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LAWS OF LARGE NUMBERS FOR PRODUCTS OF SOME MEASURES AND PARTIAL SUM PROCESSES INDEXED BY SETS

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1. Introduction

Let N and **R** denote the set of positive integers and real numbers respectively. Fix $d_1, d_2 \in \mathrm{N}$ with $d = d_1 + d_2$. Let X be a real random variable and let $\{X_{\mathbf{i}} : \mathbf{i} \in \mathrm{N}^{d_1}\}$ be a family of independent identically distributed random variables with $\mathcal{L}(X) = \mathcal{L}(X_{\mathbf{i}})$ and $0 < E|X| < \infty$. The partial sum processes S_{1n} formed from $\{X_{\mathbf{i}}\}$ and indexed by subsets of $\mathbf{I}^{d_1}$, where $\mathbf{I}^{d_1}$ denotes the d_1-dimentional unit cube, are

$$S_{1n}(X, A) := \sum_{|\mathbf{i}| \leq n} X_{\mathbf{i}} \delta_{\mathbf{i}/n}(A), \qquad A \subset \mathbf{I}^{d_1},$$

where, $\mathbf{i} = (i_1, i_2, \cdots, i_{d_1})$, $\mathbf{i}/n = (i_1/n, i_2/n, \cdots, i_{d_1}/n)$, $|\mathbf{i}| = \max_{1 \leq k \leq d_1} i_k$ and $\delta_{\mathbf{i}/n}(A) = 1$ or 0 depending on $\mathbf{i}/n \in A$ or not. For S_{1n} many authors have studied laws of large numbers, central limit theorems and laws of iterated logarithms under various conditions on a sub-family of $\mathcal{B}(\mathrm{I}^{d_1})$. See Alexander and Pyke [1], Bass and Pyke [2, 3], Giné and Zinn [5] and references therein. Denote

$$\begin{aligned} C_{n\mathbf{i}\mathbf{j}} = \{(x_1, x_2, \cdots, x_d) \in \mathbf{R}^d :& \\ &(i_k - 1)/n < x_k \leq i_k/n,\ k = 1, 2, \cdots, d_1 \\ &(j_l - 1)/n < x_{d_1+l} \leq j_l/n,\ l = 1, 2, \cdots, d_2\}. \end{aligned}$$

Assume that $\{\lambda_n\}$ is a sequence of positive Borel measures on $\mathbf{I}^d$ satisfying

$$(1.1) \qquad c_1 n^{-d} \leq \lambda_n(C_{n\mathbf{i}\mathbf{j}}) \leq c_2 n^{-d},$$

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for some $c_1 > 0$, $c_2 < \infty$ and for all $n \in \mathbf{N}$, and that $\mathcal{A}$ is a subfamily of $\mathcal{B}(\mathbf{I}^d)$. Now define the product of measures and partial sum processes corresponding to $\{X_{\mathbf{i}}\}$ and $\{\lambda_n\}$, indexed by subsets of $\mathbf{I}^d$, which is defined as

$$S_n(A) := S_n(X,1,A) := \sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} X_{\mathbf{i}}\lambda_n(A\cap C_{n\mathbf{ij}}), \qquad A\subset \mathbf{I}^d$$

where, $\mathbf{j} = (j_1, j_2, \cdots, j_{d_2})$ and $|\mathbf{j}| = \max_{1\le k\le d_2} j_k$.
If $\lambda_n = n^{-d}\sum_{|\mathbf{i}|\le n,|\mathbf{j}|\le n}\delta_{(\mathbf{i}/n,\mathbf{j}/n)}$, then

$$S_n(X,1,A) = n^{-d}\sum_{|\mathbf{i}|\le n,|\mathbf{j}|\le n} X_{\mathbf{i}}\delta_{(\mathbf{i}/n,\mathbf{j}/n)}(A), \qquad A\subset \mathbf{I}^d.$$

This product process can be viewed as not only a special case of dependent partial sum processes but also a generalization of usual partial sum processes by taking a special class of sets (i.e.$\mathcal{A} = \{A\times \mathbf{I}^{d_2} : A\subset \mathbf{I}^{d_1}\}$).
If $\lambda_n = \lambda$, the Lebesgue measure, then

$$S_n(X,\dot{1},A) = \sum_{|\mathbf{i}|\le n,|\mathbf{j}|\le n} X_{\mathbf{i}}\lambda(A\cap C_{n\mathbf{ij}}) \qquad A\subset \mathbf{I}^d$$

which is the smoothed product of Lebesgue measure and partial sum processes.

