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INCLUSION THEOREMS OF SYMMETRIC SPACES

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1. Introduction and Definitions.

Many examples in Banach space theory are symmetric spaces which consist of measurable functions in a measure space $(\Omega, \sum, \mu)$. Systematic work on symmetric spaces can be found in the long series of papers by W.A.J. Luxemburg and A.C. Zaanen. They defined a symmetric space using a norm which is defined on all positive measurable functions and imposed the Fatou property on that norm. We can find a few slightly different definitions for symmetric spaces in other literatures. For example, a symmetric space in [4] (called a r.i. space) is defined in a rather technical way. E.M. Semenov showed $L_1 \cap L_\infty \subset E \subset L_1 + L_\infty$ for every köthe function space E with some additional conditions which were taken as a part of definition in [4,p117]. We define a symmetric space below and show some imbedding theorems which include the Semenov result. Like the usual way, we restric our attention to the case in which $(\Omega, \sum, \mu)$ is a finite or infinte interval on the real line with Lebesgue measure since seperable measure spaces, in general, can be mapped to a finite or infinite interval and to a point set with the same mass by a measure isomorphism.

We say that f is equimeasurable with g on $(\Omega, \sum, \mu)$ when $\mu\{x : |f(x)| > t\} = \mu\{x : |g(x)| > t\}$ for all t. We denote f^* to be the non-increasing and right continuous function which is equimeasurable with f. For an explicit formula for f^*, we have

$$f^*(t) = \inf\{y \geq 0; \mu(\{x : |f(x)| > y\}) \leq t\}. \qquad ([3])$$

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DEFINITION 1. A Banach space E consisting of measurable functions on I with the norm $\|\cdot\|_E$ (I is either $[0,1]$ or $[0,\infty]$ with Lebesgue measure) is called a symmetric space if it satisfies the following conditions:

1) If f is in E and $|g(x)| < |f(x)|$ a.e. on I, then g is in E and $\|g\|_E \le \|f\|_E$.
2) If f is in E and $g(x)$ is equimeasurable with $f(x)$, then g is in E and $\|g\|_E = \|f\|_E$.
3) If $\{f_n\}_{n=1}^{\infty}$ and f are non-negative elements of E such that $f_n \uparrow f$ a.e., then $\|f_n\|_E \to \|f\|_E$.

We know that condition 3 in the above definition is weaker than the Fatou property (if $f_n \uparrow f$ for $f_n \in E$, then $\|f_n\|_E \to \|f\|_E$) and is equivalent to the fact that the associate space of E is a norming subspace of the dual space E^*. Condition 1 and 2 can be replaced by a single one, that is, if $f \in E$ and $g^*(t) \le f^*(t)$ for all t, then $g \in E$ and $\|g\|_E \le \|f\|_E$.

DEFINITION 2. ([1, LEMMA 2.3.1.]). 1) A Banach couple (A, B) is two Banach spaces A and B which are algebraically and topologically imbedded into a Hausdorff topological vector space Γ.

2) For a given Banach couple (A, B), the intersection $A \cap B$ consists of the elements common to A and B. Its norm is defined by

$$\|f\|_{A\cap B} = \text{Max}(\|f\|_A, \|f\|_B).$$

The sum $A + B$ consists of elements of the form $f = g + h$ where $g \in A$, $h \in B$ and is equipped with the norm

$$\|f\|_{A+B} = \inf\{\|g\|_A + \|h\|_B\},$$

where the infimum is taken over all $f = g + h$, $g \in A$ and $h \in B$.

DEFINITION 3. ([4, DEFINITION 1.F.4.]). Let $1 < p,\ q < \infty$. A symmetric space E is said to satisfy an upper p-estimate, respectively, lower q-estimate if there exists a constant $M < \infty$ such that, for every choice of pairwise disjoint elements $\{f_i\}_{i=1}^{n}$ in E, we have

$$\|\sum_{i=1}^{n} f_i\| \le M(\sum_{i=1}^{n} \|f_i\|^p)^{\frac{1}{p}},$$

respectively

$$\|\sum_{i=1}^{n} f_i\| \geq M^{-1}(\sum_{i=1}^{n} \|f_i\|^q)^{\frac{1}{q}}.$$

DEFINITION 4. ([4, DEFINITION 1.D.3.]). Let $1 \leq p,\ q \leq \infty$. A symmetric space E is said to be p-convex, respectively, q-concave if there exists $M < \infty$ such that

$$\|(\sum_{i=1}^{n} |f_i|^p)^{\frac{1}{p}}\| \leq M(\sum_{i=1}^{n} \|f_i\|^p)^{\frac{1}{p}}$$

respectively,

$$\|(\sum_{i=1}^{n} |f_i|^q)^{\frac{1}{q}}\| \geq M^{-1}(\sum_{i=1}^{n} \|f_i\|^q)^{\frac{1}{q}}.$$

When $p = \infty$, we understand the above inequality as involving the l_∞ norm of the quantities.

