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Parameter Estimation and Confidence Limits for the Weibull Distribution

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Abstract

For the three parameter Weibull distribution, the parameter estimation techniques are applied and the asymptotic variances of the quantile to obtain the confidence limits for a given return period are derived. Three estimation techniques are used for these purposes: the methods of moments, maximum likelihood and probability weighted moments. The three parameter Weibull distribution as a flood frequency model is applied to actual flood data.

요 지

본 연구에서는 Weibull 확률분포함수의 매개변수 추정방법을 적용하였으며, 재현기간별 신뢰한계를 구하기 위한 漸近分散式을 유도하였다. 각 과정은 기존의 모멘트법, 최우도법, 확률가중 모멘트법(Probability weighted moments) 개념에 기초하여 유도하였으며, 유도된 식들을 실제 홍수자료에 적용하였다.

1. Introduction

The Weibull distribution, known as the generalized extreme value type 3 distribution (GEV-3)[1], was used to describe the reliability and life testing at the beginning by Weibull[2]. In 1960s and 1970s, there were many papers especially for the maximum likelihood estimation of the parameters for the two or three parameter Weibull distribution based on complete and censored samples[3-8]. The Weibull model also has been used to fit the frequency distribution of flood and drought events in hydrology and water resources. The Weibull distribution is very flexible model because this model is close to the normal distribution or exactly exponential distribution depending on the specific values of the parameters. Rao[9] compared the two and three parameter Weibull distributions based on the statistical parameters such as mean, variance and skewness coefficient. Boes et al.[10] applied the two parameter Weibull model to the regional flood quantile estimation based on the index flood assumption and compared the simulation experiment results of the estimation techniques such as the methods of moments (MOM), maximum likelihood (ML) and probability weighted moments (PWM). Heo et al.[11] derived the asymptotic variances of the quantiles for these three estimation techniques based on the regional analysis for the two parameter Weibull model. They also applied the regional Weibull model to the annual flood data.

The basic statistical properties for the three pa-

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rameter Weibull distribution are described in this study. The details of the parameter estimation techniques based on the methods of moments, maximum likelihood and probability weighted moments are proposed. In spite of the Weibull model's flexibility, there are only few papers for the confidence limits[7,8,10,11]. Furthermore, these studies are very limited: only for the method of maximum likelihood or only for the two parameter Weibull model. Therefore, the asymptotic variances of estimator of the quantiles for each estimation technique are derived to obtain the confidence limits of the quantile for the three parameter Weibull model. Also, these estimation techniques and confidence limits are applied to actual flood data.

2. Model Description

The cumulative distribution function of the three parameter Weibull distribution is defined by Johnson and Kotz[12]

$$F(x)=1-\exp\left[-\left(\frac{x-x_o}{\alpha}\right)^{\beta}\right] \quad x\geq x_o \tag{1}$$

and the probability density function (PDF) is given by

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x-x_o}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x-x_o}{\alpha}\right)^{\beta}\right] \tag{2}$$

in which $\alpha>0$ is the scale parameter, $\beta>0$ is the shape parameter and x_o is the location parameter. Note that the three parameter Weibull distribution is related with the GEV-3 distribution[1]. If X is GEV-3 distributed with the location parameter x_o', scale parameter α' and shape parameter β' (β' is positive for the GEV-3 distribution), then $-X$ is Weibull distributed by using the Jacobian transformation[13] and by assuming that $\beta=1/\beta'$, $\alpha=\alpha'/\beta'$ and $x_o=-x_o'-\alpha'/\beta'$. If the location parameter $x_o=0$, then the PDF and CDF of two parameter Weibull distribution are given by

$$F(x)=1-\exp[-(x/\alpha)^{\beta}] \quad x\geq 0 \tag{3}$$

and

$$f(x)=\frac{\beta}{\alpha}(x/\alpha)^{\beta-1}\ \exp[-(x/\alpha)^{\beta}] \tag{4}$$

respectivly.

