J. Korean Math. Soc. 30(1993), No. 1, pp. 1–24

INTERIOR ALGEBRAS: SOME UNIVERSAL ALGEBRAIC ASPECTS

COLIN NATURMAN AND HENRY ROSE[1]

Introduction

The concept of a closure algebra, i.e. a Boolean algebra enriched with a closure operator, was introduced by McKinsey and Tarski in [4] as an algebraic generalization of a topological space. An interior algebra is a Boolean algebra enriched with an interior operator. Most modern topologists prefer to consider a topological space to be a set together with a family of open subsets and most algebraists working with lattice ordered algebras prefer to use filters instead of ideals. As a result of these trends it is more natural to work with interior algebras than with closure algebras. However closure algebras and interior algebras are essentially the same thing.

Before the introduction of closure algebras, it had already been noticed that there is a connection between topology and modal logic (see McKinsey [3]). This connection is due to the fact that topological spaces and modal logics both give rise to interior algebras. In [2] Jónsson and Tarski showed that the power algebra of a pre–ordered set is a closure algebra. This is interesting in the light of the fact that pre–ordered sets are the models for the modal logić S4. Subsequent work on closure algebras focused mainly on applications to modal logic (see for example [1] and [6]) and their universal algebraic aspects have been neglected.

Since most mathematicians are not familiar with closure algebras and since interior algebras are new we have devoted Section 1 of this paper to basic results concerning interior algebras. We have avoided restating known results which are not relevant to our work and have

Received August 24, 1991.

[1]The work of the second author was supported by a grant from the University of Cape Town Research Commitee, and the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

instead focused on unknown results and results which have not been precisely and explicitly stated elsewhere.

Section 2 deals with the connection between topology and interior algebras. The results in this section provide useful tools for studying interior algebras.

In Section 3 we investigate congruences on interior algebras, we show that congruences can be described by filters of open elements and that the congruence lattice of an interior algebra is isomorphic to the lattice of all filters in the lattice of open elements (Theorem 3.5) we also describe the fully invariant congruences in terms of full filters (Proposition 3.9) we also show that interior algebras are congruences distributive, congruence permutable and congruence extensile (Corollary 3.1 and Theorem 3.7).

In Section 4 we characterize simple, subdirectly irreducible, finitely subdirectly irreducible and directly indecomposable interior algebras (Theorem 4.1) we in fact show that the classes of such interior algebras are all finitely axiomatizable elementary classes. We characterize these interior algebras topologically (Theorem 4.7 and 4.8).

In Section 5 we look at some results concerning quotients of interior algebras by congruences determined by open elements, openly decompasable interior algebras and dissectable interior algebras.

Notation and terminology

Bold capitals such as $\mathbf{B}, \mathbf{C}, \mathbf{X}, \mathbf{Y}$ will be used to denote structures. If $\mathbf{B}, \mathbf{K}, \mathbf{X}$ are structures then B, K, X will denote their respective underlying sets. If $\mathbf{B}$ is a structure and F an operation such that $F(\mathbf{B})$ is a structure, then $F(B)$ will denote the underlying set of $F(\mathbf{B})$.

If $\mathbf{B}, \mathbf{C}$ are algebraic structures $\mathbf{B} \cong \mathbf{C}$ is used to denote that $\mathbf{B}$ is isomorphic to $\mathbf{C}$. If $\mathbf{X}$ and $\mathbf{Y}$ are topological spaces $\mathbf{X} \cong \mathbf{Y}$ denotes that $\mathbf{X}$ is homeomorphic to $\mathbf{Y}$.

If $\mathbf{X}$ is a topological space and $a \in X$ then $\mathcal{U}_a$ will denote the set of all neighborhoods of a in $\mathbf{X}$.

The term epimorphism will only be used to mean a surjective algebraic homomorphism.

Contravariant functors will be referred to as co–functors.

1. Basic results concerning interior algebras

DEFINITION 1.0.1. Let $B = \langle B, \cdot, +, ', {}^I, 0, 1\rangle$
B is an interior algebra iff:

i) $\langle B, \cdot, +, ', 0, 1\rangle$ is a Boolean algebra
ii) ^{I} is a unary operation satisfying:
a) $a^I \leq a$ for all $a \in B$
b) $a^{II} = a^I$ for all $a \in B$
c) $(ab)^I = a^I b^I$ for all $a, b \in B$
d) $1^I = 1$

From now on B, C, D will denote interior algebras. The operation ^{I} is called an interior operator and "a^I" is read as the "interior of a".

PROPOSITION 1.0.2. *Let B be an interior algebra. Then:*

i) *If $a, b \in B$ with $a \leq b$ then $a^I \leq b^I$.*
ii) $0^I = 0$.

1.1. Lattices of open elements

By a $0,1$-lattice we mean an algebraic structure $K = \langle K, \cdot, +, 0, 1\rangle$ where $\langle K, \cdot, +\rangle$ is a lattice and $0, 1$ are nullary operations such that 0 is the bottom element of K and 1 is the top element.

Let Int and $0,1$-Lat denote the categories of interior algebras and $0,1$-lattices respectively, with their homomorphisms.

DEFINITION 1.1.1. If B is an interior algebra and $a \in B$, we say that a is open iff $a^I = a$. Put $L(B) = \{a \in B : a \text{ is open}\}$. Note that $0, 1 \in L(B)$ and $L(B)$ is closed under finite meets and joins. We thus obtain a $0,1$-lattice:

$$L(B) = \langle L(B), \cdot, +, 0, 1\rangle$$

suppose $f : B \to C$ is a homomorphism. If $a \in L(B), f(a)^I = f(a^I) = f(a)$ and so $f(a) \in L(C)$. f preserves $\cdot, +, 0, 1$ and so we obtain a homomorphism $Lf : L(B) \to L(C)$ by putting $Lf = f|_{L(B)}$.

COROLLARY 1.1.2. *L : Int $\to 0,1-$ Lat is a functor.*

Let Balg denote the category of Boolean algebras and homomorphisms. Let Ba : Int $\to$ Balg be the forgetful functor.

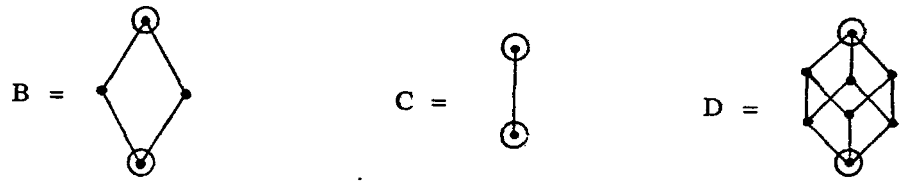

fig.2

LEMMA 1.1.3. *Let B be an interior algebra and $a \in B$. Then* $a^I = \max\{b \in L(B) : b \leq a\}$.

