J. Korean Math. Soc. 30(1993), No. 1, pp. 173–183

STRUCTURE THEOREM OF ULTRADISTRIBUTIONS WITH COMPACT SUPPORT

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0. Introduction.

Let M_p, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers satisfying $(M.1)$, $(M.2)$ and $(M.3)$ given in §1. Let K be a compact set in R^n and let $\mathcal{E}_{(M_p)}{}'(K)$ and $\mathcal{E}_{\{M_p\}}{}'(K)$ denote the space of all ultradistributions with compact support in K of Beurling type and of Roumieu type, respectively (in §1). Hereafter M_p will designate both (M_p) and $\{M_p\}$ unless stated otherwise.

In this paper, we prove the following structure theorem for ultradistributions supported by K of class M_p:

$$u \in \mathcal{E}_{M_p}{}'(K)$$

if and only if there exist an ultradifferential operator $P(d/dt)$ of class M_p (in §2) and bounded continuous functions $g(x)$ and $h(x)$ such that

$$\begin{aligned} P(d/dt) &= \sum_{q=0}^{\infty} a_q (d/dt)^q, \quad \text{with } |a_q| \le CL^q/M_q; \\ u &= P(\Delta)g + h, \end{aligned}$$

where $g(x) \in C^\infty(R^n \backslash K)$, $h(x) \in C^\infty(R^n)$ and $P(\Delta)g(x) + h(x) = 0$ in $R^n \backslash K$. Here $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ denotes the Laplace operator. We shall prove it by using the heat kernel method, introduced by T. Matsuzawa [4], [5] who proved the case of distributions. The

Received March 21, 1992.
Partially supported by Daewoo Foundation* and by the Ministry of Education**,***

similar structure theorem for $\mathcal{E}_{M_p}'(K)$ has been investigated earlier by H. Komatsu [2], [3], but using the heat kernel method we improve it in a more concrete form.

In §1 we introduce necessary definitions and their basic properties. In §2 we construct an ultradifferential operator $P(d/dt)$ of class M_p and introduce the heat kernel method, which are essential in the proof of the main theorem of this paper. Also, we extend the results of T. Matsuzawa [4] for the case of Gevrey class to the space of ultradistributions with compact support of class M_p. Finally, in §3 we prove the above-mentioned structure theorem for ultradistributions with compact support.

1. Definitions and Basic Properties.

Let M_p, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers and assume $M_0 = 1$. We will impose the following conditions on M_p:
$(M.1)$ (Logarithmic convexity)

$$M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1, 2, \cdots .$$

$(M.2)$ (Stability under ultradifferential operators)

There are positive constants A and H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q}, \quad p = 0, 1, 2, \cdots .$$

$(M.3)$ (Strong non-quasi-analyticity)

There is a positive constant A such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le ApM_p/M_{p+1}, \quad p = 1, 2, \cdots .$$

DEFINITION 1.1. We denote by $\mathcal{E}_{(M_p)}(R^n)$ (resp. $\mathcal{E}_{\{M_p\}}(R^n)$) the space of all infinitely differentiable functions φ satisfying the following condition:
For every compact subset K in R^n and for every $h > 0$ there exists $C > 0$ (resp. there exist constants $h, C > 0$) depending on φ and K such that

$$(1.1) \qquad \sup_{x \in K} |\partial^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \cdots$$

where for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}; \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Such a function φ is called an *ultradifferentiable function.* Also, we denote by $\mathcal{D}_{M_p}(R^n)$ the space of all $\varphi \in \mathcal{E}_{M_p}(R^n)$ with compact support. Then $\mathcal{D}_{M_p}(R^n)$ is a closed subspace of $\mathcal{E}_{M_p}(R^n)$. Elements of the strong dual space of $\mathcal{E}_{(M_p)}(R^n)$ (resp. $\mathcal{E}_{\{M_p\}}(R^n)$) are called *ultradistributions with compact support of class* (M_p) (resp. *of class* $\{M_p\}$) *of Beurling type* (resp. *of Roumieu type*). The space of ultradistributions supported by K is denoted by $\mathcal{E}_{M_p}{}'(K)$

DEFINITION 1.2. For each sequence M_p of positive numbers we define an *associated function* $M(\rho)$ on $(0, \infty)$ by

$$M(\rho) = \sup_{p \geq 0} \log (\rho^p / M_p).$$

and denote by $M^*(\rho)$ the associated function of $M_p^* = M_p / p!$.

