $G_\omega = (G,\ \omega)$, where $\omega : E(G) \to \{-1, 1\}$ is a function. We call G the *underlying graph* of G_ω and ω the *weight function* of G_ω. In a signed graph, the edges which are assigned the weight 1 are called *positive* and the others *negative*. We denote by $E^+(G_\omega)$ (resp. $E^-(G_\omega)$) the set of all positive (resp. negative) edges of a signed graph G_ω. Every subgraph S of the underlying graph G of a signed graph G_ω has two natural subgraph structures, as a signed and unsigned subgraph. In the unsigned case, S is merely a subgraph of G; in the other case, S is the signed graph with the weight function ω of G_ω restricted to $E(S)$. We denote it by S or S_ω respectively.

Given any signed graph G_ω, its adjacency matrix $A(G_\omega) = (a_{ij})$ is a square matrix of order $|V(G)|$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \text{ is positive} \\ -1 & \text{if } v_iv_j \text{ is negative} \\ 0 & \text{if } v_iv_j \text{ is not an edge} \end{cases}$$

for $1 \le i, j \le |V(G)|$, and its characteristic polynomial is that of $A(G_\omega)$.

Since any element of $\mathbf{Z}_2$ has itself as inverse, we can regard any voltage assignment $\phi \in C^1(G;\mathbf{Z}_2)$ as the map from the set of all undirected edges of G to $\mathbf{Z}_2$. Thus, for any $\phi \in C^1(G;\mathbf{Z}_2)$, we consider ϕ as a weight function of G. From now on, we identify the set of all $\mathbf{Z}_2$-voltage assignments of G with the set of all weight functions of G. Denote the signed graph associated with ϕ by G_ϕ. A graph S is a *basic figure* if every component of S is either a cycle or K_2. Let G_n denote the set of those n-vertex subgraphs of G which are basic figures. A cycle $C = e_1\ e_2\ \cdots e_n$ in a subgraph S of G_ϕ is negative if C has odd numbers of negative edges in S_ϕ, *i.e.*, $\prod_{i=1}^{n} \phi(e) = -1$ in $\mathbf{Z}_2$. We denote the set of negative cycles in a subgraph S of G_ϕ by $N(S_\phi)$.

For example, let $G = K_3$ and let $V(G) = \{v_1, v_2, v_3\}$. Let $\phi \in C^1(G;\mathbf{Z}_2)$ such that $\phi(v_1v_2) = -1$ and $\phi(e) = 1$ for any $e \neq v_1v_2$. Figure 2 and Table 1 show K_3, K_{3_ϕ}, G_n and $N(S_\phi)$.

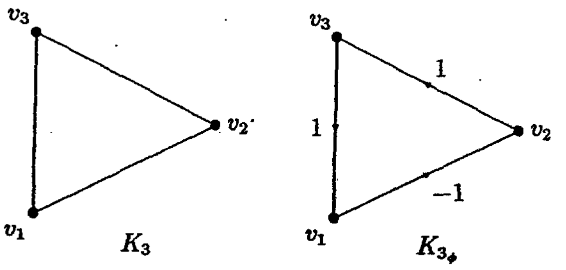


Figure 2. K_3 and K_{3_ϕ}

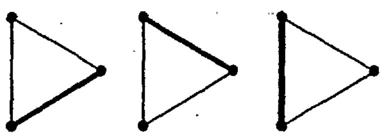
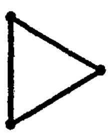
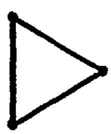
n	G_n	$N(S_\phi)$
1	empty set	empty set
2		empty set
3		

Table 1. G_n and $N(S_\phi)$

Moreover,

$$A(K_{3_\phi}) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

For any $\psi \in C^1(G; \mathbb{Z}_2)$, we denote by G_{ψ^+} (resp. G_{ψ^-}) the spanning subgraph of the underlying graph G whose edge set is $E^+(G_\psi)$ (resp. $E^-(G_\psi)$), and consider them as unsigned graphs. Then we have

$$A(G \times^\psi \overline{K_2}; \lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G_{\psi^+}) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G_{\psi^-}),$$

where $\otimes$ means the Kronecker product of two matrices. Note that

$$A(G_{\psi^+}) + A(G_{\psi^-}) = A(G)$$

and

$$A(G_{\psi^+}) - A(G_{\psi^-}) = A(G_\psi).$$

THEOREM 4. $\Phi(G \times^\phi \overline{K_2}\,; \lambda) = \Phi(G; \lambda)\Phi(G_\phi; \lambda)$ *for any* $\phi \in C^1(G; \mathbb{Z}_2)$.