We immediately have:

COROLLARY 1.1.4. *Let $f : B \to C$ be a homomorphism. Then f is an isomorphism iff $\mathrm{Ba}\, f$ and Lf are isomorphisms.*

In particular we see that in order to specify an interior algebra B it suffices to specify $\mathrm{Ba}\, B$ and $L(B)$.

We can thus represent finite interior algebras diagrammatically by giving the lattice diagram for the underlying Boolean algebra and indicating the open elements. We will use circles to indicate the open elements.

EXAMPLE 1.1.5. Let B be as in fig.1.

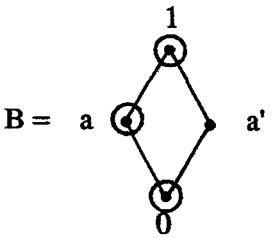

fig.1

B has $\mathrm{Ba}\, B$ equal to the four element Boolean algebra $\{1, a, a', 0\}$ and $L(B)$ equal to the three element chain $\{1, a, 0\}$.

EXAMPLE 1.1.6. Let B, C, D be as in fig.2 .

L is not full. There is an isomorphism from $L(B)$ to $L(C)$ but no homomorphism from B to C.

L is also not faithful. There are several embeddings of B into D all of which have the same image under L.

Despite the fact that L is neither full nor faithful we can still show that L has interesting preservation properties:

PROPOSITION 1.1.7. *L preserves embeddings, epimorphisms, isomorphisms, products, subdirect products, direct limits and inverse limits.*

LEMMA 1.1.8. *Let B be an interior algebra, and let $M \subseteq L(B)$. Then the join of M exists in B iff the join of M exists in $L(B)$. If these joins exist then they are equal.*

Proof. Suppose b is the join of M in B. Then for all $a \in M$ $a \leq b$ and so $a^I \leq b^I$ i.e. $a \leq b^I$. Hence $b \leq b^I$ whence $b^I = b$. Hence $b \in L(B)$ and so b is also the join of M in $L(B)$.

Conversely suppose b is the join of M in $L(B)$. Then b is an upper bound for M in B. Then for all $a \in M$ $a \leq c$ and so $a^I \leq c^I$ i.e. $a \leq c^I$. Thus c^I is an upper bound for M in $L(B)$ and so $b \leq c^I$. Hence $b \leq c$ and so b is the join of M in B.

•

Thus when we mention a join of elements of $L(B)$ it is not necessary to mention whether the join is taken in $L(B)$ or in B.

DEFINITION 1.1.9. Let B be an interior algebra and let K be a $0,1$-sublattice of $L(B)$. We say that K is a base for B iff every $a \in L(B)$ can be represented as a join of elements of K.

In particular $L(B)$ is a base for B.

PROPOSITION 1.1.10. *Let K be a $0,1$-lattice. Then K is distributive iff K is isomorphic to a base for some interior algebra B.*

Proof. The reverse direction is clear. For the forward direction: Let $\mathcal{A}$ be the set of all proper prime filters in K and consider the $0,1$-lattice $M = \langle \mathcal{P}(\mathcal{A}), \cap, \cup, \phi, \mathcal{A} \rangle$. Define a map $\varphi : K \to M$ by: $\varphi(a) = \{F \in \mathcal{A} : a \in F\}$ for all $a \in K$. Then φ is a $0,1$-lattice embedding. We define an interior operator on $\mathcal{P}(\mathcal{A})$ as follows: For all $\mathcal{B} \in \mathcal{P}(\mathcal{A})$ define: $\mathcal{B}^I = \bigcup\{\varphi(a) : a \in K \text{ and } \varphi(a) \subseteq \mathcal{B}\}$. Then $\langle \mathcal{P}(\mathcal{A}), \cap, \cup, ', {}^I, \phi, \mathcal{A} \rangle$ is an interior algebra with base $\varphi[K] \cong K$.

Of course $L(B)$ is always distributive. Note that if $L(B)$ is finite then $L(B)$ is the only base for B. Notice also that if K is a finite distributive $0,1$-lattice then the interior algebra B constructed in proposition 1.1.10 is finite.

COROLLARY 1.1.11. *Let K be a $0,1$-lattice. K is finite and distributive iff $K \cong L(B)$ for some finite interior algebra B.*

1.2. The closure operator, closed and clopen elements

DEFINITION 1.2.1. For any interior algebra B define a unary operation ^{c} on B by $a^c = a'^{I\prime}$, for all $a \in B$. The operation ^{c} is called the closure operator and "a^c" is read as the "closure of a".

Using the closure operator we can generalize the Principle of Duality for Boolean algebras to interior algebras:

Given an interior algebra sentence σ, define the dual of σ to be the sentence σ' obtained from σ by interchanging $\cdot$ and $+$, 0 and 1, and also ^{I} and ^{c}.

LEMMA 1.2.2. *If σ is a sentence holding in B then σ' also holds in B.*

PROPOSITION 1.2.3. *For any interior algebra B and $a, b \in B$ the following hold:*

(i) $a^c \geq a$
(ii) $a^{cc} = a^c$
(iii) $(a+b)^c = a^c + b^c$
(iv) $0^c = 0$
(v) $1^c = 1$
(vi) $a \geq b \Longrightarrow a^c \geq b^c$
(vii) $a'^I = a^{c\prime}$, $a^{I\prime} = a'^c$
(viii) $a^{IcIc} = a^{Ic}$, $a^{cIcI} = a^{cI}$

The dual notion of an open element is that of a closed element. More precisely:

DEFINITION 1.2.4. If B is an interior algebra and $a \in B$ we say that a is closed iff $a^c = a$.

Notice that a is closed iff a' is open. If we put

$$\begin{aligned} L'(B) &= \{a \in B : a \text{ is closed}\} \\ &= \{a' : a \in L(B)\} \end{aligned}$$

we can define another functor from Int to $0,1 - \text{Lat}$: $L'C(B) = \langle L'(B), \cdot, +, 0, 1\rangle$ is a $0,1$-lattice and if $f : B \to C$ is a homomorphism then $L'f : L'(B) \to L'(C)$ given by $L'f = f|_{L'(B)}$ is a $0,1$-lattice homomorphism.

DEFINITION 1.2.5. Let B be an interior algebra and $a \in B$. We say that a is clopen iff a is both open and closed. For an interior algebra B define :

$$\mathrm{Co}\, B = L(B) \cap L'(B).$$

$\mathrm{Co}\, B$ is thus the set of clopen elements of B. Then $\mathrm{Co}\, B = \langle \mathrm{Co}\, B, \cdot, +, ', 0, 1\rangle$ is a Boolean algebra. If $f : B \to C$ is a homomorphism defining $\mathrm{Co}\, f = f|_{\mathrm{Co}\, B}$ gives a homomorphism $\mathrm{Co}\, f : \mathrm{Co}\, B \to \mathrm{Co}\, C$.

