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A NONLINEAR ANALYSIS OF CONVECTION IN A VERTICAL POROUS LAYER

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1. Introduction

Making use of integral inequalities, Straughan [20] has studied the question of nonlinear stability for the full three dimensional nonlinear theory of convection in a vertical porous layer when a constant temperature difference is a maintained across the layer. In this work he neglected the nonlinear inertia terms of Darcy's momentum equation and assumed a Bousinessq's linear density-temperature relationship. In the present investigation we include the inertia terms and use a cubic density-temperature relationship suggested by Ruddick and Shirtcliffie [13].

We also make use of the assumptions of Kwok and Chen [9] to reduce our problem to a two dimensional layer problem. As in earlier work of Straughan [20], we establish a nonlinear stability result guaranteeing an exponential decay of a perturbation to a basic solution provided initial data are sufficiently small. We remark that although our results are derived for classical solutions, extensions to appropriately weak solutions are obvious. Throughout this paper we will make use of a comma to denote partial differentiation and adopt the summation convention of summing over repeated indices (in a term of an expression) from one to two.

For related work on energy stability and convection, see, for example, Song [15, 18, 19], Payne et al. [11], Galdi et al. [5], and Galdi and Straughan [6]. Also, as reference to work on continuous dependence on modelling and initial data, we mention the papers of Adelson [1], Ames [2], Bennett [3], and Song [15, 16, 17]. Furthermore, a similiar analysis of a micropolar fluid backward in time (an ill-posed problem) was addressed by Payne and Straughan [12], and Payne [10]. The methods we employ are based on integral inequalities.

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2. Formulation of the porous vertical convection

Consider the thermal convection of a fluid of viscosity μ, density ρ (reference density ρ_0), and specific heat c which saturates a vertical slab of a porous medium of permeability k, conductivity λ, gravity g, porosity ϵ, and slab thickness L. The governing equations of continuity, momentum, and energy are as follows (see Joseph [7] and Kwok and Chen [9]) $u_{i,i} = 0$,

(2.1)
$$\rho_0/\epsilon(u_{i,t} + u_j u_{i,j}) + (\mu/k)u_i = -P_{,i} + \mu u_{i,jj} + \rho g k_i,$$
$$(\rho c)_m T_{,t} + (\rho c)_f u_j T_{,j} = \lambda_m T_{,jj}.$$

Here u_i, T, and P are the velocity, temperature, and pressure of the fluid in the porous medium. The properties of the medium are defined as

$$\lambda_m = (1-\epsilon)\lambda_s + \epsilon\lambda_f,$$
$$(\rho c)_m = (1-\epsilon)(\rho c)_s + \epsilon(\rho c)_f.$$

The subscripts, f, m, and s denote the property of the fluid, the porous medium, and the solid matrix. The body force term has a positive sign since we take $\underline{k} = (0,0,1)$ to be the unit vector in the direction of gravity g. The momentum and energy equations are then rendered dimensionless by the following characteristic quantities: length L, temperature difference δT, time $L^2(R_a\mathcal{H}^*)^{-1}$, stream function $(R_a\mathcal{H}^*)^{-1}$, viscosity μ_0, density ρ_0, and pressure $k(R_a\mu_0\mathcal{H}^*)^{-1}$, here $\mathcal{H}^* = \lambda_m(\rho c)_f^{-1}$, and the Rayleigh number is defined as

$$R_a = \frac{g\partial_1 \delta T L}{\nu_0 \mathcal{H}^*},$$

where ∂_1, is the coefficient of thermal expansion, and ν_0 is a kinematic viscosity at the reference temperature of $25°C$. The density of the fluid in the gravity force is given by Ruddick and Shirtcliffe [13] as

$$\rho = \rho_0[1 - \partial_1(T-T_0) - \partial_2(T-T_0)^2 - \partial_3(T-T_0)^3], \tag{2.2}$$

in which

$$\begin{aligned}\partial_1 &= 2.539\times 10^{-4}(^\circ C^{-1}),\\ \partial_2 &= 4.968\times 10^{-6}(^\circ C^{-2}) = O(\partial_1\times 10^{-2}\ ^\circ C^{-1}),\\ \partial_3 &= -2.7\times 10^{-8}(^\circ C^{-3}) = O(\partial_1\times 10^{-4}\ ^\circ C^{-2}),\end{aligned}$$

at the reference temperature of 25°C.