For strong law results in terms of metric entropy the following notations and developments will be used, and we follow the work of Giné and Zinn [5], in which they studied the same problems for partial sum processes. Let $\mathbf{i} = (i_k)\in \mathbf{N}^{d_1}$, $\mathbf{j} = (j_l)\in\mathbf{N}^{d_2}$ and $n\in\mathbf{N}$. Assume that $|\mathbf{i}|\le n$ and $|\mathbf{j}|\le n$.

Now define (pseudo)metrics on $\mathcal{A}$ associated with λ_n as follows: For $A, B\in\mathcal{A}$, define

$$\mathrm{d}_{\lambda_{n,p}}(A,B) = n^{d_1(1-1/p)}\left(\sum_{|\mathbf{i}|\le n}\left|\sum_{|\mathbf{j}|\le n}[\lambda_n(A\cap C_{n\mathbf{ij}}) - \lambda_n(B\cap C_{n\mathbf{ij}})]\right|^p\right)^{1/p}$$
$$\text{if } 1\le p<\infty,$$

$$\mathrm{d}_{\lambda_{n,\infty}}(A,B) = \max_{|\mathbf{i}|\le n} n^{d_1}\left|\sum_{|\mathbf{j}|\le n}[\lambda_n(A\cap C_{n\mathbf{ij}}) - \lambda_n(B\cap C_{n\mathbf{ij}})]\right|.$$

Set for $1 \leq p \leq \infty$, and $\delta > 0$

$$N_{\lambda_n,p}(\delta, \mathcal{A}) := \inf\{m : \quad \text{there exist} \quad A_1, A_2, \cdots, A_m \in \mathcal{A} \quad \text{such that} \quad \sup_{A \in \mathcal{A}} \min_{r \leq m} \mathbf{d}_{\lambda_n,p}(A_r, A) \leq \delta\}$$

the covering number of $(\mathcal{A}, \mathbf{d}_{\lambda_n,p})$. Then we have the following: For any $1 \leq p \leq \infty$,

$$\mathbf{d}_{\lambda_n,1} \leq \mathbf{d}_{\lambda_n,p} \leq \mathbf{d}_{\lambda_n,\infty}. \tag{1.2}$$

In particular, for $\delta > 0$, $N_{\lambda_n,1}(\delta, \mathcal{A}) \leq N_{\lambda_n,p}(\delta, \mathcal{A}) \leq N_{\lambda_n,\infty}(\delta, \mathcal{A})$.

Finally, let $\{\varepsilon_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ denote, always in what follows, a family of independent symmetric Bernoulli (or Rademacher) random variables, that is, $P[\varepsilon = 1] = P[\varepsilon = -1] = 1/2$, independent of any other set of random variables that appear in the argument where they are used. We will write

$$S_{\lambda_n}(\varepsilon, 1, A) = \sum_{|\mathbf{i}| \leq n, |\mathbf{j}| \leq n} \varepsilon_{\mathbf{i}} \lambda_n(A \cap C_{n\mathbf{i}\mathbf{j}}), A \in \mathcal{A},\ n \in \mathbf{N}.$$

Let $\{X'_{\mathbf{i}}\}$ denote always an independent copy of $\{X_{\mathbf{i}}\}$ so that $\{X_{\mathbf{i}} - X'_{\mathbf{i}}\}$ is a set of independent symmetric random variables that symmetrizes $\{X_{\mathbf{i}}\}$.