COROLLARY 1.2.6. *Co : Int $\to$ Balg is a functor.*

PROPOSITION 1.2.7. Co *preserves embeddings, isomorphisms, products, direct limits and inverse limits.*

Co is neither full nor faithful but it is trivially surjective on objects : Given a Boolean algebra Q we can define an interior operator on Q by $a^I = a$ for all $a \in Q$. Then we obtain an interior algebra B with $\mathrm{Co}\, B = Q$.

2. Interior algebras and topology

2.1. Interior algebras from topological spaces

Let Tco denote the category of topological spaces and continuous open maps. From now on X, Y, Z will denote topological spaces.

DEFINITION 2.1.1. Given a topological space $X = \langle X, \mathcal{T}\rangle$ we define an interior operator on $\mathcal{P}(X)$ by: $V^I = \{a \in X : V \in \mathcal{U}a \text{ in } X\}$ for all $V \subseteq X$. We then have an interior algebra $A(X) = \langle \mathcal{P}(X), \cap, \cup, ', {}^I, \phi, X\rangle$. Let $f : X \to Y$ be a continuous open map. Define a map

$$Af : A(Y) \to A(X) \text{ by} : Af(V) = f^{-1}[V] \text{ for all } V \subseteq Y.$$

Clearly Af preserves $\cap, \cup$ and $'$. Consider $V \in \mathcal{U}a$ where $a \in X$. By continuity $f^{-1}[V] \in \mathcal{U}a$ there is an open set $W \subseteq X$ such that $a \in W \subseteq f^{-1}[V]$. Then $f(a) \in f[W] \subseteq ff^{-1}[V] \subseteq V$ and since f is open $f[W]$ is open. Hence $V \in \mathcal{U}_{f(a)}$. It follows that $Af(V^I) = Af(V)^I$ i.e. Af preserves ^{I}. $Af : A(Y) \to A(X)$ is thus a homomorphism. Clearly A acts contravariantly and preserves identity morphisms so $A : \text{Tco} \to \text{Int}$ is a co-functor.

PROPOSITION 2.1.2. *$A : Tco \to Int$ is a dual embedding.*

Proof. For faithfulness:
Suppose $f : X \to Y$ and $g : X \to Y$ are continuous open maps such that $Af = Ag$. If $a \in X$ then $f^{-1}[\{g(a)\}] = g^{-1}[\{g(a)\}]$ and so $a \in f^{-1}[\{g(a)\}]$ whence $f(a) = g(a)$. Thus $f = g$.

For injectivity on objects:
Let $X = \langle X, \mathcal{T}\rangle$ and $Y = \langle Y, \mathcal{R}\rangle$ be topological spaces such that $A(X) = A(Y)$. Then $X = \cup\mathcal{P}(X) = \cup\mathcal{P}(Y) = Y$, and $\mathcal{T} = LA(X) = LA(Y) = \mathcal{R}$ and so $X = Y$.

Notice that we have used: $LA(X) = \mathcal{T}$ for all topological spaces $X = \langle X, \mathcal{T}\rangle$.

PROPOSITION 2.1.3. *Let $f : X \to Y$ be a continuous open map.*

i) *Af is an embedding iff f is surjective.*
ii) *Af is an epimorphism iff f is injective.*

LEMMA 2.1.4. *Let X, Y be topological spaces and let $g : A(Y) \to A(X)$ be a homomorphism. Then g is an isomorphism iff there is a homeomorphism $f : X \to Y$ such that $g = Af$.*

Proof. The reverse direction is clear. For the forward direction let $g : A(Y) \to A(X)$ be an isomorphism. Then in particular g is an isomorphism between the power set Boolean algebras over Y and X.

Hence there is a bijection $f : X \to Y$ such that for all $V \subseteq Y$ $g(V) = f^{-1}[V]$. Thus it suffices to show that f is continuous and open.

Let $V \subseteq X$ and $W \subseteq Y$ be open. Then since g and g^{-1} are isomorphisms we have $f^{-1}[W]^I = f^{-1}[W^I] = f^{-1}[W]$ and $f[V]^I = f[V^I] = f[V]$ whence $f^{-1}[W]$ and $f[V]$ are open. Thus f is continuous and open as required.

COROLLARY 2.1.5. *Let X, Y be topological spaces. Then $X \cong Y$ iff $A(X) \cong A(Y)$.*

PROPOSITION 2.1.6. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Then*

$$A(\sum_{i \in I} X_i) \cong \prod_{i \in I} A(X_i)$$

Proof. The map $V \to (V \cap X_i)_{i \in I}$ is an isomorphism.

COROLLARY 2.1.7. *Let $\mathcal{D}$ be a directed family of topological spaces and continuous open maps. Then X is the direct limit of $\mathcal{D}$ iff $A(X)$ is the inverse limit of $A[\mathcal{D}]$.*

PROPOSITION 2.1.8. *Let B be an interior algebra.*

i) *B is complete and atomic iff $B \cong A(X)$ for some topological space X.*
ii) *B is finite iff $B \cong A(X)$ for some finite topological space X.*

Proof. The reverse directions are clear. For the forward directions: Let X be the set of atoms in $\mathrm{Ba}\, B$. Then $\mathrm{Ba}\, B$ is isomorphic to the power set Boolean algebra over X. Letting the interior operator on B carry over to $\mathcal{P}(X)$ gives an interior algebra C such that $B \cong C$. Putting $X = \langle X, L(C) \rangle$ gives the result.

2.2. Topological spaces from interior algebras

Let Top denote the category of topological spaces and continuous maps.

DEFINITION 2.2.1. For each interior algebra B define : $T(B) = \{F \subseteq B : F$ is an ultrafilter in $B\}$. If $a \in B$ let $\alpha(a) = \{F \in T(B) : a \in F\}$. We can consider α to be a Boolean algebra embedding from

Ba B to the power set Boolean algebra on $T(B)$. In particular, if $a, b \in L(B)$ we have $\alpha(a) \cap \alpha(b) = \alpha(ab)$. Since $ab \in L(B)$ $\alpha[L(B)]$ is closed under finite intersections. Also $1 \in L(B)$ and $\alpha(1) = T(B)$. Thus we see that $\alpha[L(B)]$ is a base for a topology on $T(B)$. Thus $\mathcal{T}(B)$ defined by:

$$\mathcal{T}(B) = \{\cup\mathcal{A} : \mathcal{A} \subseteq \alpha[L(B)]\}$$

is a topology on $T(B)$. Put $T(B) = \langle T(B), \mathcal{T}(B)\rangle$. Let $f : B \to C$ be a homomorphism. If $F \in T(C)$ then $f^{-1}[F] \in T(B)$ and so we can define a map $Tf : T(C) \to T(B)$ by:

$$Tf(F) = f^{-1}[F] \quad \text{for all} \quad F \in T(C).$$

Now let $\mathcal{F} \subseteq T(B)$ be open. Then there is a subset $S \subseteq L(B)$ such that $\mathcal{F} = \cup\alpha[S]$. Then $(Tf)^{-1}[\mathcal{Y}] = \cup\alpha[f[S]]$. Since $S \subseteq L(B)$, $f[S] \subseteq L(C)$ and so $(Tf)^{-1}[\mathcal{Y}]$ is open. Tf is thus a continuous map.