With the introduction of the stream function denoting the z axis points vertically upward, that is, $u=\Psi_{,z}, w=-\Psi_{,x}$ the continuity equation is satisfied identically. Eliminating the pressure term in the momentum equation, we obtain the following nondimensional equations

$$R_a\left[C_1\frac{\partial}{\partial t}+C_2\left(\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial z}\right)\right]\triangle\Psi+\triangle\Psi = \frac{k}{L^2}\triangle(\triangle\Psi)+\frac{1}{\partial_1\delta T}\frac{\partial\rho}{\partial x}, \tag{2.3}$$

where

$$C_1=\frac{k}{\epsilon L^2}\frac{\kappa_m}{\nu_0},\quad C_2=\frac{(\rho c)_m}{(\rho c)_f}C_1,\quad \kappa_m=\frac{\lambda_m}{(\rho c)_m}.$$

and $\triangle$ is the two dimensional Laplace operator. The energy equation becomes

$$\frac{\partial T}{\partial t}+\frac{\partial\Psi}{\partial z}\frac{\partial T}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial z}=\frac{1}{R_a}\triangle T. \tag{2.4}$$

Due to the nonslip condition at the walls, the appropriate boundary conditions for equations (2.3) and (2.4) are

$$\begin{aligned} u &= \Psi_{,z}=0\\ w &= -\Psi_{,x}=0\\ T &= \pm 1/2\end{aligned} \tag{2.5}$$

at $x=\pm 1/2$, Ψ and T are periodic in z direction.

Then the basic steady solution to (2.3) and (2.4) subject to the boundary conditions (2.5) is

$$\overline{T} = x, \tag{2.6}$$

$$\overline{\Psi} = \alpha \cosh \tilde{L}x + \beta \sinh \tilde{L}x + \sum_{j=0}^{4} \gamma_j x^j,$$

in which $\tilde{L} = L/\sqrt{k}$ and α, β, and γ_j are constants which are given in the paper of Kwok and Chen [9], that is,

$$\begin{aligned}
\alpha &= (1 - (\partial_3/\partial_1)\delta T^2(6/\tilde{L}^2 + 1/4))/(2\tilde{L}\sinh(\tilde{L}/2)),\\
\beta &= \partial_2\delta T/(12\partial_1)/((\tilde{L}/2)\cosh(\tilde{L}/2) - \sinh(\tilde{L}/2)),\\
\gamma_0 &= 1/8 - ((\partial_3\delta T^2/\partial_1)(1/4 + 6/\tilde{L}^2) + 1)/(2\tilde{L}^2 tanh(\tilde{L}/2))\\
&\quad + \partial_3\delta T^2/(4\partial_1)(1/16 + 3/\tilde{L}^2),\\
\gamma_1 &= (\partial_2\delta T/\partial_1)(1/4 - 1/6(1 - 2tanh(\tilde{L}/2)/\tilde{L})),\\
\gamma_2 &= -1/2 - 3\partial_3\delta T^2/(\tilde{L}\partial_1),\\
\gamma_3 &= -\partial_2\delta T/(3\partial_1),\\
\gamma_4 &= -\partial_3\delta T^2/(4\partial_1).
\end{aligned}$$

Considering the density-temperature relationship of the equation (2.2), we assume the nondimensional perturbation density

$$\rho = a_1\theta + a_2\theta^2 + a_3\theta^3,$$

where $a_1(x), a_2(x)$, and $a_3(x)$ are ∂_1, ∂_2, and ∂_3, and $T - T_0$ fuctional, and $\theta = T - \overline{T}$. The equations for a perturbation (Ψ, θ) to this problem are reduced to

(2.7)

$$\begin{aligned}
&R_a[C_1\frac{\partial}{\partial t}\triangle\Psi + C_2\{\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial z} - \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial z}\}\triangle\Psi\\
&= -R_aC_2\{\frac{\partial\Psi}{\partial z}\frac{d^3\overline{\Psi}}{dx^3} - \frac{\partial\overline{\Psi}}{\partial x}\frac{\partial}{\partial z}\triangle\Psi\} - \triangle\Psi + \frac{k}{L^2}\triangle(\triangle\Psi)\\
&+ \frac{1}{\partial_1\delta T}\frac{\partial\rho}{\partial x}\cdots\triangle(\triangle\Psi) + \frac{1}{\partial_1\delta T},
\end{aligned}$$

$$\frac{\partial \theta}{\partial t}+\frac{\partial \Psi}{\partial z}\frac{\partial \theta}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial \theta}{\partial z}=-\frac{d\bar{T}}{dx}\frac{\partial \overline{\Psi}}{\partial z}+\frac{d\overline{\Psi}}{dx}\frac{\partial \theta}{\partial z}+\frac{1}{R_a}\triangle\theta. \tag{2.8}$$

