J. Korean Math. Soc. 30(1993), No. 1, pp. 105–123

ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF INFINITE ORDER

KI-HO KWON

1. Introduction.

Let $f(z)$ be meromorphic in the complex plane. We will use freely the standard notations of Nevanlinna theory, including

$$T(r,f), m(r,f), N(r,f), \log M(r,f), \cdots .$$

In addition, we define $m_2(r,f)$ and $m_p^+(r,f)$, $1 < p < \infty$, by

$$m_2(r,f) = \left[\frac{1}{2\pi}\int_0^{2\pi} (\log|f(re^{i\theta})|)^2 d\theta\right]^{1/2}$$

and

$$m_p^+(r,f) = \left[\frac{1}{2\pi}\int_0^{2\pi} (\log^+|f(re^{i\theta})|)^p d\theta\right]^{1/p}.$$

For $E \subset [1,\infty)$, define the logarithmic measure of E by

$$m_l(E) = \int_E \frac{dt}{t}.$$

The upper and lower logarithmic density of E are defined by

$$\overline{\log\text{ dens}}E = \overline{\lim}_{r\to\infty}\frac{m_l(E\cap[1,r])}{\log r}$$

and

$$\underline{\log\text{ dens}}E = \underline{\lim}_{r\to\infty}\frac{m_l(E\cap[1,r])}{\log r}$$

We denote the Ahlfors-Shimizu characteristic by

Received March 4, 1992.

$$T_0(r,f)=\int_0^r \frac{A(t,f)}{t}dt,$$

where $A(t,f)$ is the average number of solutions of $f(z)=a$ in $|z|\le t$ as a varies over the Riemann sphere.

It has long been of interest to compare the sizes of $T(r,f)$ and $\log M(r,f)$ for entire functions. In 1932 R.E.A.C. Paley conjectured that an entire function $f(z)$ of order λ satisfies

$$\underline{\lim}_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}\le\begin{cases}\frac{\pi\lambda}{\sin\pi\lambda}, & \lambda\le\frac{1}{2},\\ \pi\lambda, & \lambda>\frac{1}{2}.\end{cases} \tag{1.1}$$

This conjecture was proved by Valiron [13] and Wahlund [14] for $\lambda<\frac{1}{2}$ in 1935. The first complete proof was given by Govorov [6] in 1969. Petrenko [10] has established that the inequality (1.1) remains valid if the order λ is replaced by the lower order μ and $f(z)$ is assumed to be meromorphic.

The situation is quite different for entire functions of infinite order. In fact, for such functions

$$T(r,f)=o(\log M(r,f)),\quad r\longrightarrow\infty,$$

is possible. An upper bound for $\log M(r,f)$ in terms of $T(r,f)$ was obtained by Shimizu [12] in 1929 when he showed that for each number $k>1$ and each function $f(z)$

$$\underline{\lim}_{r\to\infty}\frac{\log M(r,f)}{T(r,f)[\log T(r,f)]^k}=0. \tag{1.2}$$

We also know that the relation (1.2) is not always valid for $k=1$ [7, p.83]. In 1990, C.J. Dai, D. Drasin and B.Q. Li [5] improved (1.2) as follows.

THEOREM A. *Let $f(z)$ be a nonconstant meromorphic function and let $\varphi(x)$ be a positive increasing function defined on $[0,\infty)$ with*

$$\int^{\infty}\frac{dx}{\varphi(x)}<+\infty. \tag{1.3}$$

Then there exists a set E of logarithmic density 1 such that

$$\lim_{r\to\infty r\in E} \frac{\log M(r,f)}{T(r,f)\varphi(\log T(r,f))\log\varphi(\log T(r,f))} = 0. \tag{1.4}$$

If f is entire, then (1.4) can be improved to

$$\lim_{r\to\infty r\in E} \frac{\log M(r,f)}{T(r,f)\varphi(\log T(r,f))} = 0. \tag{1.5}$$

We note that Shimizu's result is a special case of (1.5), obtained by choosing

$$\varphi(x) = x^k, \quad k > 1.$$

Related results can be found in papers by C.T. Chuang [4], I.I. Marchenko and A.I. Shcherba [8], W. Bergweiler [2], and A.I. Shcherba [11]. In [4] and [8], the density of the sets involved is not discussed. W. Bergweiler considers only entire functions, but obtains precise comparisons between $\log M(r,f)$ and the increasing function $A(r,f) = rT_0'(r,f)$, where $T_0(r,f)$ is the Ahlfors-Shimizu characteristic, and (1.5) can be obtained from [2] in a routine manner. A.I. Shcherba [11] also considers only entire functions, but he obtains a precise result which can be interpreted in the form of (1.5), with an exceptional set of finite logarithmic measure.