Like A, T acts contravariantly and preserves indentity morphisms, so we have:

PROPOSITION 2.2.2.

$$T : \mathit{Int} \to \mathit{Top} \text{ is a co-functio.}$$

The co-functor T is a generalization of the Stone co-functor on Boolean algebras. Given an interior algebra B, $\mathcal{T}(B)$ is coarser than the Stone topology on Ba B. *Since the Stone topology is always compact we have:*

COROLLARY 2.2.3. *For any interior algebra B, $T(B)$ is compact.*

PROPOSITION 2.2.4. *T : Int $\to$ Top is faithful.*

Proof. Let $f : B \to C$ and $g : B \to C$ be homomorphisms such that $Tf = Tg$. Suppose $f \neq g$. Then there is a $b \in B$ such that $f(b) \neq g(b)$. So there is an ultrafilter $F \in T(B)$ such that F contains one of $f(b)$, $g(b)$ but not both. Suppose $f(b) \in F$. Then $b \in f^{-1}[F]$. Now $Tf(F) = Tg(F)$ i.e. $f^{-1}[F] = g^{-1}[F]$. Hence $b \in S^{-1}[F]$ and so $g(b) \in F$, a contradiction. Similarly we have a contradiction if $g(b) \in F$. Thus $f = g$.

Trivially T is not an embedding. If B and C are two distinct trivial interior algebras, then $T(B)$ and $T(C)$ are both the unique empty topological space.

PROPOSITION 2.2.4. *Let $f : B \to C$ be a homomorphism. Tf is surjective iff f is an embedding.*

Proof. If Tf is surjective it is epic, since faithful co-functors co-reflect epics, f is monic. Since Int consists of a variety and its homomorphisms, f is an embedding.

Conversely suppose f is an embedding. Let $F \in T(B)$. $0 \notin f[F]$ or else there is an $a \in F$ such that $f(a) = 0$. Since f is an embedding $a = 0$ contradicting the fact that F is an ultrafilter. Also if $a, b \in F$ then $ab \in F$ and $f(ab) = f(a)f(b)$ so $f[F]$ is closed under finite meets. Hence there is and ultrafilter $G \in T(C)$ such that $f[F] \subseteq G$. Then $F \subseteq f^{-1}[G]$, and since F and $f^{-1}[G]$ are ultrafilters $F = f^{-1}[G] = Tf(G)$. Hence Tf is surjective.

PROPOSITION 2.2.5. *Let $f : B \to C$ be an epimorphism. Then Tf is an open embedding.*

Proof. Clearly Tf is injective. Since Tf is continuous it remains to show that Tf is open. For this it suffices to show that for $a \in L(C)$. $Tf[\alpha(a)]$ is open. Now by Proposition 1.1.7 Lf is an epimorphism so there is a $b \in L(B)$ such that $f(b) = Lf(b) = a$. Thus we see that

$$Tf[\alpha(a)] = \{f^{-1}[F] : F \in T(C) \text{ and } b \in f^{-1}[F]\}.$$

Consider $G \in T(B)$. Since f is an epimorphism $f[G]$ is an ultrafilter i.e. $f[G] \in T(C)$. Also $G \subseteq f^{-1}f[G]$. Since G and $f^{-1}f[G]$ are ultrafilters $G = f^{-1}f[G]$. Thus every $G \in T(B)$ is of the form $f^{-1}[F]$ for some $F \in T(C)$. Hence $Tf[\alpha(a)] = \{G \in T(B) : b \in G\} = \alpha(b)$ and so $Tf[\alpha(a)]$ is open as required.

Since T is faithful we also have:

COROLLARY 2.2.6. *Let $f : B \to C$ be a homomorphism. Then Tf is a homeomorphism iff f is an isomorphism.*

There is no corresponding result to 2.1.5 for T. Let X be the indiscrete space on a denumerable set X. Let B be the subalgebra of $A(X)$ consisting of all finite and co-finite subsets of (X). Let C be the subalgebra of $A(X)$ generated by $B \cup \{V\}$ where $V \subseteq X$ is neither finite nor co-finite. Then we have $|T(B)| = |T(C) = \aleph_0$ and $T(B)$, $T(C)$ are both indiscrete. Hence $T(B) \cong T(C)$ but $B \cong C$.

2.3. The natural transformations α and β

Given an interior algebra B, α can be considered to be a map from B to $AT(B)$. Now given a topological space X, for all $x \in X$ let $\beta(x)$ denote the principal ultrafilter over X generated by $\{x\}$. Then β can be considered to be a map from X to $TA(X)$.

PROPOSITION 2.3.1. *Let B be an interior algebra and X a topological space.*

i) $\alpha : B \to AT(B)$ *is an embedding*
ii) $\beta : X \to TA(X)$ *is injective and continuous*

Proof. i) We already known that α is a Boolean algebra embedding. For interiors : Let $b \in B$. Suppose $F \in T(B)$ with $b^I \in F$. Then in particular there is an $a \in L(B)$ with $a \leq b$ and $a \in F$. Conversely if there is an $a \in L(B)$ with $a \leq b$ and $a \in F$ then $a = a^I \leq b^I$ and so $b^I \in F$. Thus $\alpha(B^I) = \cup\{\alpha(a) : a \in L(B) \text{ and } a \leq b\} = \alpha(a)^I$ as required.

ii) Clearly β is injective. Let $\Delta \subseteq TA(X)$ be open. Then there is an $\mathcal{A} \subseteq LA(X)$ such that $\Delta = \cup\alpha[\mathcal{A}]$. Note that for $x \in X$ and $V \in \mathcal{A}$, $V \in \beta(x)$ iff $x \in V$. Thus $\beta^{-1}[\Delta] = \cup\mathcal{A}$. Since $\mathcal{A}$ is a system of open sets, $\beta^{-1}[\Delta]$ is open. Hence β is continuous.

COROLLARY 2.3.2. *Let B be a finite interior algebra and let X be a finite topological space.*

i) $\alpha : B \to AT(B)$ *is an isomorphism*
ii) $\beta : X \to TA(X)$ *is a homeomorphism*

Proof. i) Let $\mathcal{F} \subseteq T(B)$. Since B is finite all ultrafilters in B are principal. Hence there is an $S \subseteq B$ such that $\mathcal{F} = \{[a] : a \in S\}$. Since B is finite S has a join b. Then $\alpha(b) = \mathcal{F}$. Thus α is surjective and hence an isomorphism.

ii) Since X is finite all ultrafilters over X are principal. Hence β is surjective. It remains to show that β is open. Let $V \subseteq X$ be open. Using the bijectivity of β we see that for any ultrafilter $\mathcal{F} \in TA(X)$, $V \in \mathcal{F}$ iff $\mathcal{F} \in \beta[V]$ and so $\beta[V] = \alpha(V)$ which is open. Hence β is open as required.