Proof. Let Q be a 2×2 matrix with the property that

$$Q^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
&\left(Q^{-1} \otimes I_{|V(G)|}\right) A(G \times^\phi \overline{K_2}) \left(Q \otimes I_{|V(G)|}\right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G_{\phi^+}) + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes A(G_{\phi^-}) \\
&= \begin{bmatrix} A(G_{\phi^+}) + A(G_{\phi^-}) & 0 \\ 0 & A(G_{\phi^+}) - A(G_{\phi^-}) \end{bmatrix} \\
&= \begin{bmatrix} A(G) & 0 \\ 0 & A(G_\phi) \end{bmatrix}.
\end{aligned}$$

Thus, we have $\Phi(G \times^\phi \overline{K_2}\,; \lambda) = \Phi(G; \lambda)\Phi(G_\phi; \lambda)$.

For a signed graph G_ϕ, let

$$\begin{aligned}
\Phi(G_\phi; \lambda) = c_0(G_\phi)\lambda^{|V(G)|} &+ c_1(G_\phi)\lambda^{|V(G)|-1} \\
&+ \cdots + c_{|V(G)|-1}(G_\phi)\lambda + c_{|V(G)|}(G_\phi).
\end{aligned}$$

Note that $c_0(G_\phi)$ is 1 and $c_1(G_\phi)$ is 0 for all simple graph G. Let $c(G)$ and $\kappa(G)$ denote the numbers of cycles and components in G, respectively. Then, the following theorem is Sachs's formula for a signed (weighted) graph ([2], sec. 1.4).

THEOREM 5. $c_i(G_\phi) = \sum_{S \in G_i} (-1)^{\kappa(S)+|N(S_\phi)|} 2^{c(S)}$ *for* $1 \le i \le |V(G)|$.

For convenience, we take $t \in C^1(G; \mathbb{Z}_2)$ such that $t(e)$ is the identity for all $e \in D(G)$. Then $|N(S_t)| = 0$ for all $S \in G_n$. We write $c_i(G_t) = c_i(G)$ for all $i = 0, \ldots, |V(G)|$. We shall denote $G \times^t \overline{K_2}$ by $G \times \overline{K_2}$. By using Theorems 4 and 5, we can get the characteristic polynomial of a double covering of G.

THEOREM 6. *Let $\phi \in C^1(G;\mathbb{Z}_2)$. Then*

$$\Phi(G \times^\phi \overline{K_2};\lambda) = \left(\sum_{n=0}^{|V(G)|}\sum_{S\in G_n}(-1)^{\kappa(S)}2^{c(S)}\lambda^{|V(G)|-n}\right) \times \left(\sum_{n=0}^{|V(G)|}\sum_{S\in G_n}(-1)^{\kappa(S)+|N(S_\phi)|}2^{c(S)}\lambda^{|V(G)|-n}\right).$$