I.I. Marchenko and A.I. Shcherba [8] and later Dai, Drasin and Li [5] show that the relation (1.5) is in some sense best possible by obtaining a result equivalent to the following theorem.

THEOREM B. *Let $\psi(x)$ be a twice continuously differentiable non-decreasing positive convex function on the set $\{x : x \geq 0\}$ such that*

$$\lim_{x\to\infty} \frac{x\psi'(x)}{\psi(x)} = 1 \tag{1.6}$$

and

$$\int^{\infty} \frac{dt}{\psi(t)} = \infty. \tag{1.7}$$

Then there exists an entire function $f(z)$ such that

$$\lim_{r\to\infty} \frac{\log M(r,f)}{T(r,f)\psi(\log T(r,f))} = \infty.$$

Our goal in this paper is to obtain Theorem 1, a result for $m_2(r,f)$, analogous to Theorem A obtained for $\log M(r,f)$.

THEOREM 1. *Let $f(z)$ be a nonconstant meromorphic function in the plane and let $\varphi(x)$ be a positive increasing function on $[0,\infty)$ satisfying (1.3). Then there exists a set F with finite logarithmic measure such that*

$$\lim_{r\to\infty r\notin F} \frac{m_2(r,f)}{T(r,f)[\varphi(\log T(r,f))]^{\frac{1}{2}}} = 0. \tag{1.8}$$

This theorem is easily proved by using the following lemma by J. Miles and D. Shea [9].

LEMMA C. *Let f be meromorphic in $|z| \le R$, with $f(0) = 1$. Then, for $0 < r < R$,*

$$m_2(r,f) \le \{1 + A/\sqrt{\log(R/r)}\}T(R,f),$$

where $A = 8\sqrt{\log 2}$.

The conclusion of Theorem 1 is shown to be quite precise in Theorem 2.

THEOREM 2. *Let $\tilde{\psi}(x)$ be a positive increasing continuously differentiable function on $(0,\infty)$ satisfying (1.6) and (1.7). Then there exists an entire function $f(z)$ such that*

$$\lim_{r\to\infty} \frac{m_p^+(r,f)}{T(r,f)[\tilde{\psi}(\log T(r,f))]^{\frac{p-1}{p}}} = +\infty. \tag{1.9}$$

for $1 < p < \infty$.

This is analogous to Theorem B. While our result applies to all p, $1 < p < \infty$, of particular interest is the case $p = 2$, which shows the sharpness of Theorem 1.

Our method of construction closely parallels that in [5]. We consider a positive function $\theta(x)$ on $(-\infty,\infty)$, obtained from the given function $\psi(x)$ by solving a certain differential equation (see (3.4) and (3.6)). Let Ω be the strip-like region bounded above by $y = \frac{\theta(x)}{2}$ and below by $y = -\frac{\theta(x)}{2}$. Let $w(z) = u(z) + iv(z)$ be a conformal mapping from

Ω onto the standard strip $\{z = x + iy : |y| < \pi/2\}$ chosen in such a way that $w(z) \longrightarrow +\infty$ as $x \longrightarrow +\infty$. Then for $z = x + iy \in \Omega$,

$$u(z) = \lambda + \int_0^x \frac{dt}{\theta(t)} + o(1)$$

and

$$v(z) = \frac{\pi y}{\theta(x)} + o(1)$$

as $x \longrightarrow +\infty$, by a lemma of Warschawski (Lemma D). We define a subharmonic function $F(z)$ on the plane as the real part of $\exp(w(z))$ in Ω and 0 outside Ω, and note that $F(z)$ is large in Ω and bounded outside Ω. By an approximation method of Al-Katifi [1] we then obtain an entire function f such that $\log|f(z)| \sim F(z)$ for sufficiently large $|z|$. Finally we show that the width of the region Ω is such that not only can sharp lower bounds for $\log M(r,f)$ be obtained as in [8], but also sharp lower bounds for $m_p^+(r,f)$ as well.

2. Proof of Theorem 1.

We begin with the following lemma, which goes back essentially to Borel [3].

LEMMA 3. *Let $T(r)$ be a positive increasing continuous function on $[1,\infty)$ and let $\tilde{\varphi}(x)$ be a positive increasing continuous function on $[0,\infty)$ satisfying (1.3). Then*

$$F = \{r \geq 1 : T(r + \frac{r}{\tilde{\varphi}(\log T(r))}) \geq 2T(r)\}$$

is of finite logarithmic measure.