Let Inb denote the category of interior algebras and Boolean algebra homomorphisms between interior algebras. Let I : Int $\to$ Inb and

$J : \mathrm{Tco} \to \mathrm{Top}$ be the inclusion functors. We can extend the co-functor $A : \mathrm{Tco} \to \mathrm{Int}$ to a co-functor from Top to Inb, since $f : X \to Y$ is a continued map $Af : A(Y) \to A(X)$ given by $Af(V) = f^{-1}[V]$ for all $V \subseteq Y$, is a Boolean algebra homomorphism. Then we have functors $AT : \mathrm{Int} \to \mathrm{Inb}$ and $TA : \mathrm{Tco} \to \mathrm{Top}$.

PROPOSITION 2.3.3.

i) *If $f : B \to C$ is a homomorphism then $ATf \circ \alpha = \alpha \circ f$*

ii) *If $g : X \to Y$ is a continuous open map then $TAg \circ \beta = \beta \circ g$*

Consequently α and β can be considered to be natural transformations $\alpha : I \to AT$ and $\beta : J \to TA$.

We end this section with the observation that A and T are not full: If $f : X \to Y$ is continuous and open then Af preserves arbitrary meets. However not all homomorphisms between interior algebras of the form $A(Y)$ and $A(X)$ are meet preserving.

Consider the interior algebras B and C as in fig.3.

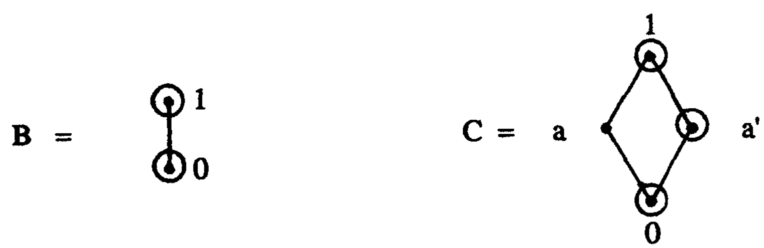

fig.3

The map $f : T(B) \to T(C)$ given by $f(\{1\}) = \{a, 1\}$ is continuous but there is no homomorphism $g : C \to B$ with $Tg = f$.

3. Congruences on interior algebras

For any interior algebra B let $\mathrm{Con}(B)$ denote the 0,1-lattice of congruences on B and let $FL(B)$ denote the 0, 1-lattice of all filters in $L(B)$.

If ψ is a congruence on B then ψ is a congruence on $\mathrm{Ba}\, B$. Now Boolean algebras are congruence permutable and all lattice ordered algebras are congruence distributive, hence we have:

COROLLARY 3.1. *Interior algebras are congruence permutable and congruence distributive.*

Let B be an interior algebra. If $G \in FL(B)$ define a relation θ_G on B by: for all $a, b \in B$ $a\theta_G b$ iff there is a $g \in G$ such that $ag = bg$.

LEMMA 3.2. *If $G \in FL(B)$ then $\theta_G \in Con(B)$.*

Proof. It is not difficult to check that θ_G is a lattice congruence. It remains to check that θ_G preserves complements and interiors. For this let $a, b \in B$ with $a\theta_G b$. Then there is a $g \in G$ such that $ag = bg$. Then $(ag + g')' = (bg + g')'$. Hence $(a' + g')g = (b' + g')g$ and so $a'g = b'g$. Also $(ag)^I = (bg)^I$ whence $a^I g^I = b^I g^I$ i.e. $a^I g = b^I g$.

If $\psi \in \mathrm{Con}(B)$ define $F_\psi = 1/\psi \cap L(B)$.

LEMMA 3.3. *If $\psi \in Con(B)$ then $F_\psi \in FL(B)$.*

LEMMA 3.4. *Let $G \in FL(B)$ and $\psi \in \mathrm{Con}(B)$ then:*

i) $F_{\psi_G} = G$
ii) $\theta_{F_\psi} = \psi$

Proof. i) Let $g \in F\theta_G$. Then $g\theta_G 1$ and so there is an $h \in G$ such that $gh = h$. Since $gh \leq g$ $g \in G$. Conversely if $g \in G$ we have $gg = g \cdot 1$ so $g\theta_G 1$. Hence $g \in F_{\theta_G}$. Thus $F_{\theta_G} = G$.

ii) Let $a\theta_{F_\psi} b$. Then there is a $g \in F_\psi$ such that $ag = bg$. Then $g\psi 1$ and so $a\psi ag = bg\psi b$. Conversely if $a\psi b$ put $c = ab + a'b'$. Then $c\psi 1$ and so $c^I\psi 1$. Hence $c^I \in F\psi$. Also $ac = ab = bc$. Then $acc^I = bcc^I$ and so $ac^I = bc^I$ whence $a\theta F\psi b$. Thus $\theta_{F_\psi} = \psi$.

THEOREM 3.5. *The maps $\psi \to F_\psi$ and $G \to \theta_G$ are inverse isomorphisms between $Con(B)$ and $FL(B)$. In particular $Con(B) \cong FL(B)$.*

Proof. That the maps are inverse bijections between $\mathrm{Con}(B)$ and $FL(B)$ follows from lemmas 3.2-4. If $\psi_1 \subseteq \psi_2$ in $\mathrm{Con}(B)$ then $F_{\psi_1} \subset F_{\psi_2}$ and if $G_1 \subseteq G_2$ in $FL(B)$ then $\theta_{G_1} \subseteq \theta_{G_2}$.
Hence the maps are isomorphisms.

(The above theorem gives an alternative proof of Corollary 3.1.)
For $a \in L(B)$ let $[a)$ denote the principal filter in $L(B)$ generated by a.

COROLLARY 3.6.

i) *The map* $a \to \theta_{[a)}$ *is a complete dense dual embedding of* $L(B)$ *into* $FL(B)$.

ii) *If* $L(B)$ *is finite then* $L(B)$ *is dually isomorphic to* $Con(B)$.

Proof. Clear from Theorem 3.5 since the map $a \to [a)$ is a complete dense dual embedding of $L(B)$ into $FL(B)$ and iff $L(B)$ is finite this map is also surjective.

If we consider a Boolean algebra to be an interior algebra satisfying the identity $x^I = x$, then the well known results concerning congruences on Boolean algebras and filters follow immediately from the above results.

THEOREM 3.7. *Interior algebras are congruence extensile*

Proof. Let B be an interior algebra and let C be a subalgebra of B. Let $\psi \in \mathrm{Con}(C)$. Put $G = \{b \in L(B) : b \geq a$ for some $a \in F_\psi\}$. Since F_ψ is a filter in $L(C)$, G is a filter in $L(B)$. Moreover $G \cap C = F_\psi$. Thus $\theta_G \in \mathrm{Con}(B)$ and $\theta_{G|C} = \psi$.