The boundary conditions are

$$\Psi_{,x}=\Psi_{,z}=\Theta=0 \quad \text{at} \quad x=\pm 1/2 \tag{2.9}$$

$$\Psi \quad \text{and} \quad \Theta \text{ are periodic in } \quad z \text{ direction.}$$

The initial conditions (data) are here given in terms of

$$\Psi_{,x}=f_1(x,z), \quad \Psi_{,z}=f_2(x,z), \quad \text{and} \quad \theta=g(x,y) \quad \text{at} \quad t=0, \tag{2.10}$$

(see Song [18,19]). Now we are primarily interested in an a *a priori* initial amplitude, establishing an asymptotic stability result, which guarantees an exponential decay of a perturbation (Ψ,θ) to the basic solution (2.6).

3. Nonlinear Energy Stability

To investigate the nonlinear stability of (2.7) and (2.8) with the appropriate periodic bounary conditions (2.9) and the prescribed initial data (2.10), we commence with the energy

$$E_\gamma(t)=1/2(\|\nabla\Psi\|^2+\gamma\|\theta\|^2), \tag{3.1}$$

in which $\gamma(>0)$ is a coupling parameter and $\|\ \|$ is the $L^2(V)$ norm, V being a disturbance cell (see Joseph [7] and Payne et al.[11]). To study a evolutionary behavior of $E_\gamma(t)$, we observe

$$\begin{aligned}\frac{dE_\gamma}{dt}&=\int\int \Psi_{,i}\Psi_{,it}\,dxdz+\gamma\int\int\theta\theta_{,t}\,dxdz\\ &\equiv\frac{dE_1}{dt}+\gamma\frac{dE_2}{dt}.\end{aligned} \tag{3.2}$$

Here the obvious definitions of E_1 and E_2 are made. Then, we have

$$\begin{aligned}\frac{dE_1}{dt}&=\int\int\Psi_{,i}\Psi_{,it}\,dxdz\\ &=-\frac{C_2}{C_1}\int\int\overline{\Psi}_{,xx}\Psi_{,z}\Psi_{,x}\,dxdz-\frac{1}{R_aC_1}\int\int\Psi_{,i}\Psi_{,i}\,dxdz\\ &\quad-\frac{1}{R_aC_1}\frac{k}{L^2}\int\int\Psi_{,ii}\Psi_{,jj}\,dxdz+\frac{1}{\partial_1\delta TR_aC_1}\int\int\Psi_{,x}\,\rho dxdz.\end{aligned}$$

Here we have carried out the obvious integration by parts and application of differential equations and boundary conditions. In a similiar way, we form

$$\frac{dE_2}{dt} = -\int\int \theta\Psi_{,z}\, dxdz - \frac{1}{R_a}\int\int \theta_{,i}\,\theta_{,i}\, dxdz.$$

Then, the evolutionary equation (3.2) of the total energy becomes

(3.3)

$$\begin{aligned}\frac{dE_\gamma}{dt} = &-\frac{C_2}{C_1}\int\int \overline{\Psi}_{,xx}\,\Psi_{,z}\,\Psi_{,x}\, dxdz - \frac{1}{R_aC_1}\int\int \Psi_{,i}\,\Psi_{,i}\, dxdz \\ &- \frac{1}{R_aC_1}\frac{k}{L^2}\int\int \Psi_{,ii}\,\Psi_{,jj}\, dxdz + \frac{1}{\partial_1\delta TR_aC_1}\int\int \Psi_{,x}\,\rho dxdz \\ &- \gamma\int\int \theta\Psi_{,z}\, dxdz - \frac{\gamma}{R_a}\int\int \theta_{,i}\,\theta_{,i}\, dxdz.\end{aligned}$$