Proof. Certainly we may presume that $T(r)$ is unbounded. Choose $r_0 > 1$ with $T(r_0) > e$, and consider the set $F_0 \subset [r_0,\infty)$ consisting of all $r \geq r_0$ such that

$$T(r + \frac{r}{\tilde{\varphi}(\log T(r))}) \geq 2T(r). \tag{2.1}$$

Let r_1 denote the least value of $r \in [r_0, \infty)$ satisfying (2.1) and write $r'_1 = r_1 + r_1/\tilde{\varphi}(\log T(r_1))$. After r_i, $r'_i (i = 1, 2, \cdots, n-1)$ have been defined, let r_n be the least value $r \in [r'_{n-1}, \infty)$ which satisfies (2.1) and let $r'_n = r_n + r_n/\tilde{\varphi}(\log T(r_n))$. Then

$$F_0 \subset \bigcup_{n=1}^{\infty} [r_n, r'_n], \tag{2.2}$$

$$r_0 \le r_1 < r'_1 \le r_2 < r'_2 \le \cdots, \tag{2.3}$$

$$T(r_n) \ge T(r'_{n-1}) \ge 2T(r_{n-1}) \ge \cdots \ge 2^{n-1}, \tag{2.4}$$

since $r_i \in F_0$. Hence by (1.3), (2.2), (2.3), and (2.4), we have

$$\begin{aligned}
\int_{F_0} \frac{dr}{r} &\le \sum_{n=1}^{\infty} \log \frac{r'_n}{r_n} \\
&= \sum_{n=1}^{\infty} \log(1 + \frac{1}{\tilde{\varphi}(\log T(r_n))}) \le \sum_{n=1}^{\infty} \frac{1}{\tilde{\varphi}(\log T(r_n))} \\
&\le \sum_{n=1}^{2} \frac{1}{\tilde{\varphi}(\log T(r_n))} + \sum_{n=3}^{\infty} \frac{1}{\tilde{\varphi}((n-1)\log 2)} \\
&\le \sum_{n=1}^{2} \frac{1}{\tilde{\varphi}(\log T(r_n))} + \frac{1}{\log 2} \int_{\log 2}^{\infty} \frac{dx}{\tilde{\varphi}(x)} < \infty.
\end{aligned}$$

Thus the proof of Lemma 3 is complete.

It is easy to show that there exists a positive increasing continuous function $\tilde{\varphi}(x)$ on $[0, \infty)$ satisfying (1.3) and

$$\lim_{x \to \infty} \frac{\tilde{\varphi}(x)}{\varphi(x)} = 0. \tag{2.5}$$

Applying Lemma 3 to $T(r,f)$ ad $\tilde{\varphi}(x)$, we obtain a set F of finite logarithmic measure such that if $r \notin F$ and

$$R = r\left(1 + \frac{1}{\tilde{\varphi}(\log T(r,f))}\right),$$

then

$$T(R,f) < 2T(r,f). \tag{2.6}$$

Since

$$\frac{1}{\tilde{\varphi}(\log T(r,f))} \longrightarrow 0, \quad r \longrightarrow \infty,$$

we have for all large r that

$$\log \frac{R}{r} > \frac{1}{2\tilde{\varphi}(\log T(r,f))}. \tag{2.7}$$

Using Lemma C, we deduce from (2.6) and (2.7) for all large $r \notin F$ that

$$m_2(r,f) \leq (1 + \sqrt{2}A[\tilde{\varphi}(\log T(r,f))]^{\frac{1}{2}})2T(r,f).$$

It follows at once that

$$\overline{\lim}_{r\to\infty\, r\notin F} \frac{m_2(r,f)}{T(r,f)[\tilde{\varphi}(\log T(r,f))]^{\frac{1}{2}}} \leq 2\sqrt{2}A,$$

and (1.8) holds by (2.5).