2. Including of a symmetric space.

For given Banach spaces E_0 and E_1, we denote $E_0 \subset E_1$ when the inclusion map is continuous.

LEMMA 1. *Let f be $\sum_{i=1}^{n} a_i \chi_{A_i}$, where A_i's are pairwise disjoint and have the same measure $1/n$.*

1) *If E is a q-concave symmetric space with constant M_1 and $\alpha = M_1\|\chi_{[0,1]}\|_E$, then $\|f\|_E \leq \alpha\|f\|_{L_q}$.*
2) *If E is a p-convex symmetric space with constant M_2 and $\beta = M_2\|\chi_{[0,1]}\|_E^{-1}$, then $\|f\|_{L_p} \leq \beta\|f\|_E$.*

Proof. 1) Let $f_k = \sum_{i=1}^{n} a_{i+k(mod\, n)}\chi_{A_i}$, for $k = 1, \cdots, n$. Then each f_k is equimeasurable with f and has the same E-norm $\|f\|_E$. By disjointness of A_i, we have $\sum_{k=1}^{n} |f_k|^q = (\sum_{i=1}^{n} |a_i|^q)\chi_{\cup A_i}$. Thus,

$$\begin{aligned}
\|f\|_E &= (n\|f\|_E^q/n)^{1/q} = (\sum_{k=1}^{n} \|f_k\|_E^q/n)^{1/q} \\
&\leq M_1\|(\sum_{k=1}^{n} |f_k|^q/n)^{1/q}\|_E \\
&= M_1\|(\sum_{i=1}^{n} |a_i|^q/n)^{1/q}\chi_{\cup A_i}\|_E \\
&= M_1(\sum_{i=1}^{n} |a_i|^q/n)^{1/q}\|\chi_{[0,1]}\|_E \\
&= \alpha\|f\|_{L_q}.
\end{aligned}$$

2) In the same way as above, we can prove 2.

LEMMA 2. *Let E be a symmetric space. Let $\{f_n\}_{n=1}^{\infty}$ be a norm cauchy sequence in E. If $f_n \uparrow f$ a.e., then f is in E and f_n norm converges to f.*

Proof. Suppose f_n converges to g in E. Then we have $f_n \leq g$ a.e. for all n. If not, then there exitst $r > 0$ and N such that the measure of the set $A = \{x : f_N(x) - g(x) > r\}$ is strictly positive. By the monotonicity of f_n, we have $f_n(x) - g(x) > r$ for all $n > N$ on the set A. Then

$$\|g - f_n\|_E = \|\ |g - f_n|\ \|_E \geq \|r\chi_A\|_E > 0.$$

This is a contradiction. Hence $f_n \leq g$ and $f \leq g$ a.e. since $f_n \to f$ a.e.. Since $g - f_n \geq g - f \geq 0$ a.e. and $g - f_n$ is in E, f is in E and $\|g - f\|_E \leq \|g - f_n\|_E \to 0$. Thus, we have $f = g$ a.e. and f_n norm converges to f.

THEOREM 3. *Let $1 \leq p \leq q \leq \infty$.*

1) *If E is a symmetric space which satisties an upper p-estimate and is q-concave with constants M_p, M_q respectively, then $L_p \cap L_q \subset E$.*
2) *If E is a symmetric space which satisfies a lower q-estimate and is p-convex with constants M_p^1, M_q^1 respectively, then $E \subset L_p + L_q$.*

Proof. 1) Suppose $f \in L_p \cap L_q$. First, we show $f \in E$. Define, for all positive integers n,

$$f_n = \sum_{i=1}^{2^n} f^*(i/2^n)\chi_{[(i-1)/2^n, i/2^n]}.$$

Then $f_n \uparrow f^*\chi_{[0,1]}$. Since L_q is σ-order continuous, we have $f_n \to f^*\chi_{[0,1]}$ in L_q and so, $\{f_n\}$ is cauchy in L_q. Since E is q-concave, lemma 1 gives that $\{f_n\}$ is cauchy in E. Thus, we have $f^*\chi_{[0,1]}$ is in E by lemma 2. Define

$$g_n = \sum_{i=1}^{n} f^*(i)\chi_{[i,i+1]}.$$

Since $\sum_{i=1}^{n} f^*(i) \cdot \chi_{[i-1,i]} \leq f^*$ and $f^* \in L_p$,

$$\|\sum_{i=1}^{\infty} f^*(i)\chi_{[i-1,i]}\|_{L_p} = [\sum_{i=1}^{\infty}\{f^*(i)\}^p]^{1/p} < \infty.$$