For a computational example, consider Figure 3 and Table 2. Here $G = K_4$

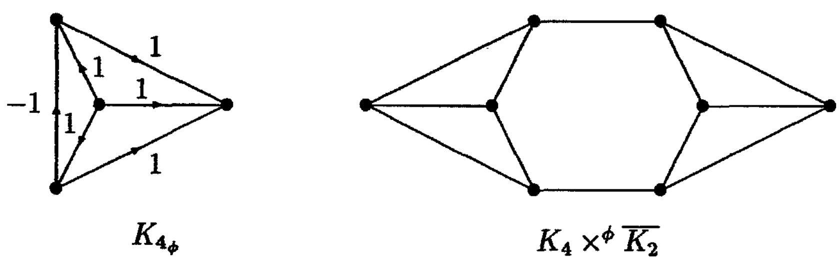


Figure 3. K_{4_ϕ} and $K_4 \times^\phi \overline{K_2}$

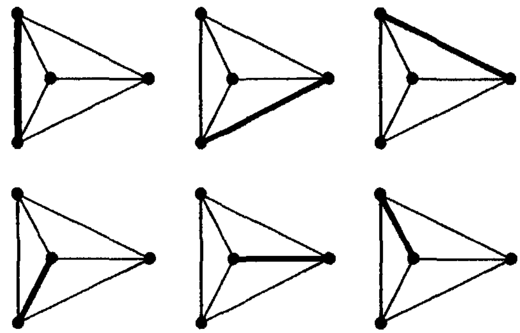
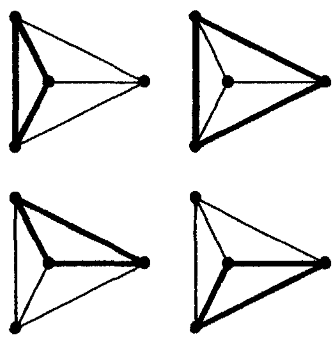
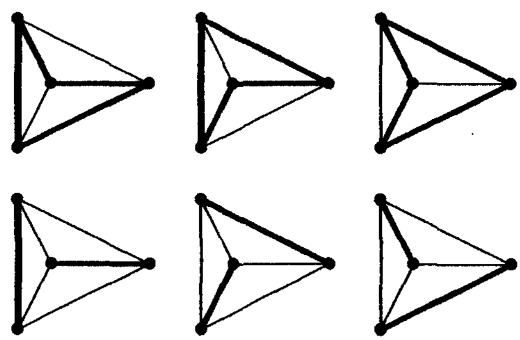
n	G_n	$c_n(G)$	$c_n(G_\phi)$
1	empty set	0	0
2		-6	-6
3		-8	0
4		-3	5

Table 2. G_n and associated coefficients

Thus

$$\begin{aligned}\Phi(K_4 \times^{\phi} \overline{K_2}; \lambda) &= (\lambda^4 - 6\lambda^2 - 8\lambda - 3)(\lambda^4 - 6\lambda^2 + 5)\\ &= (\lambda - 3)(\lambda + 1)^3(\lambda^2 - 1)(\lambda^2 - 5).\end{aligned}$$

A graph G is *integral* if every eigenvalue of G is an integer.

COROLLARY 2. *For each $\phi \in C^1(G; \mathbf{Z}_2), G \times^{\phi} \overline{K_2}$ is integral if and only if both G and G_ϕ are integral.*

For example, consider Figure 2 and Table 1. Since C_6 is isomorphic to $K_3 \times^{\phi} \overline{K_2}$, $\Phi(K_{3_\phi}; \lambda) = \lambda^3 - 3\lambda + 2 = (\lambda + 2)(\lambda - 1)^2$ and K_3 is integral, C_6 is integral.

Let $-t$ be the $\mathbf{Z}_2$-voltage assignment of G with the property that $\phi(e) = -1$ for all $e \in D(G)$. We denote $\Phi(G \times^{-t} \overline{K_2}; \lambda)$ by $\Phi(G \widetilde{\times} \overline{K_2}; \lambda)$.

COROLLARY 3.

(1) $\Phi(G \times \overline{K_2}; \lambda) = (\Phi(G; \lambda))^2$.
(2) $\Phi(G \widetilde{\times} \overline{K_2}; \lambda) = (-1)^{|V(G)|}\Phi(G; \lambda)\Phi(G; -\lambda)$.

Proof. (1) is clear.
(2) If $\phi(e) = -1$ for all $e \in D(G)$, then $A(G_\phi) = -A(G)$. Thus

$$\begin{aligned}\Phi(G_\phi; \lambda) &= \det(\lambda I + A(G)) = (-1)^{|V(G)|}\det(-\lambda I - A(G))\\ &= (-1)^{|V(G)|}\Phi(G; -\lambda).\end{aligned}$$

Note that Corollary 3.(2) is Theorem 1 of [4]. Let G be a k-regular graph. Then the number of spanning trees of G is $\tau(G) = |V(G)|^{-1}\Phi'(G; k)$, where Φ' denotes the derivative of the characteristic polynomial $\Phi(G; \lambda)$ ([1], p. 36).

COROLLARY 4. *Let G be a k-regular connected simple graph. Then,*

(1) *$G \times^{\phi} \overline{K_2}$ is connected if and only if $\Phi(G_\phi; k) \neq 0$.*
(2) *$\tau(G \times^{\phi} \overline{K_2}) = \frac{1}{2}\tau(G)\Phi(G_\phi; k)$.*

Proof. We prove (2) only.

$$\begin{aligned}\tau(G \times^{\phi} \overline{K_2}) &= (2|V(G)|)^{-1}\Phi'(G \times^{\phi} \overline{K_2}; k)\\ &= (2|V(G)|)^{-1}(\Phi'(G; k)\Phi(G_\phi; k) + \Phi(G; k)\Phi'(G_\phi; k))\end{aligned}$$

Since G is k-regular, k is an eigenvalue of G and hence $\Phi(G;k) = 0$. Thus we have, $\tau(G \times^{\phi} \overline{K_2}) = \frac{1}{2}\tau(G)\Phi(G_{\phi};k)$.