3. Proof of Theorem 2.

We first show that there is a function $\psi(x)$ on $(0,\infty)$ satisfying (1.6), (1.7), and

$$\tilde{\psi}(x) = o(\psi(x)), \quad x \longrightarrow +\infty. \tag{3.1}$$

We let

$$\psi(x) = (\tilde{\psi}(x) + x)\int_0^x \frac{dt}{\tilde{\psi}(t) + t}.$$

It is elementary that

$$\int_0^\infty \frac{dt}{\tilde{\psi}(t)+t} = +\infty,$$

and (3.1) follows. We claim that $\psi(x)$ satisfies all the hypotheses of Theorem 2. In fact, it is easy to show that $\psi(x)$ is a positive increasing continuously differentiable function on $(0,\infty)$. By substituting

$$y = \int_0^x \frac{dt}{\tilde{\psi}(t)+t},$$

we have

$$\int^\infty \frac{dx}{\psi(x)} = \int^\infty \frac{dy}{y} = +\infty.$$

Since

$$\psi'(x) = (\tilde{\psi}'(x)+1)\int_0^x \frac{dt}{\tilde{\psi}(t)+t} + 1,$$

we have

$$\frac{x\psi'(x)}{\psi(x)} = \frac{x\tilde{\psi}'(x)+x}{\tilde{\psi}+x} + \frac{x}{(\tilde{\psi}(x)+x)\int_0^x \frac{dt}{\tilde{\psi}(t)+t}}.$$

Observe that as $x \longrightarrow +\infty$,

$$\frac{x}{(\tilde{\psi}(x)+x)\int_0^x \frac{dt}{\tilde{\psi}(t)+t}} \longrightarrow 0$$

and

$$\frac{x\tilde{\psi}'(x)+x}{\tilde{\psi}(x)+x} = 1 + \frac{\frac{x\tilde{\psi}'(x)}{\tilde{\psi}(x)}}{1+\frac{x}{\tilde{\psi}(x)}} \longrightarrow 1.$$

Therefore

$$\frac{x\psi'(x)}{\psi(x)} \longrightarrow 1, \quad x \longrightarrow \infty.$$

Thus ψ does indeed satisfy (1.6), (1.7), and (3.1). Hence our claim is proved.

It follows immediately from (1.6) for all large x that

$$\frac{9}{10x} \leq \frac{\psi'(x)}{\psi(x)} \leq \frac{11}{10x} \tag{3.2}$$

and

$$x^{\frac{9}{10}} \leq \psi(x) \leq x^{\frac{11}{10}}. \tag{3.3}$$

Hence we may assume that $\psi(x)$ satisfies the inequalities (3.2) and (3.3) for all $x \geq 0$. With ψ as above, consider the differential equation

$$\frac{ds}{dr} = \frac{\pi\psi(2s)}{r}, \quad r > 1, \quad s(1) = 1. \tag{3.4}$$

We know that there exists a solution $s = s(r)$ of (3.4) on $r \geq 1$ such that s is strictly increasing and $s(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ by (1.7). By the left inequality of (3.3) and by (3.4) we have that

$$\frac{dr}{r} = \frac{ds}{\pi\psi(2s)} \leq \frac{ds}{\pi(2s)^{\frac{9}{10}}} \leq \frac{ds}{5s^{\frac{9}{10}}}, \quad r > 1,$$

and

$$s(1) = 1.$$

Integrating each side, we obtain that

$$\log r \leq 2s^{\frac{1}{10}}, \quad r > 1.$$

Thus

$$s = s(r) \geq 2^{-10}(\log r)^{10}, \quad r > 1. \tag{3.5}$$

Now set

$$\theta(r) = \pi\frac{dr}{ds} = \frac{r}{\psi(2s)}, \quad r > 1, \tag{3.6}$$

and let

$$\theta(r) = \theta(1), -\infty < r \leq 1. \tag{3.7}$$

Then $\theta(r)$ has the following properties.

LEMMA 4. *With the above notation, we have*

$$\text{(a)} \qquad \theta'(r) \longrightarrow 0, \quad r \longrightarrow +\infty,$$

and

$$\text{(b)} \qquad \int_1^{+\infty} \frac{\theta'(r)^2}{\theta(r)} dr < +\infty.$$

Proof. We get from (3.4) and (3.6) that

$$\begin{aligned}\theta'(r) &= \psi(2s)^{-2}\{\psi(2s) - r\psi'(2s)\frac{d(2s)}{dr}\} \\ &= \psi(2s)^{-2}\{\psi(2s) - r\psi'(2s)\frac{2\pi\psi(2s)}{r}\} \\ &= \psi(2s)^{-1}\{1 - 2\pi\psi'(2s)\}.\end{aligned} \tag{3.8}$$

Thus (a) of Lemma 4 follows from (1.6), (3.3), and (3.8). We next have

$$\begin{aligned}\frac{\theta'(r)^2}{\theta(r)} &= \psi(2s)^{-2}\{1 - 4\pi\psi'(2s) + 4\pi^2(\psi'(2s))^2\}r^{-1}\psi(2s) \\ &\le \frac{1}{r\psi(2s)} + 4\pi^2\frac{\psi(2s)}{r}(\frac{\psi'(2s)}{\psi(2s)})^2.\end{aligned} \tag{3.9}$$