The mean and variance of the three parameter Weibull distribution are given by

$$\mu=x_o+\alpha\Gamma(1+1/\beta) \tag{5}$$

and

$$\sigma^2=\alpha^2[\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta)] \tag{6}$$

in which $\Gamma(w)$ is the gamma function with argument w. Likewise, the skewness coefficient is given by

$$\gamma=\frac{\Gamma(1+3/\beta)-3\Gamma(1+2/\beta)\Gamma(1+1/\beta)+2\Gamma^3(1+1/\beta)}{[\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta)]^{3/2}} \tag{7}$$

where the skewness coefficient γ has a lower limit of -1.1396[14].

For $\beta>1$, the mode can be obtained from (2) as

$$\text{mode}(X)=x_o+\alpha\left(\frac{\beta-1}{\beta}\right)^{1/\beta} \tag{8}$$

and for $0<\beta\leq 1$, the mode is at zero. The median of the Weibull distribution is given by Johnson and Kotz[12]

$$\text{med}(X)=x_o+\alpha[\log(2)]^{1/\beta} \tag{9}$$

where log represents the natural logarithm. Fig. 1 shows some examples of the PDF of the Weibull distribution as a function of β for fixed values

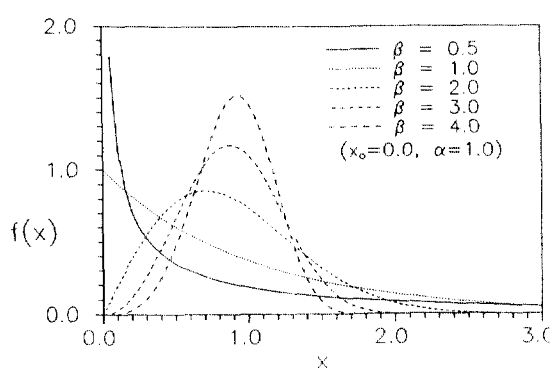

Fig. 1. Typical PDFs of the Weibull Dsitribution as a Function of β for Fixed Values of $\alpha=1.0$ and $x_o=0.0$.

of $\alpha=1.0$, $x_o=0.0$. Note that the three parameter Weibull distribution is the exponential distribution if $\beta=1.0$.

3. Estimation of Parameters

Three methods of parameter estimation for the Weibull distribution are considered here. They are: the methods of moments, maximum likelihood and probability weighted moments.

3.1 Method of Moments

The moment estimators $\hat{x}_o$, $\hat{\alpha}$, $\hat{\beta}$ can be obtained by substituting μ, σ and γ in Eqs. (5), (6) and (7) for corresponding sample estimates $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\gamma}$. The skewness coefficient is only a function of the shape parameter β. Thus, the approximate regression equations of $\hat{\beta}$ as a function of skewness coefficient are obatained as

$$\hat{\beta}=3.156997-2.282672\hat{\gamma}+.8403815\hat{\gamma}^2 -.1396762\hat{\gamma}^3+.0084155\hat{\gamma}^4 \quad (10a)$$

for $0.35<\hat{\gamma}<6.6$ and

$$\hat{\beta}=3.5569876-4.703961\hat{\gamma}+6.586978\hat{\gamma}^2 +13.3137\hat{\gamma}^3-13.86354\hat{\gamma}^4-133.588\hat{\gamma}^5 -49.61454\hat{\gamma}^6+212.2529\hat{\gamma}^7+180.3612\hat{\gamma}^8 \quad (10b)$$

for $-0.98<\hat{\gamma}\leq 0.35$ in which $\hat{\gamma}$ is the sample skewness coefficient. For a more precise solution of $\hat{\gamma}$, Eq. (10) can be used as the initial value for a numerical procedure such as Newton-Raphson method. For this purpose, Eq. (7) is rewritten as

$$G(\hat{\beta})\equiv \frac{\Gamma(1+3/\hat{\beta})-3\Gamma(1+2/\hat{\beta})\Gamma(1+1/\hat{\beta})+2\Gamma^3(1+1/\beta)}{[\Gamma(1+2/\hat{\beta})-\Gamma^2(1+1/\hat{\beta})]^{3/2}} -\hat{\gamma}=0 \quad (11)$$