We can easily see that $M(\rho)$ is non-decreasing and vanishes for sufficiently small $\rho > 0$, and we have

$$(1.2) \qquad M_p^* \leq C B^p M_p; \; M(\rho) \leq M^*(B\rho) + \log C \quad \text{for any } B > 0.$$

If $m(\lambda)$ denotes the number of $m_p = M_p/M_{p-1} \le \lambda$ then we have

$$m(\lambda) = \lambda \frac{dM(\lambda)}{d\lambda} = \frac{dM(\lambda)}{d\log\lambda}. \tag{1.3}$$

Also, if the sequence M_p satisfies $(M.1)$, then $(M.3)$ implies that for any $L > 0$ there exists a constant $C > 0$ such that

$$p! \le CL^p M_p, \quad p = 0, 1, 2, \cdots. \tag{1.4}$$

For the details of these properties we refer to [3].

2. Characterization of $\mathcal{E}_{M_p}{}'(K)$.

Hereafter we always assume that the sequence M_p satisfies conditions $(M.1)$, $(M.2)$ and $(M.3)$ given in §1.

DEFINITION 2.1. An operator of the form

$$P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha, \qquad a_\alpha \in \mathcal{C},\ |\alpha| = 0, 1, 2, \cdots$$

is called an *ultradifferential operator of class* (M_p) (resp. *of class* $\{M_p\}$) if there are positive constants L, C (resp. for every $L > 0$ there is a constant $C > 0$) such that

$$|a_\alpha| \le CL^{|\alpha|}/M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \cdots. \tag{2.1}$$

THEOREM 2.2. *There exist a function $v(t) \in C_0^\infty(R)$ and an ultradifferential operator $P(d/dt)$ of class M_p such that*

$$P(d/dt)v(t) = \delta(t) + \omega(t); \tag{2.2}$$

$$|\, v(t)\, | \le C\exp[-M^*(L/t)], \tag{2.3}$$

where $\omega(t) \in C_0^\infty(R)$ *with supp* $\omega \subset [0,2]$ *and* L *is the constant in* (2.1).

Proof. We shall only give a sketch of the proof. If we set

$$(2.4) \qquad P(\zeta) = (1+\zeta)^2 \prod_{q=1}^{\infty} [1 + l_q\zeta/m_q];$$

$$(2.5) \qquad V(z) = \frac{1}{2\pi i}\int_0^{\infty} \exp(z\zeta)P(\zeta)^{-1}d\zeta,$$

where l_q is a constant $L > 0$ (resp. a sequence of complex numbers which converges to zero as $q \to \infty$), then the formula (2.4) is an entire function which defines an ultradifferential operator $P(d/dt)$ of class (M_p) (resp. of class $\{M_p\}$) in virtue of $(M.2)$ and $(M.3)$. In fact, we note that $(M.2)$ and $(M.3)$ can be expressed in terms of the associated function $M(\rho)$ as in [3]. Also, the integral (2.5) converges absolutely for $Re\ z < 0$; and thus is a holomorphic function which can be analytically continued to Riemann domain $\{z | -\pi/2 < \arg z < 5\pi/2\}$. If we set

$$v_1(t) \ = \ V(t+i0) - V(t-i0),$$

then we have

$$|v_1^{(p)}(t)| \ \le \ M_p/2L_1^p;$$

$$P(d/dt)v_1(t) \ = \ \delta(t);$$

$$v_1(t) = 0 \ \text{ for } t < 0; \quad v_1(t) \ge 0 \ \text{ for } t \ge 0 \quad \text{ and}$$

$$\int_{-\infty}^{\infty} v_1(t)dt \ = \ P(i\eta)^{-1}|_{\eta=0} \ = \ 1,$$

where $L_1 = L$ (resp. L_1 is any positive constant). Also, we have for $t > 0$

$$v_1(t) \ \le \ \inf_p \ [M_p^*/(L_1/t)^p] \ = \ \exp[-M^*(L_1/t)].$$

Hence, multiplying $v_1(t)$ by a suitable C_0^∞ function, we obtain $v(t)$ satisfying (2.2) and (2.3). For the details of the proof we refer to [1], [3] and [4].

Now, let $E(x,t)$ be the n-dimensional heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right), & t > 0 \\ 0 & , \quad t < 0. \end{cases}$$

Then $E(\cdot\,,t)$ is an entire function for each $t > 0$ satisfying

$$\int_{R^n} E(x,t)dx = 1, \qquad t > 0 \tag{2.6}$$

and there exist positive constants C and a such that

$$|\partial_x^\alpha E(x,t)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} (\alpha!)^{1/2} \exp\left(-a|x|^2/4t\right), \quad t > 0 \tag{2.7}$$

where a can be taken as close as desired to 1 and $0 < a < 1$.