DEFINITION 3.8. Let B be an interior algebra and let $G \in FL(B)$. We say that G is **full** iff for every endomorphism f on B we have $f[G] \subseteq G$. The full filters of $L(B)$ form a complete $0,1$-sublattice $F_f L(B)$ of $FL(B)$.

PROPOSITION 3.9. *ψ is a fully invariant congruence on B iff F_ψ is a full filter. (Equivalently G is a full filter in $L(B)$ iff θ_G is fully invariant.)*

Proof. Suppose ψ is fully invariant. Let $f : B \to B$ be an endomorphism. Let $a \in F_\psi$. Then $a\psi 1$ and $a \in L(B)$. Hence $f(a)\psi 1$ and $f(a) \in L(B)$ whence $f(a) \in F_\psi$. Thus F_ψ is full. Conversely suppose F_ψ is full. Let $a, b \in B$ with $a\psi b$ and let $f : B \to B$ be an endomorphism. There is a $g \in F_\psi$ such that $ag = bg$. Then $f(a)f(g) = f(b)f(g)$. Since F_ψ is full $f(g) \in F_\psi$ and so $f(a)\psi f(B)$. Thus ψ is fully invariant.

If $\mathrm{Con}_f(B)$ denotes the $0,1$-lattice of fully invariant congruences on B we have:

COROLLARY 3.10. *$Con_f(B) \cong F_f L(B)$.*

4. Simple, subdirectly irreducible, finitely subdirectly irreducible and directly indecomposable interior algebras

The results of Section 3 allow us to characterize simple, subdirectly irreducible (S.I.), finitely subdirectly irreducible (F.S.I.) and directly indecomposable (D.I.) interior algebras.

THEOREM 4.1. *Let B be a non-trivial interior algebra.*

i) *B is simple iff $L(B) = \{0, 1\}$.*
ii) *B is S.I. iff 1 is completely join irreducible in $L(B)$, equivalently $L(B)$ has a largest non-1 element.*
iii) *B is F.S.I. iff 1 is join irreducible in $L(B)$.*
iv) *B is D.I. iff $Co\, B = \{0, 1\}$.*

Proof.

i) By Corollary 3.6(i), if Con(B) is a two element chain than $L(B) = \{0, 1\}$. Conversely by Corollary 3.6(ii) if $L(B) = \{0, 1\}$ then Con(B) is a two element chain.
ii) By Corollary 3.6(i), Con(B) has a smallest non-Δ element iff $L(B)$ has a largest non-1 element.
iii) By Corollary 3.6(i) Δ is meet irreducible in Con(B) iff 1 is join irreducible in $L(B)$.
iv) If B is not D.I. then by congruence permutability Con(B) contains a pair of non-trivial complementary congruences θ, ψ. Then F_θ and F_ψ are non-trivial and complementary in $FL(B)$. Then $0 \in F_\theta + F_\psi$ and so there are $a \in F_\theta$ and $b \in F_\psi$ such that $0 = ab$. Now $[a+b) = [a) \cap [b) \subseteq F_\theta \cap F_\psi = \{1\}$ and so $a+b = 1$. Hence $a \in \mathrm{Co}\, B$ but $0 < a < 1$. Thus if $\mathrm{Co}\, B = \{0, 1\}$ then B is D.I. Conversely if there is an $a \in \mathrm{Co}\, B$ with $0 < a < 1$ then $\theta_{[a)}$ and $\theta_{[a')}$ are complementary non-trivial congruences in Con(B) and so B cannot be D.I.

COROLLARY 4.2. *The classes of all simple, F.S.I. and D.I. interior algebras are all finitely axiomatizable elementary classes defined by a universal sentence.*

Proof. Let B be an interior algebra. Then by theorem 4.1 we have:

B is simple iff $B \models \neg(0 = 1) \wedge (\forall x)(x < 1 \Rightarrow x^I = 0)$

B is F.S.I. iff $B \models \neg(0 = 1) \wedge (\forall x)(\forall y)(x^I + y^I = 1 \Rightarrow (x = 1 \vee y = 1))$

B is D.I.iff $B \models \neg(0 = 1) \wedge (\forall x)((x^I = x \wedge x^c = x) \Rightarrow (x = 0 \vee x = 1))$

COROLLARY 4.3. *The classes of simple, F.S.I. and D.I. interior algebras are closed under ultraproducts and subalgebras. Their complements are closed under ultraproducts and extensions.*

COROLLARY 4.4. *The classes of S.I. interior algebras is a finitely axiomatizable elementary class defined by an existential-universal sentence.*

Proof. Let B be an interior algebra. Then by theorem 4.1 B is S.I. iff $B \models (\exists x)(\forall y)(x^I < 1 \wedge (y^I < 1 \Rightarrow y^I < x^I))$.

COROLLARY 4.5. *The class of S.I. interior algebras is closed under ultraproducts and algebras which are sandwiched by its members. Its complement is closed under ultraproducts, algebras which sandwich its members, and unions of chains.*

We can characterize simple, S.I., F.S.I. and D.I. interior algebras topologically.

First we need to define some topological concepts with which the reader is probably not familiar.

DEFINITION 4.6. Let X be a topological space. We say that X is supercompact iff X satisfies one (and hence allk) of the following equivalent conditions:

i) Every filter over X converges in X.
ii) Every net in X converges in X.
iii) Every open cover of X contains X.
iv) X is non-empty and the intersection of any family of non-empty closed subsets of X is non-empty.

Notice that X is supercompact iff X has a largest proper open subset, $0[X]$.

We say that X is ultra-connected iff for any open subsets V, W of $X : V \cup W = X$ implies $V = X$ or $W = X$.

Any indiscrete space is supercompact. Any supercompact space is compact and ultra-connected. Any ultra-connected space is (pathwise) connected.

THEOREM 4.7. *Let X be a non-empty topological space.*

i) *$A(X)$ is simple iff X is indiscrete.*
ii) *$A(X)$ is S.I. iff X is supercompact.*
iii) *$A(X)$ is F.S.I. iff X is ultra-connected.*
iv) *$A(X)$ is D.I. iff X is connected.*

Proof. Clear from theorem 4.1 and the definitions.

THEOREM 4.8. *Let B be a non-trivial interior algebra.*

i) *B is simple iff $T(B)$ is indiscrete.*
ii) *B is S.I. iff $T(B)$ is supercompact with $0[T(B)] = \alpha(a)$ for some $a \in L(B)$.*
iii) *B is F.S.I. iff $T(B)$ is ultra-connected.*
iv) *B is D.I. iff $T(B)$ is connected.*

Proof. By Proposition 2.3.1 B is embeddable in $AT(B)$ and so the reverse directions of (i), (iii) and (iv) follow from theorem 4.7 and corollary 4.3.