For example, if ϕ is the same one as in Figure 3, then

$$\begin{aligned}\tau(K_4 \times^{\phi} \overline{K_2}) &= \frac{1}{2}\tau(K_4)\Phi(K_{4_{\phi}};3)\\ &= \frac{1}{2} \cdot 16 \cdot 8 \cdot 4 = 256.\end{aligned}$$

It is also known ([1], p. 51) that a graph G is bipartite if and only if $\lambda_s = -\lambda_1$, where

$$\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \cdots & \lambda_s \\ m(\lambda_1) & \cdots & m(\lambda_s) \end{pmatrix}.$$

Moreover, the above statements are equivalent to the following:

$$\Phi(G;-\lambda) = (-1)^{|V(G)|}\Phi(G;\lambda) \text{ ([1], Prop. 8.2).}$$

COROLLARY 5. *Every double covering of a bipartite graph G is also bipartite.*

Proof. Let ϕ be a $\mathbf{Z_2}$-voltage assignment of G. Then, the bipartiteness of G implies that the number of vertices of every subgraphs of G, which is a basic figure, is even. This fact implies that $\Phi(G_{\phi};-\lambda) = (-1)^{|V(G)|}\Phi(G_{\phi};\lambda)$. Clearly,

$$\Phi(G \times^{\phi} \overline{K_2};-\lambda) = \Phi(G \times^{\phi} \overline{K_2};\lambda).$$

This completes the proof.

COROLLARY 6. *Let G be a graph which is not bipartite and let ϕ be a $\mathbf{Z}_2$-voltage assignment of G. Then the following are equivalent:*

(1) $G \times^{\phi} \overline{K_2}$ *is bipartite.*
(2) $G \times^{\phi} \overline{K_2}$ *is isomorphic to* $G\widetilde{\times}\overline{K_2}$.
(3) $\Phi(G_{\phi};\lambda) = (-1)^{(|V(G)|)}\Phi(G;-\lambda)$.
Moreover, if G is a k-regular, then the above three statements are equivalent to the following statement:
(4) $\Phi(G_{\phi};-k) = 0$.

Proof. Let G be a graph which is not bipartite. Then G must contain an odd cycle, say C_o. Take an edge e_o in C_o. Let T be a spanning tree of G containing all edges of C_o except e_o. Then, for each $e \in E(G)-E(T)$, e determines a cycle C_e such that $E(C_e)-\{e\}$ is a subset of $E(T)$. Note that C_{e_o} is the cycle C_o. By Theorem 2, we may assume that $\phi(e) = 1$ for any $e \in E(T)$

(1) $\Rightarrow$ (2) Let $e \in E(G) - E(T)$. Then the length of C_e is either even or odd. If it is odd, then C_e must be negative because $G \times^\phi \overline{K_2}$ is bipartite. This gives $\phi(e) = -1$. If it is even, then C_e must be non-negative. Indeed, if C_e is negative, then $S \times^\phi \overline{K_2}$ contains an odd cycle, where S is the spanning subgraph of G whose edge set $E(S)$ is $\{e_o, e\} \cup E(T)$. This contradicts the bipartiteness of $G \times^\phi \overline{K_2}$. This gives $\phi(e) = 1$. By Theorem ıtree, $G \times^\phi \overline{K_2}$ is isomorphic to $G\tilde{\times}\overline{K_2}$.

The two implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. Moreover, the two implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are clear if G is k-regular.

Note that Corollary 6 gives that there exists only one bipartite double covering, up to isomorphism, if the base graph is not bipartite.

A signed graph is *balanced* if every cycle of it is non-negative. Note that every disconnected double cover of a connected graph G is isomorphic to the disjoint union of two copies of G. It is known that a graph G is connected if and only if its largest eigenvalue both has multiplicity one and has an eigenvector whose components are all positive ([2], Theorem 3.34). We also note that $G \times \overline{K_2}$ is isomorphic to two disjoint copies of G. By using these facts and Theorem 2, we have the following corollary.