This shows that $\{g_n\}$ is a cauchy seqence in E. Indeed, upper p-estimateness of E gives that

$$\begin{aligned}\|g_n - g_m\|_E &= \|\sum_{i=m+1}^{n} f^*(i)\chi_{[i,i+1]}\|_E \\ &\leq M_p \left[\sum_{i=m+1}^{n} \{f^*(i)\}^p\right]^{1/p} \|\chi_{[0,1]}\|_E.\end{aligned}$$

Note that $g_n \uparrow \sum_{i=1}^{\infty} f^*(i)\chi_{[i,i+1]}$. Therefore, $\sum_{i=1}^{\infty} f^*(i)\chi_{[i,i+1]}$ is in E by lemma 2. This implies that $f^*\chi_{[1,\infty]}$ is also in E, since $f^*\chi_{[1,\infty]} \leq \sum_{i=1}^{\infty} f^*(i)\chi_{[i,i+1]}$. Thus we have $f^* \in E$ and hence $f \in E$.

Now, let $\|f^*\chi_{[0,1]}\|_E = I_1$ and $\|f^*\chi_{[1,\infty]}\|_E = I_2$. In order to show that $I_1 \leq c_1\|f\|_{L_q}$ and $I_2 \leq c_2\|f\|_{L_p}$, for some constant c_1 and c_2, we first show $\|f^*\chi_{[\frac{1}{n},1-\frac{1}{n}]}\|_E \le c_1\|f\|_{L_q}$ and $\|f^*\chi_{[1,n]}\|_E \le \|c_2\|f\|_{L_p}$, for all positive n. Since E is q-concave, by applying lemma 1, we have, for all positive n,

$$
\begin{aligned}
(1)\qquad \|f^*\chi_{[\frac{1}{n},1-\frac{1}{n}]}\|_E &\le \|\sum_{i=1}^{n-2} f^*(\frac{i}{n})\chi_{[\frac{i}{n},\frac{i+1}{n}]}\|_E \\
&\le M_q\|\chi_{[0,1]}\|_E\|\sum_{i=1}^{n-2} f^*(\frac{i}{n})\chi_{[\frac{i}{n},\frac{i+1}{n}]}\|_{L_q} \\
&= c_1\{\sum_{i=1}^{n-2} f^*(\frac{i}{n})^q/n\}^{1/q} \\
&= c_1\|\sum_{i=1}^{n-2} f^*(\frac{i}{n})\chi_{[\frac{i-1}{n},\frac{i}{n}]}\|_{L_q} \\
&\le c_1\|f^*\chi_{[0,1,]}\|_{L_q} \\
&\le c_1\|f\|_{L_q}.
\end{aligned}
$$

Here we take c_1 as $M_q \cdot \|\chi_{[0,1]}\|_E$. For $I_2 \le c_2\|f^*\chi_{[1,\infty]}\|_E$, we have

$$
\begin{aligned}
(2)\qquad \|f^*\chi_{[1,n]}\|_E &\le \|\sum_{i=1}^{n-1} f^*(i)\chi_{[i,i+1]}\|_E \\
&\le M_p(\sum_{i=1}^{n-1} |f^*(i)|^p\|\chi_{[i,i+1]}\|_E^p)^{1/p} \\
&= M_p(\sum_{i=1}^{n-1} |f^*(i)|^p)^{1/p}\|\chi_{[0,1]}\|_E \\
&= c_2\|\sum_{i=1}^{n-1} f^*(i)\chi_{[i-1,i]}\|_{L_p} \\
&\le c_2\|f^*\chi_{[0,n-1]}\|_{L_p} \le c_2\|f\|_{L_p}.
\end{aligned}
$$

Here we take c_2 as $M_p\|\chi_{[0,1]}\|_E$. Since $f^*\chi_{[\frac{1}{n},1-\frac{1}{n}]} \uparrow f^*\chi_{[0,1]}$, $f^*\chi_{[1,n]} \uparrow f^*\chi_{[1,\infty]}$ and $f \in E$, (1) and (2) imply that $I_1 \le c_1\|f\|_{L_q}$ and $I_2 \le$

$c_2\|f\|_{L_p}$ by the condition 3) in definition 1. Therefore, we have

$$\begin{aligned}\|f\|_E = \|f^*\|_E &\le I_1 + I_2 \\ &\le \max(c_1, c_2)\{\|f\|_{L_q} + \|f\|_{L_p}\} \\ &\le 2\max(c_1, c_2)\|f\|_{L_p\cap L_q}.\end{aligned}$$