and the first derivative of Eq. (11) with respect to $\hat{\beta}$ is given by

$$G'(\hat{\beta})=\frac{1}{\hat{\beta}^2[\Gamma(1+2/\hat{\beta})-\Gamma^2(1+1/\hat{\beta})]^{5/2}}\times \{[-3\Gamma'(1+3/\beta')+6\Gamma'(1+2/\hat{\beta})\Gamma(1+1/\hat{\beta}) +3\Gamma'(1+1/\hat{\beta})\Gamma(1+2/\hat{\beta})-6\Gamma'(1+1/\hat{\beta})\Gamma^2(1 +1/\hat{\beta})][\Gamma(1+2/\hat{\beta})-\Gamma^2(1+1/\hat{\beta})]+[\Gamma(1+ 3/\hat{\beta})-3\Gamma(1+2/\hat{\beta})\Gamma(1+1/\hat{\beta})+2\Gamma^3(1+1/ \hat{\beta})][3\Gamma'(1+2/\hat{\beta})-3\Gamma'(1+1/\hat{\beta})\Gamma(1+1/\hat{\beta})]\} \quad (12)$$

where $\Gamma'(w)$ is the first derivative of the gamma function with argument w. Therefore, the estimate of $\hat{\beta}$ at the iteration i+1 is updated by

$$\hat{\beta}_{i+1}=\hat{\beta}_i-G(\hat{\beta}_i)/G'(\hat{\beta}_i) \quad (13)$$

until satisfying the error criterion

$$\left|\frac{\hat{\beta}_{i+1}-\hat{\beta}_i}{\hat{\beta}_i}\right|<\varepsilon \quad (14)$$

in which ε is a specified relative error.

Once $\hat{\beta}$ is obtained, $\hat{\alpha}$ and $\hat{x}_o$ are determined from Eqs. (6) and (5) as

$$\hat{\alpha}=\hat{\sigma}/[\Gamma(1+2/\hat{\beta})-\Gamma^2(1+1/\hat{\beta})]^{1/2} \quad (15)$$

and

$$\hat{x}_o=\hat{\mu}-\hat{\alpha}\Gamma(1+1/\hat{\beta}) \quad (16)$$

For a two parameter Weibull distribution, $\hat{\beta}$ can be obtained numerically by combining Eqs. (5) and (6) and $\hat{\alpha}$ is determined from Eq. (5) by letting $x_o=0$.

3.2 Method of Maximum Likelihood

The log-likelihood function of the three parameter Weibull distribution is given by

$$LL(\underline{x};x_o,\ \alpha,\ \beta)=N\log(\beta)-N\beta\log(\alpha) +(\beta-1)\sum_{i=1}^{N}\log(x_i-x_o) -\sum_{i=1}^{N}\left[\frac{x_i-x_o}{\alpha}\right]^{\beta} \quad (17)$$

where log represents natural logarithm. The derivatives of Eq. (17) with respect to x_o, α and β are

$$\partial LL/\partial x_o=-(\beta-1)\sum_{i=1}^{N}(x_i-x_o)^{-1} +\frac{\beta}{\alpha}\sum_{i=1}^{N}\left[\frac{x_i-x_o}{\alpha}\right]^{\beta-1}=0 \quad (18a)$$

$$\partial LL/\partial\alpha=-\frac{N\beta}{\alpha}+\frac{\beta}{\alpha}\sum_{i=1}^{N}\left[\frac{x_i-x_o}{\alpha}\right]^{\beta}=0 \quad (18b)$$

$$\partial LL/\partial\beta=\frac{N}{\beta}-N\log(\alpha)+\sum_{i=1}^{N}\log(x_i-x_o)-\sum_{i=1}^{N}\left[\frac{x_i-x_o}{\alpha}\right]^{\beta}\log\left[\frac{x_i-x_o}{\alpha}\right]=0 \quad (18c)$$

respectively. Equation (18) must be solved simultaneously to find the estimators of the parameters x_o, α and β. For a Newton-Raphson method, the increments of x_o, α and β can be written as

$$\begin{bmatrix}\Delta x_o\\ \Delta\alpha\\ \Delta\beta\end{bmatrix}=\begin{bmatrix}-\partial^2LL/\partial x_o^2 & -\partial^2LL/\partial x_o\partial\alpha & -\partial^2LL/\partial x_o\partial\beta\\ -\partial^2LL/\partial\alpha\partial x_o & -\partial^2LL/\partial\alpha^2 & -\partial^2LL/\partial\alpha\partial\beta\\ -\partial^2LL/\partial\beta\partial x_o & -\partial^2LL/\partial\beta\partial\alpha & -\partial^2LL/\partial x_o\partial\beta^2\end{bmatrix}^{-1}\begin{bmatrix}\partial LL/\partial x_o\\ \partial LL/\partial\alpha\\ \partial LL/\partial\beta\end{bmatrix} \quad (19)$$