COROLLARY 7. *Let ϕ be a $\mathbf{Z}_2$-voltage assignment of G. Then the following are equivalent.*

(1) *G_ϕ is balanced.*
(2) *$G \times^\phi \overline{K_2}$ is isomorphic to $G \times \overline{K_2}$.*
(3) $\Phi(G_\phi; \lambda) = \Phi(G; \lambda)$.

5. Characteristic polynomials of K_2-bundles

We observe that for any $\phi \in C^1(G; \mathbf{Z}_2)$, the construction of $G \times^\phi K_2$ gives

$$A(G\times^\phi K_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G_{\phi^+}) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G_{\phi^-}) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{|V(G)|}.$$

THEOREM 7. *$\Phi(G \times^{\phi} K_2; \lambda) = \Phi(G; \lambda - 1)\Phi(G_{\phi}; \lambda + 1)$ for any $\phi \in C^1(G; \mathbf{Z}_2)$.*

Proof. Using a method similar to the proof of Theorem 4, we have that $A(G \times^{\phi} K_2)$ is similar to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A(G_{\phi^+}) + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes A(G_{\phi^-}) + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I_{|V(G)|},$$

which is equal to

$$\begin{bmatrix} A(G_{\phi^+}) + A(G_{\phi^-}) + I_{|V(G)|} & 0 \\ 0 & A(G_{\phi^+}) - A(G_{\phi^-}) - I_{|V(G)|} \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} \Phi(G \times^{\phi} K_2; \lambda) = \det\left(\lambda I_{|V(G)|} - A(G) - I_{|V(G)|}\right) \\ \det\left(\lambda I_{|V(G)|} - A(G_{\phi}) + I_{|V(G)|}\right) \\ = \Phi(G; \lambda - 1)\Phi(G_{\phi}; \lambda + 1). \end{aligned}$$

For example, let $G = C_4$ and $\phi : D(G) \to \mathbf{Z}_2$ defined as in Figure 4.

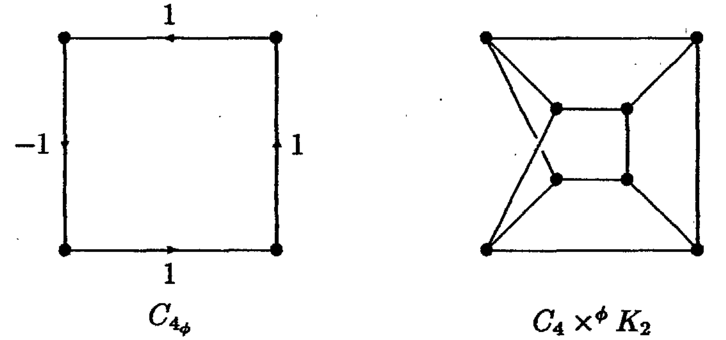


Figure 4. C_{4_ϕ} and the associated K_2-bundle

By Theorem 5, we have $\Phi(C_{4_\phi};\lambda) = \lambda^4 - 4\lambda^2 + 4 = (\lambda^2-2)^2$ and $\Phi(C_4;\lambda) = (\lambda-2)\lambda^2(\lambda+2)$. Thus $\Phi(C_4 \times^\phi K_2;\lambda) = (\lambda-3)(\lambda-1)^2(\lambda+1)(\lambda^2+2\lambda-1)^2$.

Note that the 4-runged Möbius ladder M_4 is isomorphic to $C_4 \times^\phi K_2$ and hence $\Phi(M_4;\lambda) = (\lambda-3)(\lambda-1)^2(\lambda+1)(\lambda^2+2\lambda-1)^2$. Moreover, $\tau(M_4) = \frac{1}{8}\cdot 2^2 \cdot 4 \cdot (14)^2 = 392$.

Note that $A(G_t) = A(G)$ and $A(G_{-t}) = -A(G)$. By Theorem 7, we have the following corollary.

COROLLARY 8.

(1) $\Phi(G \times K_2;\lambda) = \Phi(G;\lambda-1)\Phi(G;\lambda+1)$.
(2) $\Phi(G\tilde{\times}K_2;\lambda) = (-1)^{|V(G)|}\Phi(G;\lambda-1)\Phi(G;-\lambda-1)$.

COROLLARY 9. *For every graph G, $G \times^\phi K_2$ is bipartite if and only if $G \times^\phi K_2$ is isomorphic to $G\tilde{\times}K_2$.*

Moreover, if G is bipartite, then $G \times^\phi K_2$ is bipartite if and only if $G \times^\phi K_2$ is isomorphic to $G \times K_2$.