In section 2 it will be proved a law of large numbers for a sequence of products of some measures and partial sum processes. This result includes that of Giné and Zinn [5] as a corollary by taking a special class of subsets of $\mathbf{I}^d$(i.e.$\mathcal{A} = \{A \times \mathbf{I}^{d_2} : A \subset \mathbf{I}^{d_1}\}$).

2. Main results

Throughout this section we assume that $0 < E|X| < \infty$. Before proceeding to the strong law results, we first make an observation about measurability whose proof is similiar to that of the corresponding result of Giné and Zinn [5]. Let $\|\cdot\|_{\mathcal{A}}$ denote the sup-norm defined by $\|\psi\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} |\psi(A)|$.

LEMMA 2.1. *Let $\mathcal{A}$ be any collection of measurable subsets of $\mathbf{I}^d$, let $\{\lambda_n\}$ be a family of finite positive Borel measures on $\mathbf{I}^d$ satisfying (1.1),* *and let* $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ *be independent identically distributed random variables with the law of* X. *Then for all* $n \in \mathbf{N}$, $\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}$ *is a Borel measurable function of the* $\mathbf{R}^{n^d}$*-valued random vectors* $(X,1)_n = \{X_{\mathbf{i}} : |\mathbf{i}| \le n, |\mathbf{j}| \le n\}$.

We now prove that, independently of the law of X as long as X is integrable, the sequences $\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}$ and $\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}}$ both either converge to 0 or do not converge to 0, where the convergence is either a.s. or in probability. We begin with the case when $EX = 0$.

THEOREM 2.2. *Let* $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ *be independent identically distributed random variables with the same law of* X *such that* $0 < E|X| < \infty$ *and* $EX = 0$. *Let* $\{\lambda_n\}$ *be a sequence of positive Borel measures on* $\mathbf{I}^d$ *satisfying condition (1.1) and let* $\mathcal{A}$ *be a class of Borel subsets of* $\mathbf{I}^d$. *Then the following are equivalent;*

(i) $\lim_{n\to\infty} \|S_{\lambda_n}(X,1)\|_{\mathcal{A}} = 0$ *a.s.. (respectively, in probability)*

(ii) $\lim_{n\to\infty} \|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}} = 0$ *a.s.. (respectively, in probability)*

Proof. Since $\{X_n\}$ converges in probability to X if and only if every subsequence $\{X_{n_k}\}$ contains a further subsequence that converges a.s. to X, we need only to check the a.s. statements. For $0 < M < \infty$ and $\mathbf{i} \in \mathbf{N}^{d_1}$. Let $X_{\mathbf{i}}^M = X_{\mathbf{i}} 1_{(|X_{\mathbf{i}}| > M)}$ and $X_{\mathbf{i},M} = X_{\mathbf{i}} - X_{\mathbf{i}}^M$. Then, by the classical law of large numbers,

$$\begin{aligned} \text{a.s.-} \lim_{M\to\infty} \limsup_{n\to\infty} \|S_{\lambda_n}(X^M - EX^M, 1)\|_{\mathcal{A}} &\le c_2 \lim_{M\to\infty} \limsup_{n\to\infty} n^{-d_1} \sum_{|\mathbf{i}| \le n} |X_{\mathbf{i}}^M - EX_{\mathbf{i}}^M| \\ &= c_2 \lim_{M\to\infty} E|X^M - EX^M| = 0. \end{aligned}$$

Hence, by considering $(X_{\mathbf{i},M} - EX_{\mathbf{i},M})/2M$ instead of $X_{\mathbf{i}}$, $\mathbf{i} \in \mathbf{N}^{d_1}$, we may assume that the random variables $X_{\mathbf{i}}$ in (i) are centered and bounded by one.