where -1 represents the inverse and the second partial derivatives of the log-likelihood function of the Weibull distribution are given in Appendix A. The new estimates at the iteration $(i+1)$ are computed by

$$\lambda_{i+1}=\lambda_i+\Delta\lambda_i \quad (20)$$

until satisfying the error criterion

$$|\Delta\lambda_i/\lambda_{i-1}|<\varepsilon \quad (21)$$

in which λ represents one of the parameters x_o, α and β, and ε is a specified relative error.

As an alternative suggested by Jenkinson[15], the second derivatives in the inverse of Eq. (19) can be replaced by corresponding expected values (see Appendix B) and then the inverse of the information matrix, say II^{-1}, can be written as

$$\Pi^{-1}=\frac{1}{ND}\begin{bmatrix}\frac{\alpha^2 b}{(\beta-1)^2} & -\frac{\alpha^2 h}{\beta(\beta-1)} & \frac{\alpha\beta f}{(\beta-1)}\\ -\frac{\alpha^2 h}{\beta(\beta-1)} & \frac{\alpha^2 a}{\beta^2} & \alpha g\\ \frac{\alpha\beta f}{(\beta-1)} & \alpha g & \beta^2 c\end{bmatrix} \quad (22)$$

where

$$a=\Gamma(1-2/\beta)[1+\Gamma''(2)]-\Gamma^2(1-1/\beta)[1+\psi(1-1/\beta)]^2 \quad (23a)$$

$$b=1+\Gamma''(2)-[\Gamma'(2)]^2=\pi^2/6 \quad (23b)$$

$$c=\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta) \quad (23c)$$

$$f=\Gamma(1-2/\beta)[1-\Gamma'(2)+\psi(1-1/\beta)] \quad (23d)$$

$$g=\Gamma(1-2/\beta)\Gamma'(2)-\Gamma^2(1-1/\beta)[1+\psi(1-1/\beta)] \quad (23e)$$

$$h=\Gamma(1-1/\beta)\{1+\Gamma''(2)-\Gamma'(2)[1+\psi(1-1/\beta)]\} \quad (23f)$$

$$D=bc+f^2 \quad (23g)$$

where $\Gamma''(2)$ and $\psi(w)=\Gamma'(w)/r(w)$ are the second derivative of the gamma function and a digamma function with argument 2. From Eqs. (19), (22) and (23), the increments of the parameters at ith iteration may be written as

$$\Delta x_{oi}=\frac{1}{ND}\left\{\frac{\alpha^2 b}{(\beta-1)^2}\left[\frac{\partial LL}{\partial x_o}\right]_i-\frac{\alpha^2 h}{\beta(\beta-1)}\left[\frac{\partial LL}{\partial\alpha}\right]_i+\frac{\alpha\beta f}{(\beta-1)}\left[\frac{\partial LL}{\partial\beta}\right]_i\right\} \quad (24a)$$

$$\Delta\alpha_i=\frac{1}{ND}\left\{-\frac{\alpha^2 h}{\beta(\beta-1)}\left[\frac{\partial LL}{\partial x_o}\right]_i+\frac{\alpha^2 a}{\beta^2}\left[\frac{\partial LL}{\partial\alpha}\right]_i+\alpha g\left[\frac{\partial LL}{\partial\beta}\right]_i\right\} \quad (24b)$$

$$\Delta\beta_i=\frac{1}{ND}\left\{\frac{\alpha\beta f}{(\beta-1)}\left[\frac{\partial LL}{\partial x_o}\right]_i+\alpha g\left[\frac{\partial LL}{\partial\alpha}\right]_i+\beta^2 c\left[\frac{\partial LL}{\partial\beta}\right]_i\right\} \quad (24c)$$

These recursive equations are repeated until satisfying Eq. (20). Note that the Jenkinson procedure for the three parameter Weibull distribution is only valid for $\beta>2$.