Furthermore we obtain from (3.2), (3.3), and (3.5) that

$$\int_1^\infty \frac{dr}{r\psi(2s)} = \int_1^\infty \frac{dr}{r\psi(2s(r))} \le \int_1^\infty \frac{dr}{r(2s(r))^{\frac{9}{10}}} \le 2^9 \int_1^\infty \frac{dr}{r(\log r)^9} < \infty,$$

and

$$\begin{aligned}\int_1^\infty \frac{\psi(2s)}{r}(\frac{\psi'(2s)}{\psi(2s)})^2 dr &\le \int_1^\infty \frac{(2s(r))^{\frac{11}{10}}}{r}(\frac{11}{20s(r)})^2 dr \\ &\le \int_1^\infty \frac{dr}{rs(r)^{\frac{9}{10}}} < \infty.\end{aligned}$$

This proves (b) of Lemma 4 by (3.9).

We now need the following lemma of S.E. Warschawski [15, p.296 and p.323].

LEMMA D. *Let $\phi_1(x) > \phi_2(x)$ be real-valued differentiable functions on $(-\infty, \infty)$ such that*

$$\lim_{x_1, x_2 \to \infty_{x_2 > x_1}} \frac{\phi_i(x_2) - \phi_i(x_1)}{x_2 - x_1} = 0, \quad i = 1, 2,$$

and

$$\int_{x_0}^{\infty} \frac{[\phi_i'(x)]^2}{\phi_1(x) - \phi_2(x)} dx < +\infty, \quad i = 1, 2.$$

Suppose that $\tilde{\Omega}$ is the strip domain defined by

$$\tilde{\Omega} = \{z = x + iy : \phi_2(x) < y < \phi_1(x)\},$$

and that $\tilde{w}(z) = \tilde{u}(z) + i\tilde{v}(z)$ maps $\tilde{\Omega}$ conformally onto the strip $|y| < \frac{\pi}{2}$ in such a manner that

$$\lim_{x \to +\infty} \tilde{w}(z) = +\infty.$$

Then

(a) there exists a constant $\tilde{\lambda}$ such that for $z = x + iy \in \tilde{\Omega}$,

$$\tilde{u}(z) = \tilde{\lambda} + \pi \int_{x_0}^{x} \frac{dt}{\phi_1(t) - \phi_2(t)} + o(1), \quad as \quad x \longrightarrow +\infty,$$

uniformly with respect to y;

(b) uniformly for $z \in \tilde{\Omega}$,

$$\tilde{v}(z) = \pi \frac{y - \phi_0(x)}{\phi_1(x) - \phi_2(x)} + o(1), \quad as \quad x \longrightarrow +\infty,$$

where $\phi_0(x) = \frac{1}{2}(\phi_1(x) + \phi_2(x))$.

Now let us consider the strip domains

$$\Omega = \{z = x + iy : |y| < \frac{\theta(x)}{2}\} \tag{3.10}$$

and

$$D = \{w = u + iv : |v| < \frac{\pi}{2}\}.$$

Let $w(z) = u(z) + iv(z)$ be the conformal mapping of Ω onto D such that

$$\lim_{x \to +\infty} u(z) = +\infty, \quad \lim_{x \to -\infty} u(z) = -\infty$$

and

$$w(i\frac{\theta(0)}{2}) = i\frac{\pi}{2}, \quad w(-i\frac{\theta(0)}{2}) = -i\frac{\pi}{2}.$$

In view of Lemma 4, we may apply Lemma D with the choices $\phi_1(x) = \theta(x)/2$ and $\phi_2(x) = -\theta(x)/2$ to obtain the following facts about the mapping $w(z)$.

FACT 1. There exists a constant λ such that for $z = x + iy \in \Omega$,

$$u(z) = \lambda + \pi \int_0^x \frac{dt}{\theta(t)} + o(1), \quad as \quad x \longrightarrow +\infty, \tag{3.11}$$

uniformly with respect to y;

FACT 2. Uniformly for $z = x + iy \in \Omega$,

$$v(z) = \frac{\pi y}{\theta(x)} + o(1), \quad as \quad x \longrightarrow +\infty. \tag{3.12}$$

We also need the following approximation theorem of W. Al-Katifi [1, Theorem 2.1].