Suppose (i) holds. Then, a.s.-

$$\lim_{n\to\infty} \|S_{\lambda_n}(X - X', 1)\|_{\mathcal{A}} \le \lim_{n\to\infty} \|S_{\lambda_n}(X,1)\|_{\mathcal{A}} + \lim_{n\to\infty} \|S_{\lambda_n}(X',1)\|_{\mathcal{A}} = 0.$$

Hence, $\lim_{n\to\infty}\sup_{n\geq N}\|S_{\lambda_n}(X-X',1)\|_{\mathcal{A}} = 0$ in probability. By Lemma 2.1, for all $N\in\mathbf{N}$,

$$\mathcal{L}\left(\sup_{n\geq N}\|S_{\lambda_n}(X-X',1)\|_{\mathcal{A}}\right)=\mathcal{L}\left(\sup_{n\geq N}\|S_{\lambda_n}(\varepsilon|X-X'|,1)\|_{\mathcal{A}}\right).$$

So that $\lim_{n\to\infty}\sup_{n\geq N}\|S_{\lambda_n}(\varepsilon|X-X'|,1)\|_{\mathcal{A}}=0$ in probability. Put

$$Y_{\mathbf{i},N}=\left\{\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\left(\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{j}})\right):n\geq\mathbf{i}\vee\mathrm{N},A\in\mathcal{A}\right\}$$

Then, using this notation, we rewrite

$$\begin{aligned}
&\lim_{N\to\infty}E\left\{\sup_{n\geq N}\|S_{\lambda_n}(\varepsilon|X-X'|,1)\|_{\mathcal{A}}\right\}\\
&\qquad=\lim_{N\to\infty}E\left\{\sup_{n\geq N,A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\leq n}\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\cdot\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{ij}})\right|\right\}\\
&\qquad=\lim_{N\to\infty}E\left\{\left\|\sum_{\mathbf{i}}Y_{\mathbf{i},N}\right\|_{\ell^\infty(\mathbf{N}\times\mathcal{A})}\right\}.
\end{aligned}$$

By applying Hoffmann-Jørgensen's inequality (Hoffmann and Jørgensen[7, p.164-165]) to $Y_{\mathbf{i},N}$,

$$\begin{aligned}
&E\left\{\sup_{n\geq N,A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\leq n}\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\cdot\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{ij}})\right|\right\}\\
&\qquad\leq 6E\left\{\max_{|\mathbf{i}|\leq n}\sup_{n\geq N,A\in\mathcal{A}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\left(\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{ij}})\right)\right\}+24t_0,
\end{aligned}$$

where

$$\begin{aligned}
t_0=\inf&\left\{t>0:P\left(\sup_{n\geq\mathrm{N},A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\leq n}\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\cdot\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{ij}})\right|>t\right)\right.\\
&\left.\qquad\leq 1/24\right\}.
\end{aligned}$$

Now

$$E\left\{\max_{|\mathbf{i}|\le n}\sup_{n\ge N,A\in\mathcal{A}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\left(\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right)\right\}$$
$$\le 2c_2n^{d_2}/n^{d_1+d_2}\longrightarrow 0,$$

as $N\to\infty$ or $n\to\infty$. Also we have

$$t_0=\inf\left\{t>0:P\left(\sup_{n\ge \mathrm{N},A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\le n}\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\cdot\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right|>t\right)\right.$$
$$\left.\le 1/24\right\}\longrightarrow 0,$$

as $n\longrightarrow\infty$ because $\lim_{N\to\infty}\sup_{n\ge\mathrm{N}}\|S_{\lambda_n}(\varepsilon|X-X'|)\|_{\mathcal{A}}=0$ in probability. But by Fubini's theorem and Jensen's inequality ,

$$E\left(\sup_{n\ge \mathrm{N},A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\le n}\varepsilon_{\mathbf{i}}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\cdot\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right|\right)$$
$$\ge E|X_{\mathbf{i}}-X'_{\mathbf{i}}|E\left(\sup_{n\ge \mathrm{N},A\in\mathcal{A}}\left|\sum_{|\mathbf{i}|\le n}\varepsilon_{\mathbf{i}}\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right|\right)$$
$$=E|X_{\mathbf{i}}-X'_{\mathbf{i}}|E\left(\sup_{n\ge\mathrm{N}}\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}}\right),$$

which shows that $E\left(\sup_{n\ge\mathrm{N}}\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}}\right)\to 0$ as $n\to\infty$.