Proof. We first prove that $G\tilde{\times}K_2$ is bipartite. Let $\Phi(G;\lambda) = \prod_{i=1}^{s}(\lambda-\lambda_i)^{m(\lambda_i)}$. Then $\Phi(G\tilde{\times}K_2;\lambda) = \prod_{i=1}^{s}(\lambda-(\lambda_i+1))^{m(\lambda_i)}(\lambda+(\lambda_i+1))^{m(\lambda_i)}$, which implies that $G\tilde{\times}K_2$ is bipartite. Conversely, suppose that $G \times^\phi K_2$ is bipartite. Without loss of generality, we may assume that the value of $\phi = 1$ for all edges of a fixed spanning tree T of G. Let e be an edge of the cotree $G - T$. Then e determines a cycle C_e. If the length of C_e is odd, then, by using the method of proof of Corollary 6, we have $\phi(e) = -1$. If the length of C_e is even and $\phi(e) = -1$, then the subgraph $C_e \times^\phi K_2$ of the bipartite graph $G \times^\phi K_2$ has an odd cycle (Figure 4). This contradicts the fact that every subgraph of a bipartite graph is bipartite. Thus $\phi(e)$ must be 1. If we use the method of the proof of Corollary 6 once more, we have that $G \times^\phi K_2$ and $G \tilde{\times} K_2$ are isomorphic.

Clearly, $G \times^\phi K_2$ is $(k+1)$-regular for any k-regular graph G and any $\phi \in C^1(G; \mathbf{Z}_2)$. This fact implies the following corollary.

COROLLARY 10. *Let G be a k-regular connected simple graph. Then*

$$\tau(G \times^\phi K_2) = \frac{1}{2}\tau(G)\Phi(G_\phi; k+2),$$

In particular,

$$\tau(G \times K_2) = \frac{1}{2}\tau(G)\Phi(G; k+2)$$

and

$$\tau(G \tilde{\times} K_2) = (-1)^{|V(G)|}\frac{1}{2}\tau(G)\Phi(G; -k-2).$$

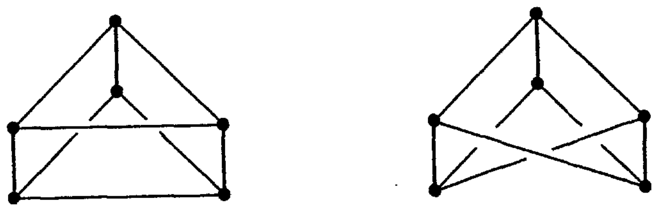
The above corollary gives that every K_2-bundle over a connected graph G is also connected because $\Phi(G_\phi; k+2)$ is not 0.

Now we compute the number of spanning trees of some graphs. Let T_3 be the 3-prism and let M_3 be the 3-runged Möbius ladder. Note that T_3 and M_3 are isomorphic to $C_3 \times K_2$ and $C_3 \tilde{\times} K_2$ respectively. Since $\Phi(C_3; \lambda) = (\lambda - 2)(\lambda + 1)^2$,

$$\tau(T_3) = \tau(C_3 \times K_2) = \frac{1}{2}\tau(C_3)\Phi(C_3; 4) = \frac{1}{2}3 \cdot 2 \cdot 25 = 75$$

and

$$\tau(M_3) = \tau(C_3 \tilde{\times} K_2) = (-1)^3\frac{1}{2}\tau(C_3)\Phi(C_3; -4) = \frac{1}{2} \cdot 3 \cdot 6 \cdot 9 = 81.$$



T_3 M_3

Figure 5. T_3 and M_3

Note that the complete bipartite graph $K_{n,n}$ is isomorphic to $K_n \tilde{\times} K_2$. Thus,

$$\Phi(K_{n,n};\lambda) = (\lambda - n)\lambda^{n-1}(\lambda + n)\lambda^{n-1} = (\lambda - n)(\lambda + n)\lambda^{2n-2},$$

and

$$\tau(K_{n,n}) = n^{2n-2}.$$

Let $G \times^1 K_2 = G \times K_2$ and let $G \times^n K_2 = (G \times^{n-1} K_2) \times K_2$ for $n \geq 2$. Similarly, we define $G \tilde{\times}^n K_2$.

Corollary 11.

(1) $\Phi(G \times^n K_2;\lambda) = \prod_{k=0}^{n} \Phi(G;\lambda - n + 2k)^{\binom{n}{k}}$ *for* $n \geq 1$.