For a two parameter Weibull distribution, $x_o=0$ in Eqs. (17), (18b) and (18c). Equating Eqs. (18b) and (18c) to zero and combining them give

$$N+\beta\sum_{i=1}^{N}\log(x_i)-\frac{N\beta\sum_{i=1}^{N}x_i^{\beta}\log(x_i)}{\sum_{i=1}^{N}x_i^{\beta}}=0 \quad (25)$$

This equation is only a function of β. Solving Eq. (25) for β gives the maximum likelihood estimator $\hat{\beta}$ and then $\hat{\alpha}$ can be obtained from Eq. (18b).

3.3 Method of Probability Weighted Moments

The general form of the probability weighted moments (PWM) of the three parameter Weibull distribution is given by Greenwood et al.[16]

$$A_r = E[X(1-F(x))^r] = \frac{1}{r+1}[x_o + \alpha(r+1)^{-1/\beta}\Gamma(1+1/\beta)] \quad (26)$$

in which r is a nonnegative integer. The first three PWMs are given from Eq. (26) as

$$A_0 = x_o + \alpha\Gamma(1+1/\beta) \quad (27)$$

$$A_1 = [x_o + \alpha 2^{-1/\beta}\Gamma(1+1/\beta)]/2 \quad (28)$$

$$A_2 = [x_o + \alpha 3^{-1/\beta}\Gamma(1+1/\beta)]/3 \quad (29)$$

By substituting the first three population PWMs for the corresponding sample PWMs, $\hat{A}_0$, $\hat{A}_1$ and $\hat{A}_2$, the PWM estimator of the shape parameter β is a solution for

$$\frac{1-3^{-1/\beta}}{1-2^{-1/\beta}} = \frac{3\hat{A}_2 - \hat{A}_0}{3\hat{A}_1 - \hat{A}_0} \quad (30)$$

where $\hat{\beta}$ can be obtained by the Newton-Raphson method using the value of $\hat{\beta}$ from Eq. (10) as an initial value. Then, the PWM estimator of the parameter $\hat{\alpha}$ may be obtained by combining Eqs. (27) and (28) as

$$\hat{\alpha} = (\hat{A}_0 - 2\hat{A}_1)/[(1-2^{1/\hat{\beta}})\Gamma(1+1/\hat{\beta})] \quad (31)$$

Finally, the estimator of x_o is given by

$$\hat{x}_o = \hat{A}_o - \hat{\alpha}\Gamma(1+1/\hat{\beta}) \quad (32)$$

in which the sample PWM, $\hat{A}_r$ is given by Landwehr et al.[17]

$$\hat{A}_0 = \frac{1}{N}\sum_{j=1}^{N} x_j \qquad \text{for } r=0 \quad (33a)$$

$$\hat{A}_r = \frac{1}{N}\sum_{j=1}^{N} x_j \frac{(N-j)(N-j-1)\cdots(N-j-r+1)}{(N-1)(N-2)\cdots(N-r)} \qquad \text{for } r>1 \quad (33b)$$

where x_j is the order statistic such that $x_1 \leq x_2 \leq \cdots \leq x_N$.

For a two parameter Weibull distribution ($x_o = 0$), the PWM estimators are given from Eqs. (27) and (28)

$$\hat{\beta} = \log(2)/[\log(\hat{A}_0/\hat{A}_1) - \log(2)] \quad (34)$$

and

$$\hat{\alpha} = \hat{A}_0/\Gamma(1+1/\hat{\beta}) \quad (35)$$

Note that the method of PWM does not need any iterative procedure for a two parameter Weibull distribution.

4. Confidence Limits on Quantiles

The $\gamma = (1-\alpha)$ confidence limits X_1 on the population quantiles may be determined by

$$X_1 = \hat{X}_T \pm u_{1-\alpha/2} S_T \quad (36)$$

where $u_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution, $\hat{X}_T$ is the quantile estimator corresponding to return period T, and S_T is the standard deviation of $\hat{X}_T$. The quantile estimator $\hat{X}_T$ of the Weibull distribution can be obtained from Eq. (1) as

$$\hat{X}_T = \hat{x}_o + \hat{\alpha}[-\log(1/T)]^{1/\hat{\beta}} \quad (37)$$

where F(x) is replaced by $1-1/T$. Also, Chow[18] expressed the quantile estimator $\hat{X}_T$ as

$$\hat{X}_T = \hat{\mu} + K_T\hat{\sigma} \quad (38)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the sample mean and standard deviation and K_T is the frequency factor.