For the converse, assume now that (ii) holds. Since λ_n satisfies (1.1) and

$$\sup_{n\ge N,\mathcal{A}}E\left\{\sum_{|\mathbf{i}|\le n}X_{\mathbf{i}}\left(\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right)\right\}^2\le\sup_{n\ge N,\mathcal{A}}n^{-d_1}E(X_{\mathbf{i}})^2=0.$$

By a symmetrization lemma (for example, Giné and Zinn [4, Lemma 2.5]), we have

$$E\left(\sup_{n\ge N,\mathcal{A}}|S_{\lambda_n}(X,1)|\right)\le 2E\left(\sup_{n\ge N,\mathcal{A}}|S_{\lambda_n}(X-X',1)|\right)$$

But, applying Hoffmann-Jørgensen's inequality,

$$E\left\{\sup_{n\geq \mathrm{N},\mathcal{A}}|S_{\lambda_n}(X-X',1)|\right\}$$
$$\leq 2E\left(\max_{|\mathbf{i}|\leq n}|X_{\mathbf{i}}-X'_{\mathbf{i}}|\right)E\left\{\sup_{n\geq \mathrm{N},\mathcal{A}}\left|\sum_{|\mathbf{i}|\leq n}\varepsilon_{\mathbf{i}}\left(\sum_{|\mathbf{j}|\leq n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right)\right|\right\}$$
$$=4E\left\{\sup_{n\geq \mathrm{N}}\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}}\right\}.$$

Hence $\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}\to 0$ a.s., as $n\to\infty$.

The next result gives a necessary and sufficient condition for laws of large numbers for products of some measures and partial sum processes in terms of the metric entropy defined in section 1. Notice that our metric, which will be used in the next Theorem, is $\mathbf{d}_{\lambda_n,p}$.

THEOREM 2.3. *Let X, $\{X_{\mathbf{i}}:\mathbf{i}\in\mathrm{N}^{d_1}\}$, $\{\lambda_n\}$, and $\mathcal{A}$ be as in Theorem 2.2. Then the following are equivalent:*

(i) $\lim_{n\to\infty}\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}=0$ *a.s.*.

(ii) $\lim_{n\to\infty}\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}=0$ *in probability.*

(iii) $\lim_{n\to\infty}n^{-d_1}\left[\ln N_{\lambda_n,p}(\tau,\mathcal{A})\right]=0$ *for some* $p\in[1,\infty]$ *and all* $\tau>0$.

(iv) $\lim_{n\to\infty}n^{-d_1}\left[\ln N_{\lambda_n,p}(\tau,\mathcal{A})\right]=0$ *for every* $p\in[1,\infty]$ *and all* $\tau>0$.

Proof. Due to (1.2) it suffices to prove the following two statements:

(I) $\lim_{n\to\infty}n^{-d_1}\left[\ln N_{\lambda_n,1}(\tau,\mathcal{A})\right]=0$ for all $\tau>0$,
$\Longrightarrow \lim_{n\to\infty}\|S_{\lambda_n}(X,1)\|_{\mathcal{A}}=0$ a.s..

and

(II) $\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}}=0$ in probability,
$\Longrightarrow \lim_{n\to\infty}n^{-d_1}\left[\ln N_{\lambda_n,\infty}(\tau,\mathcal{A})\right]=0$ for all $\tau>0$.

Proof of. (I). Given $\tau>0$, let $\mathcal{A}_{\tau/2}\subset\mathcal{A}$ be the family of centers of a minimal covering of $\mathcal{A}$ by $\mathbf{d}_{\lambda_n,1}$-balls of radius not larger than $\tau/2$ and with center in $\mathcal{A}$. Then $\sharp\mathcal{A}_{\tau/2}=N_{\lambda_n,1}(\tau/2,\mathcal{A})$, and by hypothesis, for all $\tau>0$, there exists N_τ such that if $n\geq N_\tau$, then, for some suitable constant K,

$$N_{\lambda_{n,1}}(\tau/2, \mathcal{A}) \leq \exp\left(\frac{\tau^2 n^{d_1}}{32K}\right). \tag{2.1}$$