THEOREM E. *Let C_1 and C_2 be two Jordan arcs going from $z = 0$ to $z = \infty$ and intersecting at no other point, and let C_1 and C_2 have continuously turning tangents except at isolated points. Suppose that the arc length of the pair C_1 and C_2 inside any circle of radius R is less than $K_1 R$ for some constant K_1, that T is one of the two plane domains complementary to $C_1 \cup C_2$, and that $\zeta = \zeta(z)$ maps T one-to-one and conformally onto the right half-plane in such a way that $z = \infty$ and $\zeta = \infty$ correspond. Consider the subharmonic function*

$$\tilde{F}(z) = \begin{cases} Re[\zeta(z)], & z \in T, \\ 0, & z \notin T. \end{cases}$$

Then there exists an entire function $\tilde{f}(z)$ such that for all sufficiently large $|z|$,

$$|\log|\tilde{f}(z)| - \tilde{F}(z)| < K_2[\log|z| + \log^+ \frac{1}{\tilde{\delta}(z)}],$$

where $\tilde{\delta}(z) = dist(z, C_1 \cup C_2)$, and K_2 is a constant depending on C_1 and C_2.

FACT 3. Theorem E remains valid if we replace the domain T by the strip domain Ω (see (3.10)).

To prove Fact 3, we only have to replace C_1 and C_2 in the proof of Theorem E in [1] by the two boundary curves of Ω.

Consider the function $F(z)$ on the plane given by

$$F(z) = \begin{cases} (\exp(u(z)) \cos v(z), & z \in \Omega, \\ 0, & z \notin \Omega. \end{cases}$$

Then we deduce from Theorem E and Fact 3 that there exists an entire function $f(z)$ such that for sufficiently large r

$$|\log|f(z)| - F(z)| < K(\log r + \log^+ \frac{1}{\delta(z)}, \tag{3.13}$$

where $|z| = r, \delta(z) = \text{dist}(z, \partial\Omega)$, and K is a constant which depends only on Ω.

Recall from Lemma 4(a) that $\theta'(r) \longrightarrow 0$ and from (3.6) that $\theta(r)/r \longrightarrow 0$ as $r \longrightarrow \infty$. Hence we can choose a number $r_0 \geq 10$ so large that

$$\theta(r) < \frac{r_0}{10}, \quad r \leq r_0, \tag{3.14}$$

and

$$|\theta'(r)| < \frac{1}{10}, r > r_0. \tag{3.15}$$

If $r \geq 2r_0$, then $|z| = r$ cannot intersect $\partial\Omega$ at any point in the first quadrant with real part less than r_0 by (3.14). The combination of (3.15) with the Mean Value Theorem then shows $|z| = r$ meets $\partial\Omega$ at a unique point of the first quadrant, which we denote by $re^{ih_0(r)}$, $0 < h_0(r) < \frac{\pi}{2}$. Note for $r \geq 2r_0$ that

$$\Omega \cap \{z : |z| = r \text{ and } Re(z) > 0\} = \{re^{ih} : |h| \leq h_0(r)\}.$$

If we let

$$r_1 = r \cos h_0(r),$$

it is immediate from the fact that $re^{ih_0(r)} \in \partial\Omega$ that $r_1 \longrightarrow \infty$ as $r \longrightarrow \infty$. Since

$$h_0(r) = \tan^{-1} \frac{\theta(r_1)}{2r_1},$$

we conclude $h_0(r) \longrightarrow 0$ as $r \longrightarrow \infty$. For $r_1 = r\cos h_0(r) \leq t \leq r$, we have

$$\begin{aligned}
|\theta(r) - \theta(t)| &\leq \int_t^r |\theta'(u)| du \leq (r - r_1) \max_{r_1 \leq u \leq r} |\theta'(u)| \\
&= o(r(h_0(r))^2) = o(r(\frac{\theta(r_1)}{r})^2) = o(\frac{r}{r_1}\theta(r_1)) \\
&= o(\theta(r_1)), r \longrightarrow \infty.
\end{aligned}$$

We conclude for $r \cos h_0(r) \leq t \leq r$ that

$$\theta(t) = (1 + o(1))\theta(r), \quad r \longrightarrow \infty, \tag{3.16}$$

and

$$h_0(r) = (1 + o(1))\frac{\theta(r_1)}{2r_1} = (1 + o(1))\frac{\theta(r)}{2r}, \quad r \longrightarrow \infty. \tag{3.17}$$