LEMMA 2.3. *For every $\varphi \in \mathcal{D}_{M_p}(R^n)$, let*

$$\varphi_t(x) = \chi(x) \int_{R^n} E(x-y,t)\varphi(y)dy,$$

where $\chi \in \mathcal{D}_{M_p}(R^n)$ such that $\chi = 1$ in an open neighborhood of supp φ. Then $\varphi_t \to \varphi$ in $\mathcal{D}_{M_p}(R^n)$ as $t \to 0^+$.

Proof. It can be shown without any difficulty by considering:

$$\begin{aligned}\partial_x^\alpha [\varphi_t(x) - \varphi(x)] &= \int_{|y|\le\delta} E(y,t)\partial_x^\alpha [\chi(x)\{\varphi(x-y) - \varphi(x)\}]\, dy \\ &+ \int_{|y|>\delta} E(y,t)\partial_x^\alpha [\chi(x)\{\varphi(x-y) - \varphi(x)\}]\, dy\end{aligned}$$

In fact, the first integral can be estimated by using the mean value theorem and $(M.2)$, and the second one can be estimated by the Leibniz' formula and the fact that $\int_{|y|>\delta} E(y,t)dt \to 0$ as $t \to 0^+$.

Let $u \in \mathcal{E}_{M_p}'(K)$. Then the function

$$U(x,t) = u_y(E(x-y,t)), \quad x \in R^n,\ t > 0$$

is well-defined, since $E(x - \cdot\,, t)$ is an entire function in C^n for every $(x,t) \in R_+^{n+1} = \{(x,t) |\ x \in R^n, 0 < t < \infty\}$; and thus $E(x - \cdot\,, t) \in \mathcal{E}_{M_p}(R^n)$ in virtue of (1.4). Then we shall call it a *defining function* of $u \in \mathcal{E}_{M_p}'(K)$. Then using Lemma 2.3 we characterize $u \in \mathcal{E}_{M_p}'(K)$ by the asymptotic behavior of the defining function $U(\cdot\,, t)$ as $t \to 0^+$ due to T. Matsuzawa [4].

THEOREM 2.4. *Let $u \in \mathcal{E}_{M_p}'(K)$. Then $U(x,t) = u_y(E(x-y,t))$ is infinitely differentiable in $R_+^{n+1} = \{(x,t) | x \in R^n, 0 < t < \infty\}$ and satisfies the following conditions:*

$$(2.8) \qquad (\partial/\partial t - \Delta)U(x,t) = 0 \quad \text{in } R_+^{n+1};$$

For every $\delta > 0$ there exist positive constants ε, $C = C(\delta)$ (resp. For every ε, $\delta > 0$ there exists a constant $C = C(\varepsilon, \delta) > 0$) such that we have

$$(2.9) \qquad |U(x,t)| \leq C \exp\left[M(\varepsilon/t) - \operatorname{dist}(x, K_\delta)^2/8t\right] \quad \text{in } R_+^{n+1},$$

where $K_\delta = \{x \in R^n |\ \operatorname{dist}(x,K) \leq \delta\}$;
$U(\cdot\,, t) \to u$ in $\mathcal{E}_{M_p}'(K)$ as $t \to 0^+$ in the following sense:

$$(2.10) \qquad u(\varphi) = \lim_{t \to 0+} \int_\Omega U(x,t)\varphi(x)dx, \quad \varphi \in \mathcal{E}_{M_p}(R^n),$$

where Ω is any bounded open neighborhood of K.

Proof. It is clear that for $u \in \mathcal{E}_{M_p}'(K)$

$$U(x,t) = u_y(E(x-y,t)) \in C^\infty(R_+^{n+1})$$

and satisfies the heat equation (2.8). First, let $u \in \mathcal{E}_{(M_p)}'(K)$. Then for any $\delta > 0$ there exist positive constants h, $C = C(\delta)$ such that

$$|u(\varphi)| \leq C \sup_{\substack{x \in K_\delta \\ \alpha}} |\partial^\alpha \varphi(x)|/(h^{|\alpha|} M_{|\alpha|}), \quad \varphi \in \mathcal{E}_{(M_p)}(R^n).$$

Then using (2.7) and (1.4), we have for each $(x,t) \in R^{n+1}_+$

$$\begin{aligned} |U(x,t)| &\leq C \sup_{\substack{y \in K_\delta \\ \alpha}} |\partial_y^\alpha E(x-y,t)|/(h^{|\alpha|} M_{|\alpha|}) \\ &\leq C_1 \sup_\alpha \left[\frac{((C/h)^2/t)^{|\alpha|}}{M_{|\alpha|}} \cdot \frac{\alpha!}{M_{|\alpha|}} \right]^{1/2} \sup_{y \in K_\delta} \exp\left(-\frac{|x-y|^2}{8t} \right) \\ &\leq C_2 \exp\left[M(\varepsilon/t) - \mathrm{dist}(x, K_\delta)^2/8t \right]. \end{aligned}$$

We thus obtain (2.9) with $\varepsilon = LC^2/h^2$ for the same $L > 0$ as in (1.4). A simple modification of the above proof gives the result (2.9) for $\mathcal{E}_{\{M_p\}}'(K)$.