In order to derive a differential inequality for E_γ, using (2.2), and substituting for ρ, we may deduce

$$\frac{dE_\gamma}{dt} = -\frac{1}{R_a}D_\gamma + I_\gamma + N. \tag{3.4}$$

Here we define

$$\begin{aligned}D_\gamma &= -\frac{1}{C_1}\int\int \Psi_{,i}\,\Psi_{,i}\, dxdz + \frac{k}{C_1L^2}\int\int \Psi_{,ii}\,\Psi_{,jj}\, dxdz + \gamma\int \theta_{,i}\,\theta_{,i}\, dxdz, \\ I_\gamma &= -\frac{C_2}{C_1}\int\int \overline{\Psi}_{,xx}\,\Psi_{,z}\,\Psi_{,x}\, dxdz \\ &\quad + \frac{1}{\partial_1\delta TR_aC_1}\int\int a_1\Psi_{,x}\,\theta dxdz - \gamma\int\int \theta\Psi_{,z}\, dxdz, \\ N &= \frac{1}{\partial_1\delta TR_aC_1}\{\int\int a_2\Psi_{,x}\,\theta^2 dxdz + \int\int a_3\Psi_{,x}\,\theta^3 dxdz\}.\end{aligned}$$

In fact we group together the terms on the right hand side of (3.3), according to the positive terms of the energy dissipation (D_γ), the indefinite terms of the energy production (I_γ), and nonlinear terms (N) (see Song [14] and Galdi and Straughan [6]).

Following Joseph's idea and procedure [7], we can rewrite (3.4) as

$$\frac{dE_\gamma}{dt} \le -D_\gamma\left(\frac{1}{R_a} - \frac{1}{R_\epsilon}\right) + N, \tag{3.5}$$

where

$$\frac{1}{R_\epsilon} = \max_H \frac{I_\gamma}{D_\gamma}, \tag{3.6}$$

H being the obvious space of admissible functions. Since we are here interested primarily in a nonlinear stability criteria of the prescribed initial amplitude in conjunction with R_ϵ, we just mention a computable sharper number of R_ϵ numerically by using the compound matrix method and the golden section technique (see Drazin and Reid [4], Payne et al. [11] and Song [15]).

Actually, R_ϵ turns out to be the first eigenvalue of these coupled evolutionary equations (2.7) and (2.8) with the periodic boundary conditions (2.9) and the initial data (2.10), indicating the maximum decay rate of the energy E_γ (see Payne et al. [11] and Song [15]). For a nonlinear stability we require $R_a < R_\epsilon$, for then

$$\sigma = R_a^{-1} - R_\epsilon^{-1} > 0. \tag{3.7}$$

To see how this together with (3.5) yields a nonlinear stability, we have to bound D_γ by E_γ and to estimate nonlinear terms of N. Using the one dimensional Poincaré inequality (see Hardy et al. [8]), we then arrive at

$$D_\gamma \ge \frac{1}{C_1}[1 + \frac{2k}{L^2}\pi^2] \int\int \Psi_{,i}\Psi_{,i}\, dx dz + \pi^2 \int\int \theta^2 dx dz > E_\gamma.$$

provided if $C_1 \le 1$, ($C_1 = O\ (10^{-4})$ in the work of Kwok and Chen [9]), and we choose $\gamma = L^2$. Turning to nonlinear terms N, we proceed to estimate N as follows

$$N \le \tilde{a}_2 \int\int \Psi_{,x}\theta^2 dx dz + \tilde{a}_3 \int\int \Psi_{,x}\theta^3 dx dz,$$

in which $\tilde{a}_2 = \frac{1}{\partial_1 \delta T R_a C_1} max|a_2(x)|$, $\tilde{a}_3 = \frac{1}{\partial_1 \delta T R_a C_1} max|a_3(x)|$. Then using the Cauchy-Schwarz inequality, we see

$$N \le \tilde{a}_2 \Big(\int\int \theta^4 dxdz\Big)^{\frac{1}{2}} \Big(\int\int \Psi_{,x}^2\, dxdz\Big)^{\frac{1}{2}} + \tilde{a}_3 \Big(\int\int \theta^6 dxdz\Big)^{\frac{1}{2}} \Big(\int\int \Psi_{,x}^2\, dxdz\Big)^{\frac{1}{2}}.$$