We shall need the second geometric property of Ω. For large r, let S_r^+ be the union of the two sectors with common vertex at $re^{ih_0(r)}$ defined by

$$S_r^+ = \{z : |\tan(\arg(z - re^{ih_0(r)}))| < \frac{1}{10}\}.$$

Let S_r^- be the reflection of S_r^+ in the real axis, and let $S_r = S_r^+ \cup S_r^-$. Notice for large r by (3.15) that the entire boudary of Ω in the right half plane lies in S_r. For $z \in \mathbf{C}$, let

$$\begin{aligned}\delta_1(z) &= \operatorname{dist}(z, \partial S_r),\\ \delta_2(z) &= \operatorname{dist}(z, \partial\Omega \cap \{|z| = r\}).\end{aligned}$$

For $Re(z) > 0$ and $r = |z|$ large, it is clear that

$$\delta(z) \geq \delta_1(z) \geq \frac{1}{10}\delta_2(z).$$

For $Re(z) < 0$ and $|z|$ large, it is also trivial that

$$\delta(z) > \frac{1}{10}\delta_2(z).$$

Therefore we have by [7, Lemma 2.2] for large r that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} \log^+\left(\frac{1}{\delta(re^{it})}\right)dt \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left(\frac{10}{\delta_2(re^{it})}\right)dt\\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left(\frac{r}{\delta_2(re^{it})}\right)dt = 2\log 4 + \frac{1}{2}.\end{aligned} \tag{3.18}$$

Now let $z = re^{ih} \in \Omega$. Suppose that $r\cos h \leq \tilde{r} \leq r$ and

$$\theta(\tilde{r}) = \min\{\theta(t) : r\cos h \leq t \leq r\}.$$

Then we obtain from (3.11) and (3.17) that

(3.19)

$$\begin{aligned}|u(z)-u(r)| &= \pi\int_{r\cos h}^{r}\frac{dt}{\theta(t)}+o(1)\\ &\le \pi(r-r\cos h)\frac{1}{\theta(\tilde r)}+o(1)\le \pi r h^2\frac{1}{\theta(\tilde r)}+o(1)\\ &\le \frac{(\pi+o(1))}{4}\left(\frac{r}{\tilde r}\right)\left(\frac{\theta(r)}{r}\right)^2\left(\frac{\tilde r}{\theta(\tilde r)}\right)+o(1)\\ &\le \frac{(\pi+o(1)}{4\cos h}\left\{\frac{\psi(2s(\tilde r))}{(\psi(2s(r)))^2}\right\}+o(1)=o(1), \quad r\longrightarrow\infty.\end{aligned}$$

On the other hand, we have from (3.12) and (3.16) that

$$\begin{aligned}|v(z)| &= \left|\frac{\pi r\sin h}{\theta(r\cos h)}+o(1)\right| = \pi|\tan h|\frac{r\cos h}{\theta(r\cos h)}+o(1)\\ &= (\pi+o(1))\frac{|h|r}{\theta(r)}+o(1), \quad r\longrightarrow\infty.\end{aligned} \tag{3.20}$$

Recall that $ds/dr = \pi/\theta(r)$. Hence we conclude from (3.11) that there exists a constant l such that for $z = x+iy\in\Omega$,

$$u(z) = l + s(x) + o(1) \quad \text{as} \quad x\longrightarrow+\infty, \tag{3.21}$$

uniformly with respect to y. Hence by (3.3), (3.6), (3.13), (3.17), (3.18), (3.19), and (3.21) we obtain that, with $z = re^{it}$,

$$\begin{aligned}T(r,f) &\le \frac{1}{2\pi}\int_0^{2\pi}[F(z)]^+dt + K\log r + o(1)\\ &\le \left(\frac{1}{2\pi}+o(1)\right)\frac{\theta(r)}{r}e^{u(r)}, \quad r\longrightarrow\infty.\end{aligned} \tag{3.22}$$

Now let $z = re^{ih}$ with

$$|h| \le \frac{\theta(r)}{3r}. \tag{3.23}$$

Then $z \in \Omega$ for all large r and so, by (3.19) and (3.20),

(3.24)
$$\begin{aligned} F(z) = e^{u(z)} \cos v(z) &\geq e^{u(z)} \cos[(\pi + o(1))\frac{|h|r}{\theta(r)} + o(1)] \\ &\geq e^{(u(r)+o(1))} \cos(\frac{\pi}{3} + o(1)) \\ &\geq (\frac{1}{2} - o(1))e^{u(r)}, \quad r \longrightarrow \infty. \end{aligned}$$