Now, to prove (2.10) let $\Omega \subset R^n$ be any bounded open set containing K. It suffices to prove that (2.10) holds for $\varphi \in \mathcal{E}_{M_p}(R^n)$ with compact support in K. We set

$$G(y,t) = \chi(y) \int_\Omega E(x-y,t)\varphi(x)dx,$$

where $\chi \in \mathcal{D}_{M_p}(R^n)$ such that $\chi = 1$ in an open neighborhood of supp φ. Then Lemma 2.3 implies that

$$(2.11) \qquad G(\cdot\,,t) \longrightarrow \varphi \ \text{ in } \mathcal{D}_{M_p}(R^n) \quad \text{as } t \longrightarrow 0^+.$$

Also we have

$$u_y(G(y,t)) = \int_\Omega U(x,t)\varphi(x)dx.$$

Finally taking the limit $t \to 0^+$ in both sides, we obtain (2.10) from (2.11). This completes the proof of Theorem 2.4.

REMARK. The estimate (2.9) in Theorem 2.4 is a generalization of Theorem 2.1 of T. Matsuzawa [4] for ultradistributions with compact support of class M_p.

3. Structure Theorem for $\mathcal{E}_{M_p}{}'(K)$.

Now we are in a position to state and prove the main theorem in this paper.

THEOREM 3.1. *If $u \in \mathcal{E}_{M_p}{}'(K)$, there exist an ultradifferential operator P of class M_p and bounded continuous functions $g(x)$ and $h(x)$ such that*

$$u = P(\Delta)g + h \tag{3.1}$$

where $g(x) \in C^\infty(R^n \backslash K)$, $h(x) \in C^\infty(R^n)$, $P(\Delta)g(x) + h(x) = 0$ in $R^n \backslash K$ and $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ denotes the Laplace operator in R^n.

Proof. Consider the ultradifferential operator $P(d/dt)$ and the corresponding functions $v(t)$ and $\omega(t)$ obtained in Theorem 2.2:

$$P(d/dt)v(t) = \delta(t) + \omega(t).$$

For the defining function $U(x,t)$ of $u \in \mathcal{E}_{M_p}{}'(K)$ we set

$$\tilde{U}(x,t) = \int_0^\infty U(x, t+s)v(s)ds, \quad (x,t) \in R_+^{n+1}.$$

Then, in virtue of (2.3) and (2.9) $\tilde{U}(x,t)$ is a bounded continuous function in $\overline{R_+^{n+1}}$ satisfying the heat equation:

$$(\partial/\partial t - \Delta)\tilde{U}(x,t) = 0 \quad \text{in } R_+^{n+1}.$$

In fact, it follows from (1.2), (2.3) and (2.9) that

$$\begin{aligned} |\tilde{U}(x,t)| &\le C\int_0^2 \exp\left[M\left(\frac{\varepsilon}{(t+s)}\right) - \frac{\operatorname{dist}(x,K_\delta)^2}{8(t+s)}\right]\exp\left[-M^*\left(\frac{L}{s}\right)\right] ds \\ &\le C'\int_0^2 \exp[M^*(B\varepsilon/(t+s)) - M^*(L/s)]. \end{aligned}$$

Then choosing B so that $0 < B < L/\varepsilon$, $\tilde{U}(x,t)$ is uniformly bounded on $\overline{R^{n+1}_+}$. Thus $\tilde{U}(x,t)$ is continuous in $\overline{R^{n+1}_+}$. Hence, put $g(x) = \tilde{U}(x,0)$. Then $g(x)$ is bounded and continuous on R^n and we obtain

$$\tilde{U}(x,t) = \int_{R^n} E(x-y,t)g(y)dy, \qquad t > 0,$$

from the uniqueness of solutions in the Cauchy problem.

On the other hand, we have for $t > 0$

$$P(-\Delta)\tilde{U}(x,t) = P(-\partial/\partial t)\tilde{U}(x,t) = U(x,t) + \int_0^\infty U(x,t+s)\omega(s)ds.$$

Rewriting $P(-d/dt)$ by $P(d/dt)$ and $h(x) = -\int_0^\infty U(x,s)\omega(s)ds \in C^\infty(R^n)$, we have

$$u = \lim_{t\to 0+} U(\cdot\ ,t) = P(\Delta)g + h.$$

where $P(\Delta)g(x) + h(x) = 0$ for $x \notin K$ in virtue of (2.9). The proof is complete.

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