We also estimate that

$$\int\int \theta^4 dxdz \le C \int\int \theta^2 dxdz \int\int \theta_{,i}\, \theta_{,i}\, dxdz \le \frac{C}{\pi}\Big\{\int\int \theta_{,i}\, \theta_{,i}\, dxdz\Big\}^2.$$

where $C = 2 + 4/(\pi p)$, p being a periodicity in z direction (see Galdi et al. [5]), and we used the Poincaré inequality. On the other hand, we now find (see Appendix)

$$\int\int \theta^6 dxdz \le \{(\frac{3}{2p\pi} + \frac{9}{4})C \int\int \theta^2 dxdz\}\{\int\int \theta_{,i}\, \theta_{,i}\, dxdz\}^2, \tag{3.8}$$

Using these estimations for nonlinear terms N, we now further rewrite (3.5) as

$$\frac{dE_\gamma}{dt} \le -\sigma D_\gamma + \left[\tilde{a}_2 (\frac{2C}{\pi} E_\gamma)^{\frac{1}{2}} + \tilde{a}_3 \left((\frac{3}{2p\pi} + \frac{9}{4}) C \frac{4E_{\gamma^2}}{\gamma}\right)^{\frac{1}{2}}\right] \frac{D_\gamma}{\gamma}$$
$$\le -\left[\gamma\sigma - \sqrt{2}\tilde{a}_2 (\frac{C}{\pi} E_\gamma)^{\frac{1}{2}} - \tilde{a}_3 \left(\frac{6}{p\pi} + 9\right)^{\frac{1}{2}} \left(\frac{C}{\gamma}\right)^{\frac{1}{2}} E_\gamma\right] \frac{D_\gamma}{\gamma},$$

provided

$$\gamma\sigma - \sqrt{2}\tilde{a}_2 (\frac{C}{\pi} E_\gamma(0))^{\frac{1}{2}} - \tilde{a}_3 \left(\frac{6}{p\pi} + 9\right)^{\frac{1}{2}} \left(\frac{C}{\gamma}\right)^{\frac{1}{2}} E_\gamma(0) \ge 0. \tag{3.9}$$

For then $E_\gamma(t)$ decays exponentially in time. The above results show that if the restriction of (3.9) on the prescribed initial amplitude meaning that $E_\gamma(0)$ is actually $\int\int (f_1^2 + f_2^2 + \gamma g^2) dxdz$ is met and $R_a < R_\epsilon$, then the solution is also nonlinearly stable.

4. Conclusions

In this paper we readdress the thermal convection problem in the vertical porous layer, studied by Kwok and Chen [9], by using integral inequality estimates on the full nonlinear theory of the cubic density-temperature relationship and the nonlinear inertia terms over the whole of the two dimensional cell. We believe that this mathematical analysis adds insight into the physical phenomena, although we have not computed the critical Rayleigh number R_ϵ numerically. Futhermore, as in earlier work of Straughan [20], in the present investigation of a nonlinear situation, we have shown that we have established the nonlinear stability criteria (3.7), guaranteeing an exponential decay of disturbances to the basic solution, provided the prescribed initial amplitude are sufficiently small enough to satisfy the Rayleigh number hypothesis (3.9).

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Appendix

Here we will briefly indicate a means for obtaining a bound for $||\theta^3||^2$ defined in (3.8), over the two dimensional cell. Since $\theta(-1/2, z) = \theta(1/2, z) = 0$, we have

$$\theta^3(x,z) = 3\int_{-1/2}^{x} \theta^2(\xi,z)\theta_{,\xi}(\xi,z)d\xi$$
$$= -3\int_{x}^{1/2} \theta^2(\xi,z)\theta_{,\xi}(\xi,a)d\xi,$$
$$\theta^3(x,z) \leq 3\int_{-1/2}^{x} \theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi,$$

$$\theta^3(x,z) \le 3\int_x^{1/2} \theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi,$$

$$\text{and } |\theta|^3(x,z) \le \frac{3}{2}\int_{-1/2}^{1/2} \theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi.$$

In the same way, since $\theta(x,z_1) = \theta(x,z_2)$ where $z_2 = z_1 + p$, p being a periodicity in z direction, we find

$$|\theta|^3(x,z) \le |\theta|^3(x,z_1) + \frac{3}{2}\int_{z_1}^{z_2} \theta^2(x,\eta)|\theta_{,\eta}(x,\eta)|d\eta.$$