Then, since, for each $A \in \mathcal{A}$, there is $B \in \mathcal{A}_{\tau/2}$ such that

$$|S_{\lambda_n}(\varepsilon, 1, A) - S_{\lambda_n}(\varepsilon, 1, B)| \leq \mathbf{d}_{\lambda_{n,1}}(A, B) \leq \tau/2,$$

we have

$$\{\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}} > \tau\} \subset \{\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}_{\tau/2}}\},$$

and

(2.2)

$$\begin{aligned} P(\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}} > \tau) \leq & P(\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}_{\tau/2}} > \tau/2) \\ \leq & N^2_{\lambda_{n,1}}(\tau/2, \mathcal{A}) \sup_{A \in \mathcal{A}} P(|S_{\lambda_n}(\varepsilon, A)| > \tau/2). \end{aligned}$$

Since ε is sub-Gaussian, by the standard sub-Gaussian estimate ([4], inequality (2.17)), (2.2) may be bounded as follow;

$$P(|S_{\lambda_n}(\varepsilon, A)| > \tau/2) \leq \exp(-\tau^2 n^{d_1}/16K). \tag{2.3}$$

Hence, combining (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned} P(\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}} > \tau) &\leq N_{\lambda_{n,1}}(\tau/2, \mathcal{A}) \exp\left(-\frac{\tau^2}{16K} n^{d_1}\right) \\ &\leq \exp\left(-\frac{\tau^2}{32K} n^{d_1}\right). \end{aligned}$$

Therefore, for all $\tau > 0$,

$$\sum_n P^*(\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}} > \tau) < \infty.$$

Applying the Borel-Cantelli lemma for outer measure version gives the statement (I).

Proof of. (II). First notice that if $\|S_{\lambda_n}(\varepsilon, 1)\|_{\mathcal{A}} \to 0$ in probability, then, since $S_{\lambda_n}(\varepsilon, 1)$ is bounded,

$$\lim_{n\to\infty} E\|S_{\lambda_n}(\varepsilon,1)\|_{\mathcal{A}} = 0. \tag{2.4}$$

By adjusting by c_2, if necessary, we may assume that $n^{d_1}\sum_{|\mathbf{j}|\le n}\lambda_n(C_{n\mathbf{i}\mathbf{j}}) \le 1$. Let $L_{n,\mathcal{A}}$ be the convex hull of $\{\mathbf{x} \in \mathbf{I}^{n^{d_1}} : \mathbf{x} = (x_{\mathbf{j}}),\ \ |\mathbf{i}| \le n,\ \ x_{\mathbf{i}} = n^{d_1}(\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}}))\ \ A \in \mathcal{A}\}$. Then (2.4) becomes

$$\begin{aligned} \lim_{n\to\infty} E\left\|\sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n}\varepsilon_{\mathbf{i}}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})\right\|_{\mathcal{A}} \\ = \lim_{n\to\infty} E\left\{\sup_{x\in \mathrm{L}_{n,\mathcal{A}}} n^{-d_1}\left|\sum_{|\mathbf{i}|\le n}\varepsilon_{\mathbf{i}}x_{\mathbf{i}}\right|\right\} = 0. \end{aligned}\tag{2.5}$$

Let $N_{n,\infty}(\tau, \mathrm{L}_{n,\mathcal{A}})$ be the covering number of $L_{n,\mathcal{A}}$ for the distance $\rho(\mathbf{x},\mathbf{y}) = \max_{|\mathbf{i}|\le n}|x_{\mathbf{i}} - y_{\mathbf{i}}|,\ \ \mathbf{x},\mathbf{y}\in\mathbf{I}^{n^{d_1}}$. Then, for ρ, by Vapnik and Červonenkis [7, Lemma 4], there is $t(\tau) < \infty$, independent of n, such that, if

$$\mathrm{N}(3\tau/2, \mathrm{L}_{n,\mathcal{A}}) > \exp\left(2^{n^{d_1}}\ln(1+\tau)\right), \tag{2.6}$$

then,

$$E\left\{\sup_{x\in\mathrm{L}_{n,\mathcal{A}}} n^{-d_1}\left|\sum_{|\mathbf{i}|\le n}\varepsilon_{\mathbf{i}}x_{\mathbf{i}}\right|\right\} \ge \tau\left[(1+\tau)^{1/3}-1\right](t(\tau)-n^{-d_1})/2.$$