Furthermore, since

$$\delta(z) \geq (\frac{1}{6} - o(1))\theta(r), \quad r \longrightarrow \infty,$$

by (3.10), (3.16), and (3.23), we obtain from (3.3), (3.6), and (3.21) that

$$\begin{aligned} \log^+ \frac{1}{\delta(z)} &\leq \log(6 + o(1)) + \log^+ \frac{1}{\theta(r)} \\ &= \log^+ \frac{\psi(2s(r))}{r} + o(1) \\ &= o(e^{s(r)}) = o(e^{u(r)}), \quad r \longrightarrow \infty. \end{aligned} \tag{3.25}$$

Hence we conclude from (3.5), (3.13), (3.21), (3.23), (3.24), and (3.25) that if $z = re^{ih}, |h| \leq \theta(r)/3r$, then

$$\log |f(re^{ih})| \geq (\frac{1}{2} - o(1))e^{u(r)}, \quad r \longrightarrow \infty.$$

Thus we have

$$m_p^+(r, f) \geq \left(\frac{\theta(r)}{3\pi r}\right)^{\frac{1}{p}} \left(\frac{1}{2} - o(1)\right)e^{u(r)}, \quad r \longrightarrow \infty. \tag{3.26}$$

We also obtain from (3.21) and (3.22) that

(3.27)
$$\begin{aligned} \psi(\log T(r, f)) &\leq \psi(u(r) + o(1)) \\ &= \psi(l + s(r) + o(1)) \leq \psi(2s), \quad r \longrightarrow \infty, \end{aligned}$$

since $\psi(x)$ is increasing. Hence we deduce from (3.6), (3.22), (3.26), and (3.27) that

$$(3.28)\qquad \underline{\lim}_{r\to+\infty}\frac{m_p^+(r,f)}{T(r,f)[\psi(\log T(r,f))]^{\frac{p-1}{p}}} \geq \underline{\lim}_{r\to+\infty}\frac{\pi(\theta(r)r^{-1})^{\frac{1-p}{p}}}{(3\pi)^{\frac{1}{p}}[\psi(2s)]^{\frac{p-1}{p}}} = \frac{\pi}{(3\pi)^{\frac{1}{p}}}.$$

Hence (1.9) follows at once from (3.1) and (3.28) and the proof of Theorem 2 is complete.

References

1. W. Al-Katifi, *On the asymptotic values and paths of certain integral and meromorphic functions*, Proc. London Math. Soc. (3) **16** (1966), 599–634.
2. W. Bergweiler, *Maximum modules, characteristic, and area on the sphere*, Analysis **10** (1990), 163–176.
3. E. Borel, *Sur les zéros des fonctions entières*, Acta Math. **20** (1897), 357–396.
4. C.T. Chuang, *Sur la croissance des fonctions*, Kexue Tongbao **26** (1981), 677–684.
5. C.J. Dai, D.Drasin and B.Q. Li, *On the growth of entire and meromorphic functions of infinite order*, J. D'anal. Math. **55** (1990), 217–228.
6. N.V. Govorov, *On Paley's problem*, Funk. Anal. **3** (1969), 35–40.
7. W.K. Hayman, *"Meromorphic functions "*, Oxford University Press, Oxford, 1964.
8. I.I. Marchenko and A.I. Shcherba, *Growth of entire functions*, Siberian Math. J. (Engl. Transl.) **25** (1984), 598–606.
9. J.Miles and D.F. Shea, *On the growth of meromorphic functions having at least one deficient value*, Duke Math. J. (1) **43** (1976), 171–185.
10. V.P. Petrenko, *The growth of meromorphic functions of finite lower order*, Izv. Ak. Nauk U.S.S.R. **33** (1969), 414–454.
11. A.I. Shcherba, *Growth characteristics of entire functions*, J.Soviet Math. (3) **48** (1990), 358–362.
12. T. Shimizu, *On the theory of meromorphic functions*, Japan J. Math. **6** (1929), 119–171.
13. G. Valiron, *Sur un theorème de M. Wiman*, Opuscula Math. A. Wiman dedicata, 1930.
14. A. Wahlund, *Über einen zusammenhang zwischer dem maximalbetrage der ganzen funktion und seiner unteren grenze nach dem Jensensche Theoreme*, Arkiv Math. **21A** (1929), 1–34.

15. S.E. Warschawski, *On conformal mapping of infinite strips*, Trans. Amer. Math. Soc. **51** (1942), 280–335.

Department of Mathematics
Korea Military Academy
Seoul 139-799, Korea