2) We know $\|f\|_{L_p+L_q}$ is equivalent to $\|f^*\chi_{[0,1]}\|_{L_p} + \|f^*\chi_{[1,\infty]}\|_{L_q}$ ([1]). Let $f \in E$. Define $g_n = \sum_{i=1}^{2^n} f^*(\frac{i}{2^n}) \cdot \chi_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}$. Since $g_n \le f^*\chi_{[0,1]}$ and E is p-convex, we have by lemma 1

$$\begin{aligned}\|g_n\|_{L_p} &\le M_p^1 \|\chi_{[0,1]}\|_E^{-1} \|g_n\|_E \\ &\le M_p^1 \|\chi_{[0,1]}\|_E^{-1} \|f^*\chi_{[0,1]}\|_E.\end{aligned}$$

Since g_n monotomically converges to $f^*\chi_{[0,1]}$ a.e., we have

$$(3) \qquad \|f^*\chi_{[0,1]}\|_{L_p} \le M_p^1 \|\chi_{[0,1]}\|_E^{-1} \|f^*\|_E.$$

Now, let $h_n = \sum_{i=1}^{n} f^*(i)\chi_{[i,i+1]}$. Note that $h_n^* \ge f^*\chi_{[1,n+1]}$ and

$$\begin{aligned}M_q^1 \|h_n\|_E &\ge \{\sum_{i=1}^{n} f^*(i)^q\}^{1/q} \|\chi_{[0,1]}\|_E \\ &= \|\chi_{[0,1]}\|_E \ \|h_n\|_{L_q},\end{aligned}$$

since E satifies a lower q-esimate. Thus,

$$\begin{aligned}\|f^*\chi_{[1,n+1]}\|_{L_q} &\le \|h_n\|_{L_q} \\ &\le M_q^1 \|\chi_{[0,1]}\|_E^{-1} \|h_n\|_E \\ &\le M_q^1 \|\chi_{[0,1]}\|_E^{-1} \|f\|_E.\end{aligned}$$

Since $f^*\chi_{[1,N+1]}$ monotonically converges to $f^*\chi_{[1,\infty]}$,

$$(4) \qquad \|f^*\chi_{[1,\infty]}\|_{L_q} \le M_q^1 \|\chi_{[0,1]}\|_E^{-1} \|f\|_E.$$

Therefore, there exists a constant c by (3) and (4) such that

$$\|f\|_{L_p+L_q} \le c\|f\|_E.$$

COROLLARY 4. *Let $1 \leq p \leq q \leq \infty$. Let E be a symmetric space which is p-convex and q-concave. Then we have $L_p \cap L_q \subset E \subset L_p + L_q$.*

Proof. Since p-convexity and q-concavity imply an upper p-estimate and lower q-estimate, respectively, we have $L_p \cap L_q \subset E \subset L_p + L_q$ by the theorem 3.

COROLLARY 5. *Let $I = [0,1]$ and $p \leq q$. Let E be a symmetric space which is p-convex and q-concave. Then we have $L_q \subset E \subset L_p$. In particular when $p = q$, we have $E = L_p$ as vector spaces and an isomorphism from the symmetric space E onto the Banach space L_p.*

Proof. When $I = [0,1]$, we know $L_q \cap L_q = L_q$ and $L_p + L_q = L_p$. Thus, we have $L_q \subset E \subset L_p$ by the corollary 4.

REMARK. We can not strengthen theorem 3 by replacing q-concavity by lower q-estimate. It is known that $L_{q,p}$ space with $p < q$ satisfies a lower q-estimate and an upper p-estimate but is not q-concave([2]). Futhermore, it is also known that $L_q[0,1] \not\subseteq L_{q,p}[0,1]$. By the same reason, we can not expect $E \subset L_p + L_q$ when E satisfies an upper p-estimate, since $L_{p,q}$ with $p < q$, satisfies an upper p-estimat but $L_{p,q}[0,1] \not\subseteq L_p[0,1]$.

THEOREM 6. ([4, THEOREM 1.F.7]). *If a Banach lattice E satisfies an upper, respectively, lower r-estimate for some $1 < r < \infty$, then it is p-convex, respectively, q-concave, for every $1 < p < r < q < \infty$.*

COROLLARY 7. *Let E satisfy an upper r-estimate and lower s-estimate with $r \leq s$. then $L_q \cap L_p \subset E \subset L_q + L_p$ for every $1 < p < r \leq s < q < \infty$.*

Proof. The proof is clear by theorem 3 and 6.

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