(2) $\Phi(G \tilde{\times}^n K_2;\lambda)$
$$= (-1)^{2^{n-1}|V(G)|} \prod_{k=0}^{n-1} \Phi(G;\lambda - n + 2k)^{\binom{n-1}{k}} \Phi(G; -\lambda + n - 2(k+1)^{\binom{n-1}{k}}$$ *for* $n \geq 1$.

Proof. We prove (1) by induction on n. Suppose it is true for $n = \ell$.

Then

$$
\begin{aligned}
&\Phi(G \times^{\ell+1} K_2; \lambda) \\
&=\Phi((G \times^{\ell} K_2) \times K_2; \lambda) \\
&=\Phi(G \times^{\ell} K_2; \lambda - 1)\Phi(G \times^{\ell} K_2; \lambda + 1) \\
&= \prod_{k=0}^{\ell} \Phi(G; \lambda - 1 - \ell + 2k)^{\binom{\ell}{k}} \prod_{k=0}^{\ell} \Phi(G; \lambda + 1 - \ell + 2k)^{\binom{\ell}{k}} \\
&= \prod_{k=0}^{\ell} \Phi(G; \lambda - (1 + \ell) + 2k)^{\binom{\ell}{k}} \\
&\qquad\qquad \times \prod_{k=0}^{\ell} \Phi(G; \lambda - (\ell + 1) + 2(k + 1))^{\binom{\ell}{k}} \\
&= \prod_{k=0}^{\ell} \Phi(G; \lambda - (1 + \ell) + 2k)^{\binom{\ell}{k}} \\
&\qquad\qquad \times \prod_{k=0}^{\ell+1} \Phi(G; \lambda - (\ell + 1) + 2k)^{\binom{\ell}{k-1}} \\
&=\Phi(G; \lambda - (\ell + 1))\Phi(G; \lambda + \ell + 1) \\
&\qquad\qquad \times \prod_{k=1}^{\ell} \Phi(G; \lambda - (\ell + 1) + 2k)^{\binom{\ell}{k}+\binom{\ell}{k-1}} \\
&= \prod_{k=0}^{\ell+1} \Phi(G; \lambda - (\ell + 1) + 2k)^{\binom{\ell+1}{k}}.
\end{aligned}
$$

(2) By Corollary 9, both $G\widetilde{\times}K_2$ and $(G\widetilde{\times}K_2) \times K_2$ are bipartite. If we use Corollary 9 once more, then we have $(G\widetilde{\times}K_2)\widetilde{\times}K_2$ is isomorphic

to $(G\tilde{\times}K_2)\times K_2$. Thus, we have

$$\begin{aligned}
&\Phi(G\tilde{\times}^n K_2;\lambda)\\
&=\Phi((G\tilde{\times}K_2)\times^{n-1}K_2;\lambda)\\
&=\prod_{k=0}^{n-1}\Phi(G\tilde{\times}K_2;\lambda-n+1+2k)^{\binom{n-1}{k}}\\
&=\prod_{k=0}^{n-1}\Phi(G;\lambda-n+1+2k-1)^{\binom{n-1}{k}}\\
&\qquad\times(-1)^{|V(G)|}\Phi(G;-\lambda+n-1-2k-1)^{\binom{n-1}{k}}\\
&=(-1)^{2^{(n-1)}|V(G)|}\prod_{k=0}^{n-1}(\Phi(G;\lambda-n+2k)\Phi(G;-\lambda+n-2(k+1))^{\binom{n-1}{k}}.
\end{aligned}$$

For example, let Q_n be the n-cube. Then Q_n is isomorphic to $K_1\times^n K_2$ for $n\geq 1$. Since $\Phi(K_1;\lambda)=\lambda$,

$$\Phi(Q_n;\lambda)=\prod_{k=0}^{n}\Phi(K_1;\lambda-n+2k)^{\binom{n}{k}}=\prod_{k=0}^{n}(\lambda-n+2k)^{\binom{n}{k}}$$

for $n\geq 1$ and

$$\begin{aligned}
\tau(Q_n)&=2^{2^n-n-1}\prod_{k=1}^{n}k^{\binom{n}{k}}\quad\text{for } n\geq 1\\
&=2^{2^n+\frac{1}{2}(n^2-3n-2)}\prod_{k=3}^{n}k^{\binom{n}{k}},\ \text{for } n\geq 3
\end{aligned}$$