For the forward directions:

i) If B is simple $L(B) = \{0, 1\}$ and so $\mathcal{T}(B) = \{\phi, T(B)\}$ whence $T(B)$ is indiscrete.
iii) If $T(B)$ is not ultra-connected there are proper open subsets $\mathcal{D}, \sigma$ of $T(B)$ such that $\mathcal{D} \cup \sigma = T(B)$. Then there are $R, S \subseteq L(B)$ such that $\mathcal{D} = \bigcup\alpha[R]$ and $\sigma = \bigcup\alpha[S]$. Then $T(B) = \bigcup\alpha[R \cup S]$. By compactness of $T(B)$ there is a finite $K \subseteq R \cup S$ such that $T(B) = \bigcup\alpha[K]$. Let b be the join of K. Then $T(B) = \alpha(b)$ and so $b = 1$. However $1 \notin K$ (since $R, S \neq T(B)$) and so 1 is not join irreducible in $L(B)$ whence B is not F.S.I.
iv) If $T(B)$ is not connected then $T(B)$ has a proper non-empty clopen subset $\mathcal{D}$. Then there are $R, S \subseteq L(B)$ such that $\mathcal{D} = \bigcup\alpha[R]$ and $\mathcal{D} = \bigcup\alpha[S]$. Then $T(B) = \bigcup\alpha[R \cup S]$. By compactness of $T(B)$ there are finite subsets $K \subseteq R$ and $M \subseteq S$ such that $T(B) = \bigcup\alpha[K \cup M]$. Let a, b be the joins of K, M respectively. Then $T(B) = \alpha(a + b)$ and so $a + 1 = 1$. Also

$\alpha(ab) = \alpha(a) \cap \alpha(b) \subseteq \mathcal{D} \cap \mathcal{D} = \phi$ and so $ab = 0$. Hence $b = a'$ and so $a \in \mathrm{Co}\, B$. Now $\alpha(a)' = \alpha(a') \subseteq \mathcal{D}$ and so $\mathcal{D} \subseteq \alpha(a)$. But $\alpha(a) \subseteq \mathcal{D}$ and so $\alpha(a) = \mathcal{D}$. Hence $0 < a < 1$ and so $\mathrm{Co}\, B \neq \{0, 1\}$.

For (ii): B is S.I.

iff There is an $a \in L(B)$ such that $a < 1$ and for all $b \in L(B)$ with $b < 1$, $b \leq a$.

iff There is an $a \in L(B)$ such that $\alpha(a) \neq T(B)$ and for all $b \in L(B)$ with $\alpha(b) \neq T(B)$,

$$\alpha(b) \subseteq \alpha(a).$$

iff There is an $a \in L(B)$ such that $T(B)$ is supercompact with $O[T(B)] = \alpha(a)$.

EXAMPLE 4.9. The condition that $O[T(B)] = \alpha(a)$ for some $a \in L(B)$ cannot be dropped in theorem 4.8(ii).

Consider the topological space $\omega = \langle \omega, \omega^+ \rangle$. Then $\omega = \cup_{n<\omega} n$ and so ω is not completely join irreducible in $LA(\omega)$. Hence $A(\omega)$ is not S.I. However $TA(\omega)$ is supercompact : suppose it is not. Then there is an open cover $\{\mathcal{F}_i : i \in I\}$ of $TA(\omega)$ not containing $TA(\omega)$. For all $i \in I$ there is an $S_i \subseteq \omega^+$ such that $\mathcal{F}_i = \cup \alpha\{S_i]$. Put $S = \cup_{i \in I} S_i$. Then $TA(\omega) = \cup \alpha[S]$. Now for all $i \in I$ $\mathcal{F}_i \neq TA(\omega)$ so $\omega \notin S_i$. Hence $\omega \notin S$. Put $R = \{\omega \backslash n : n < \omega\}$. Then $\phi \notin R$ and R is closed under finite intersections. Hence there is an ultrafilter $F \in TA(\omega)$ such that $R \subseteq F$. Then there is an $n \in S$ such that $F \in \alpha(n)$ i.e. $n \in F$. Now $n < \omega$ and so $\omega \backslash n \in F$ contradicting the fact that F is an ultrafilter.

Hence $TA(\omega)$ is supercompact but $A(\omega)$ is not S.I.

$A(\omega)$ is an example of a subspace of an S.I. interior algebra which is not S.I. itself. ($A(\omega)$ is embeddable in $ATA(\omega)$ by Proposition 2.3.1 and $ATA(\omega)$ is S.I. by theorem 4.1.)

Notice that theorem 4.1 tells us that we can construct arbitrary large simple, S.I., F.S.I. and D.I. interior algebras.

5. Quotients by open elements, dissectable and openly decomposable interior algebras

DEFINITION 5.1. Let B be an interior algebra and let $a \in L(B)$. Let B/a denote the principal ideal in B generated by a. $0 \in B/a$ and

B/a is closed under $\cdot, +$ and ^{I}. Define complementation on B/a by $b'^a = ab'$ for all $b \in B/a$.

Then $B/a = \langle B/a, \cdot, +, '^a, ^I, 0, a \rangle$ is an interior algebra : the quotient of B by a.

Notice that this is basically the same concept as a "relativised subalgebra" (see [4]). However we only consider $a \in L(B)$ and allow $a = 0$.

Let X be a topological space and let V be an open subspace of X. Then the inclusion map $i : V \to X$ is an open embedding. Hence by proposition 2.1.3 $A_i : A(X) \to A(V)$ is an epimorphism. Now $A(V) = A(X)/V$ and so $A(X)/V$ is an epimorphic image of $A(X)$. We can generalize this to arbitrary interior algebras:

PROPOSITION 5.2. *Let B be an interior algebra and let $a \in L(B)$. Then B/a is an epimorphic image of B, in fact $B/a \cong B/\theta_{[a)}$.*

Proof. The map $b \to ab$ is an epimorphism from B to B/a. Let ψ be the kernel of this epimorphism. Since $ab = a$ iff $a \leq b$ we see that $F\psi$ is the principal filter $[a)$ in $L(B)$ and so $\theta_{[a)} = \psi$.

If $L(B)$ is finite all congruences on B are of the form $\theta_{[a)}$ and so the above proposition allows us to compute all the epimorphic images of B.

EXAMPLE 5.3. Consider the interior algebra in fig.4.

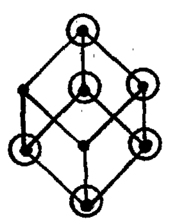

fig.4

Then by proposition 5.2 the epimorphic images of this interior algebra are those given in fig.5.