Since,

$$\rho(\mathbf{x},\mathbf{y}) = \max_{|\mathbf{i}|\le n} n^{d_1}\left|\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}}) - \sum_{|\mathbf{j}|\le n}\lambda_n(B\cap C_{n\mathbf{i}\mathbf{j}})\right|,$$

for some A and B in $\mathcal{A}$, we can embed $\mathcal{A}$ into $L_{n,\mathcal{A}}$. [The correspondence $A \to n^{d_1}\sum_{|\mathbf{j}|\le n}\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}})$ is one-to-one and isometric]. Hence we have, for all $\tau > 0$,

$$N_{n,\infty}(\tau, \mathrm{L}_{n,\mathcal{A}}) \ge N_{\lambda_n,\infty}(\tau, \mathcal{A}).$$

Suppose that (iii) does not hold with $p = \infty$. Then for some $\tau > 0$ there is a natural number N'_τ such that (2.6) is true for $n \geq N'_\tau$. Then for these values of n, (2.4) is true and it contradicts to (2.5).

Now consider the special case when $\lambda_n = n^{-d} \sum_{|\mathbf{i}|\leq n, |\mathbf{j}|\leq n} \delta_{(\mathbf{i}/n, \mathbf{j}/n)}$. In this case, Theorem 2.2 gives:

COROLLARY 2.4. *Under the same assumptions as in Theorem 2.2, the following are equivalent:*

(i) $\|n^{-d} S_n(X,1)\|_{\mathcal{A}} \longrightarrow 0$ a.s., as $n \longrightarrow \infty$.

(ii) $\|n^{-d} S_n(X,1)\|_{\mathcal{A}} \longrightarrow 0$ *in probability, as* $n \longrightarrow \infty$.

(iii) for all $\tau > 0$ *and for some (all)* $p \in [1,\infty)$, $n^{-d_1}[\ln N(\tau, \mathcal{A}, \mathrm{d}_{n,p})] \longrightarrow 0$, *as* $n \longrightarrow \infty$.

From this we obtain the following result which covers the case when $EX \neq 0$.

COROLLARY 2.5. *Let* $X, X_{\mathbf{i}}, \mathbf{i} \in \mathbf{N}^{d_1}$, *be independent identically distributed random variables such that* $0 < E|X| < \infty$. *Let* $\mathcal{A}$ *be a family of Borel subsets of* $\mathbf{I}^d$ $(d = d_1 + d_2)$. *Then*

$$\lim_{n\to\infty} \|n^{-d} S_n(X,1) - (EX)\lambda\|_{\mathcal{A}} = 0 \tag{2.7}$$

a.s. *(or in probability) if and only if both*

(i) *any of the conditions* (i)-(iii) *in Corollary 2.4 holds for* $X - EX$ *and*

(ii) $\lim_{n\to\infty} \|\lambda - n^{-d} \sum_{|\mathbf{i}|\leq n, |\mathbf{j}|\leq n} \delta_{(\mathbf{i}/n, \mathbf{j}/n)}\|_{\mathcal{A}} = 0$. *where* λ *denotes the Lebesque measure on* $\mathbf{I}^d$.

Proof. Notice that, for $A \in \mathcal{A}$

$$n^{-d} S_n(X,1,A) - (EX)\lambda(A) = n^{-d} S_n(X - EX, 1, A)$$
$$+ (EX)\left(n^{-d} \sum_{|\mathbf{i}|\leq n, |\mathbf{j}|\leq n} \delta_{(\mathbf{i}/n, \mathbf{j}/n)}(A) - \lambda(A)\right)$$

Sufficiency is obvious. Let us assume that (2.7) holds with the convergence in probability. Then (2.7) also holds with $X_{\mathbf{i}}$ replaced by its symmetrization $X_{\mathbf{i}} - X'_{\mathbf{i}}$. i.e.

$$\lim_{n\to\infty} \|n^{-d} S_n(X - X', 1)\|_{\mathcal{A}} = 0 \qquad \text{in} \quad \text{probability}.$$

Now since $E|X - X'| \neq 0$ we can use Corollary 2.4, in particular, Corollary 2.4 is true for $X_{\mathbf{i}} - EX_{\mathbf{i}}$. So $\|n^{-d}S_n(X - EX)\|_{\mathcal{A}} \to 0$ in probability and almost surely, as $n \to \infty$. The triangle inequality and this observation imply (ii).