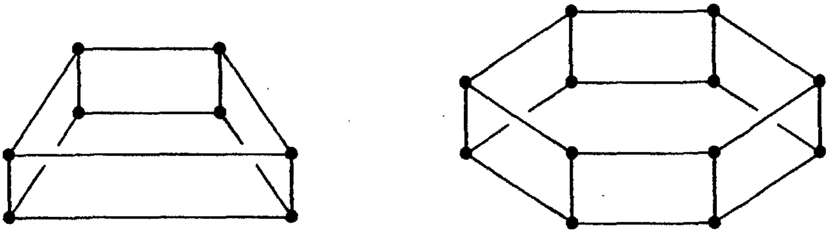
Finally, we characterize the integrality of the n-prism T_n and the n-runged Möbius ladder M_n.

COROLLARY 12.

(1) *The n-prism T_n is integral if and only if $n=3,4$ or 6.*
(2) *The n-runged Möbius ladder M_n is integral if and only if $n=3$.*

Proof. (1) We observe that T_n is isomorphic to $C_n \times K_2$. Thus T_n is integral if and only if C_n is integral. It is known that C_n is integral if and only if $n = 3, 4$ or 6 ([3]).

(2) We also observe that M_n is isomorphic to $C_n \times^{\phi} K_2$ where $\phi \in C^1(G; \mathbf{Z}_2)$ such that $\phi(e) = -1 = \phi(e^{-1})$ for a fixed edge $e \in D(G)$ and $\phi(e) = 1$ for any $e \in D(G) - \{e, e^{-1}\}$. Then M_n is integral if and only if $C_n \times^{\phi} K_2$ is integral. Moreover, $C_n \times^{\phi} K_2$ is integral if and only if $C_n \times^{\phi} \overline{K_2}$ is integral. Since $C_n \times^{\phi} \overline{K_2}$ is isomorphic to C_{2n}, M_n is integral if and only if $n = 3$.



T_4 T_6

Figure 6. Integral graphs T_4 and T_6

Note that T_4 is the 3-cube Q_3.

6. Further remarks

For each $\phi \in C^1(G; \mathbf{Z}_2)$, we define $-\phi \in C^1(G; \mathbf{Z}_2)$ by $(-\phi)(e) = -\phi(e)$. It is easy to see the following theorem.

THEOREM 8. *For each* $\phi \in C^1(G; \mathbf{Z}_2)$,

$$\Phi(G_{-\phi}; \lambda) = (-1)^{|V(G)|}\Phi(G_{\phi}; -\lambda).$$

Now, we give two examples. Let $\phi \in C^1(G; \mathbf{Z}_2)$ be the $\mathbf{Z}_2$-voltage assignment in Figure 4. Then, $\Phi(G_{-\phi}; \lambda) = (-1)^4((-\lambda)^2 - 2)^2 = (\lambda^2 - 2)^2$. Thus the characteristic polynomial of $C_4 \times^{\phi} \overline{K_2}$ and that of $C_4 \times^{-\phi} \overline{K_2}$ are equal. Note that, by Theorem 2, $C_4 \times^{\phi} \overline{K_2}$ and $C_4 \times^{-\phi} \overline{K_2}$ are isomorphic with respect to the identity automorphism.

Let $\phi \in C^1(G; \mathbf{Z}_2)$ be the $\mathbf{Z}_2$-voltage assignment in Figure 3. Then $-\phi$ is the one in Figure 7.

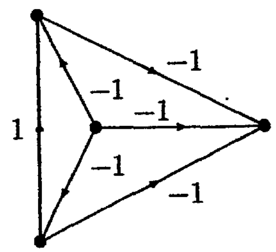


Figure 7. $K_{4_{-\phi}}$

By Theorem 2, $K_4 \times^{-\phi} \overline{K_2}$ and the graph in Figure 8 are isomorphic with respect to the identity automorphism.

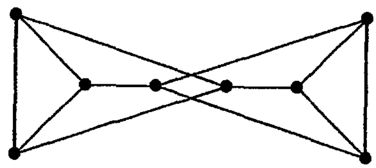


Figure 8.

Moreover, the characteristic polynomial of the graph in the Figure 8 is $(\lambda-3)(\lambda+1)^3(\lambda^2-1)(\lambda^2-5)$ and hence $K_4 \times^{\phi} \overline{K_2}$ and $K_4 \times^{-\phi} \overline{K_2}$ have the same characteristic polynomial. The above two K_2-bundles are isomorphic with respect to $\mathrm{Aut}(K_4)$ by Theorem 1.

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