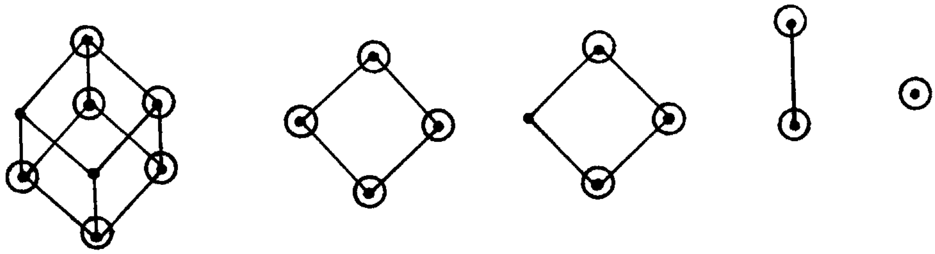

fig.5

COROLLARY 5.3. *Let B be an interior algebra and let $S \subseteq L(B)$. Then the map $b \mapsto (ab)_{a \in S}$ is a subdirect embedding of B into $\Pi_{a \in S} B/a$*

iff $\sum S = 1$.

Suppose X is a topological space and $\{V_i : i \in I\}$ is a family of open subspaces of X. Then by proposition 2.1.6 we have: $A(X) \cong \Pi_{i \in I} A(V_i)$ iff $X \cong \sum_{i \in I} V_i$ iff $\{V_i : i \in I\}$ is a partition of X. We can generalize this to arbitrary interior algebras in the case of finite partitions.

COROLLARY 5.4. *Let B be an interior algebra and let $a_1, \ldots, a_n \in L(B)$. Then the map $b \mapsto (a_1 b, \ldots, a_n b)$ is an isomorphism from B to $\Pi_{i=1}^{n} B/a_i$ iff $\{a_1, \ldots, a_n\}$ is a partition of 1.*

DEFINITION 5.5. (cf. McKinsey and Tarski [4]). An interior algebra B is called dissectable iff for every $a \in L(B)$ with $a > 0$ and every pair of integers $m \geq 0$ and $n > 0$ there is a partition $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ of a in B with $\{a_1, \ldots, a_m\} \subseteq L(B)$, ${b_1}^c = \cdots = {b_n}^c$ and $a^c a' \leq {b_i}^c \leq {a_j}^c$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

EXAMPLE 5.6. If X is a second countable, normal, dense-in-itself topological space than $A(X)$ is dissectable. In particular if $n < \omega$ then $A(\mathbf{R}^n)$ is dissectable. [4].

PROPOSITION 5.7. *(cf. McKinsey and Tarski [4]). If B is dissectable and $a \in L(B)$ then B/a is dissectable.*

PROPOSITION 5.8. *Let B be non-trivial and dissectable. Then every finite S.I. interior algebra is embeddable in B.*

Proof. The finite S.I. interior algebras are exactly the finite F.S.I. interior algebras. By corollary 4.2 and duality these are just the finite interior algebras satisfying the sentence $\neg(1 = 0) \wedge (\forall x)(\forall y)(x^c y^c = 0 \Rightarrow (x = 0 \vee y = 0)$. By theorem 3.7 of [4] such interior algebras are embeddable in B.

PROPOSITION 5.9. *(cf. McKinsey and Tarski [4]). Let B be non-trivial and dissectable. Then every finite interior algebra is embeddable in B/a for some $a \in L(B)$.*

DEFINITION 5.10. An interior algebra B is called openly decomposable iff for all $a \in L(B)$ with $a > 0$ B/a is directly decomposable.

PROPOSITION 5.11. *B is openly decomposable iff for all $a \in L(B)$ with $a > 0$ there are $b, c \in L(B)$ with $b, c > 0$, $b + c = a$ and $bc = 0$.*

Proof. Clear from theorem 4.1 (iv) since $b \in L(B/a)$ iff $b \in L(B)$ and $b \leq a$.

PROPOSITION 5.12. *Let X be a topological space. Then $A(X)$ is openly decomposable iff no non-empty open subspace of X is conneected.*

PROPOSITION 5.14. *Let B be non-triaial, openly decomposable and dissectable. Then every finite non-trivial interior algebra is embeddable in B.*

Proof. This follows from theorem 3.8 of [4] and proposition 5.11.

EXAMPLE 5.15. If $\mathbf{D}$ is Cantor's discontinnum then $A(\mathbf{D})$ satisfies the conditions of proposition 5.14 and so every finite non-trivial interior algebra is embeddable in $A(\mathbf{D})$.

LEMMA 5.16. *(McKinsey and Tarski [4]). Let σ be a universal sentence. Then σ holds in all non-trivial interior algebras iff σ holds in all finite non-trivial interior algebras.*

COROLLARY 5.17. *Let B be non-trivial, openly decomposable and dissectable.*

i) *If σ is a universal sentence then σ holds in all non-trivial interior algebras iff σ holds in B.*
ii) *Every non-trivial interior algebra is embeddable in an ultrapower of B.*

Proof.
i) Immediate from proposition 5.11 and lemma 5.16.
ii) Immediate from (i).

Thus any non-trivial openly decomposable dissectable interior algebra is **ultra-universal** in the elementary class of non-trivial algebras. (see [5]). In particular if X is Cantor's discontinuum or a denumerable space with the co-finite topology, then $A(X)$ is ultra-universal.

COROLLARY 5.18.

i) *If σ is an identity then σ holds in all interior algebras iff σ holds in all finite S.I. interior algebras.*
ii) *Let B be non-trivial and dissectable. If σ is an identity then σ holds in all interior algebras iff σ holds in B.*

Proof. (i) follows from lemma 5.16 and (ii) follows from (i) and proposition 5.8.

Let $\mathcal{T}, \mathcal{F}, \mathcal{D}$ denote the classes of all interior algebras, all finite S.I. interior algebras and all non-trivial dissectable interior algebras respectively.

COROLLARY 5.19.

i) $\mathcal{T} = HSP\ \mathcal{F}$
ii) $\mathcal{T} = HPs\ \mathcal{F}$
iii) $\mathcal{T} = PsHSPu\ \mathcal{F}$

LEMMA 5.20. *A product of dissectable interior algebras is dissectable. In particular $\mathcal{D}$ is closed under products.*

COROLLARY 5.21. *Let B be non-trivial and dissectable.*

i) $\mathcal{T} = HSP\{B\}$
ii) $\mathcal{T} = HPs\{B\}$
iii) $\mathcal{T} = PsHSPu\{B\}$
iv) $\mathcal{T} = HS\mathcal{D}$

References

1. G. E. Hughes & M. J. Cresswell, *An introduction to model logic*, Methuen London (1968).

2. B. Jónsson & A. Tarski, *Boolean algebras with operators*, I, Amer. J. Math. 73(1951), 891–939.
3. J. C. C. McKinsey, *A solution of the decision problem for the lewis systems S2 and S4 with an application to topology*, Journal of Symbolic Logic 6(1941), 117–134.
4. ———& A. Tarski, *The algebra of topology*, Ann. of Math. 45(1944), 141–191.
5. C. Naturman & H. Rose, *Ultra-universal models*, Theories and Elementary Classes, Preprint.
6. H. Rasiowa, *Algebraic treatment of the functional calculi of heyting and lewis*, Fundamenta Mathematicae, 38(1951), 99–126.

Department of Mathematics
University of Cape Town
Rondebosch 7700, South Africa