We will show in what follows that the smooth boundary condition, which is invented to show the strong law of large numbers for partial sum processes by Bass and Pyke, is also a sufficient condition for our product processes. Given a set A, let $A(\delta) = \{\rho(x, \partial A) < \delta\}$ be the δ-annulus of ∂A, where $\rho(\cdot,\cdot)$ is the Euclidean distance and ∂A is the Euclidean boundary of A. We say that $\mathcal{A}$ satisfies the smooth boundary condition, Assumption SBC, if $r(\delta) = \sup_{A\in\mathcal{A}} |A(\delta)| \longrightarrow 0$ as $\delta \longrightarrow 0$. Now define (pseudo)metrics on $\mathcal{A}$ associated with λ_n as follows:

$$\mathbf{d}'_{\lambda_{n,p}}(A,B) = n^{d(1-1/p)} \left(\sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} |\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}}) - \lambda_n(B\cap C_{n\mathbf{i}\mathbf{j}})|^p \right)^{1/p} \quad \text{if } 1 \le p < \infty,$$

$$\mathbf{d}'_{\lambda_{n,\infty}}(A,B) = \max_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} n^d |\lambda_n(A\cap C_{n\mathbf{i}\mathbf{j}}) - \lambda_n(B\cap C_{n\mathbf{i}\mathbf{j}})|,$$

If $\lambda_n = n^{-d}\sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} \delta_{(\mathbf{i}/n,\mathbf{j}/n)}$, and if we write $\mathbf{d}'_{n,p}$ for $\mathbf{d}'_{\lambda_{n,p}}$, then

$$\mathbf{d}'_{n,p}(A,B) = \left(\sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} \delta_{(\mathbf{i}/n,\mathbf{j}/n)}(A\Delta B)/n^d \right)^{1/p} \qquad 1 \le p < \infty,$$

$$\mathbf{d}'_{n,\infty}(A,B) = \max_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} \delta_{(\mathbf{i}/n,\mathbf{j}/n)}(A\Delta B),$$

where $A\Delta B = (A\setminus B)\cup(B\setminus A)$. It is easy to see that $\mathbf{d}_{n,1} \le \mathbf{d}'_{n,1}$. Denote $N'_{n,1}(\tau, \mathcal{A})$ the covering number of $(\mathcal{A}, \mathbf{d}'_{n,p})$. The following lemma shows that Assumption SBC on $\mathcal{A}$ implies (i) and (ii) of Corollary 2.5.

LEMMA 2.6 (GINÉ AND ZINN [4, PROPOSITION 1]). *Let $\mathcal{A}$ be a class of Borel subsets of $\mathbf{I}^d$ satisfying Assumption SBC. Then*

(i) *for all $\tau > 0$, $\sup_n N'_{n,1}(\tau, \mathcal{A}) < \infty$, and*

(ii) $\lim_{n\to\infty} \|\lambda - n^{-d}\sum_{|\mathbf{i}|\le n, |\mathbf{j}|\le n} \delta_{(\mathbf{i}/n,\mathbf{j}/n)}\|_{\mathcal{A}} = 0,$ *where λ denotes the Lebesgue measure on $\mathbf{I}^d$.*

COROLLARY 2.7. *It $\mathcal{A}$ satisfies Assumption SBC, then, with probability 1, as $n \to \infty$,*

$$\lim_{n\to\infty} \|n^{-d}S_n(X,1) - (EX)\lambda\|_{\mathcal{A}} = 0.$$

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