Combining these expression and integrating over $(z_1,z_2)\times(-1/2,1/2)$, we thus have

$$\begin{aligned}\int_{z_1}^{z_2}\int_{-1/2}^{1/2} \theta^6(x,z)dxdz &\le \int_{z_1}^{z_2}\int_{-1/2}^{1/2} \{|\theta|^3(x,z_1)\frac{3}{2}\int_{-1/2}^{1/2} \theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi\}dxdz\\ &+\int_{z_1}^{z_2}\int_{-1/2}^{1/2}\{\frac{3}{2}\int_{-1/2}^{1/2}\theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi\frac{3}{2}\int_{z_1}^{z_2}\theta^2(x,\eta)|\theta_{,\eta}(x,\eta)|d\eta\}dxdz\\ &= I_1 + I_2.\end{aligned}$$

Here the obvious definitions of I_1 and I_2 are made. First, we see

$$\begin{aligned}I_1 &= \frac{3}{2}\int_{z_1}^{z_2}\int_{-1/2}^{1/2}\theta^2(\xi,z)|\theta_{,\xi}(\xi,z)|d\xi dz\int_{-1/2}^{1/2}|\theta^3(x,z_1)|dx\\ &\le \frac{3}{2}(\int\int\theta^4 d\xi dz)^{\frac{1}{2}}(\int\int\theta_{,\xi}\theta_{,\xi}d\xi dz)^{\frac{1}{2}}\int_{-1/2}^{1/2}|\theta^3(x,z_1)|dx\\ &\le \frac{3}{2}(\int\int\theta^4 d\xi dz)^{\frac{1}{2}}(\int\int\theta_{,\xi}\theta_{,\xi}d\xi dz)^{\frac{1}{2}}\frac{1}{p}\int_{z_1}^{z_1+p}\int_{-1/2}^{1/2}|\theta^3(x,z_1)|dxdz\\ &\le \frac{3}{2p}(\int\int\theta^4 d\xi dz)^{\frac{1}{2}}(\int\int\theta_{,\xi}\theta_{,\xi}d\xi dz)^{\frac{1}{2}}(\int\int\theta^4 dxdz)^{\frac{1}{2}}(\int\int\theta^2 dxdz)^{\frac{1}{2}}\\ &\le \frac{3}{2p}\int\int\theta^4 d\xi dz(\int\int\theta_{,\xi}\theta_{,\xi}d\xi dz)^{\frac{1}{2}}(\int\int\theta^2 dxdz)^{\frac{1}{2}}\\ &\le \frac{3}{2p}\frac{C}{\pi}\int\int\theta^2 d\xi dz(\int\int\theta_{,\xi}\theta_{,\xi}d\xi dz)^2.\end{aligned}$$

Here we have used the Poincaré inequality, and the fact

$$\int\int \theta^4 dx dz \le C \int\int \theta^2 dx dz \int\int \theta_{,i}\, \theta_{,i}\, dx dz,$$

where $C = 2 + 4/(\pi p)$ (see Galdi et al. [4]). Second, in the similar way we estimate

$$\begin{aligned} I_2 \le \frac{9}{4}(\int_{z_1}^{z_2}\int_{-1/2}^{1/2} \theta^4(\xi,z) d\xi dz)^{\frac{1}{2}}(\int_{z_1}^{z_2}\int_{-1/2}^{1/2} \theta_{,\xi}(\xi,z)\theta_{,\xi}(\xi,z) d\xi dz)^{\frac{1}{2}} \\ (\int_{-1/2}^{1/2}\int_{z_1}^{z_2} \theta^4(x,\eta) d\eta dx)^{\frac{1}{2}}(\int_{-1/2}^{1/2}\int_{z_1}^{z_2} \theta,\eta(x,\eta)\theta,\eta(x,\eta) d\eta dx)^{\frac{1}{2}} \\ \le \frac{9}{4} C \int\int \theta^2 d\xi dz (\int\int \theta_{,\xi}\, \theta_{,\xi}\, d\xi dz)^2. \end{aligned}$$

Using these estimations for I_1 and I_2, we are then end up with

$$\int_{z_1}^{z_2}\int_{-1/2}^{1/2} \theta^6(x,y) dx dy \le (\frac{3}{2p\pi} + \frac{9}{4}) C \int\int \theta^2 d\xi dz (\int\int \theta_{,\xi}\, \theta_{,\xi}\, d\xi dz)^2.$$

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