The derivations of S_T for each method of estimation are described here to obtain the confidence limits in Eq. (36).

4.1 Standard Error by Moments

The variance of $\hat{X}_T$ based on the moments are given by Kite[19]

$$S_T^2 = Var(\hat{X}) = (\mu_2/N)\{1 + K_T\gamma + K_T^2(\gamma_2 - 1)/4 + (\partial K_T/\partial\gamma)[2\gamma_2 - 3\gamma^2 - 6 + K_T(\gamma_3 - 6\gamma\gamma_2/4 - 10\gamma/4)] + (\partial K_T/\partial\gamma)^2[\gamma_4 - 3\gamma\gamma_3 - 6\gamma_2 + 9\gamma^2\gamma_2/4 + 35\gamma^2/4 + 9]\} \quad (39)$$

where the cumulants are given by

$$\gamma = \mu_3/\mu_2^{3/2} \quad (40a)$$

$$\gamma_2 = \mu_4/\mu_2^2 \tag{40b}$$

$$\gamma_3 = \mu_5/\mu_2^{5/2} \tag{40c}$$

$$\gamma_4 = \mu_6/\mu_2^3 \tag{40d}$$

and the rth central moments,

$$\mu_2 = \alpha^2[D_2 - D_1^2] \tag{41a}$$

$$\mu_3 = \alpha^3[D_3 - 3D_2D_1 + 2D_1^3] \tag{41b}$$

$$\mu_4 = \alpha^4[D_4 - 4D_3D_1 + 6D_2D_1^2 - 3D_1^4] \tag{41c}$$

$$\mu_5 = \alpha^5[D_5 - 5D_4D_1 + 10D_3D_1^2 - 10D_2D_1^3 + 4D_1^5] \tag{41d}$$

$$\mu_6 = \alpha^6[D_6 - 6D_5D_1 + 15D_4D_1^2 - 20D_3D_1^3 + 15D_2D_1^4 - 5D_1^6] \tag{41e}$$

The derivative of of K_T with respect to γ can written as

$$(\partial K_T/\partial\gamma) = (\partial K_T/\partial\beta)(\partial\beta/\partial\gamma) \tag{42}$$

Substituting Eqs. (5) and (6) into Eq. (38) and solving for K_T yield

$$K_T = \frac{[-\log(1/T)]^{1/\beta} - \Gamma(1+1/\beta)}{[\Gamma(1+2/\beta) - \Gamma^2(1+1/\beta)]^{1/2}} \tag{43}$$

then the derivative of K_T with respect to β is given by

$$\left(\frac{\partial K_T}{\partial\beta}\right) = [(D_1\psi_1 - \log(B)B^{1/\beta})(D_2 - D_1^2) - (B^{1/\beta} - D_1)(-D_2\psi_2 + D_1^2\psi_1)]/\beta^2(D_2 - D_1^2)^{3/2} \tag{44}$$

where $D_r = \Gamma(1+r/\beta)$, $\psi_r = \psi(1+r/\beta)$ and $B = -\log(1/T)$. The derivative of γ with respect to β can be obtained from Eq. (7) as

$$(\partial\gamma/\partial\beta) = 3[(D_2 - D_1^2)(-D_3\psi_3 + 2D_2D_1\psi_2 + D_2D_1\psi_1 - 2D_1^3\psi_1) - (D_3 - 3D_2D_1 + 2D_1^3) - D_2\psi_2 + D_1^2\psi_1)]/[\beta^2(D_2 - D_1^2)^{5/2}] \tag{45}$$

Finally, the asymptotic variance S_T^2 of Eq. (39) is obtained from Eqs. (40) through (45).

4.2 Standard Error by Maximum Likelihood

The asymptotic variance of the maximum likelihood estimator of quantile, $\hat{X}T$, the three parameter Weibull distribution can be written as[5]

$$S_T^2 = (\partial X_T/\partial x_o)^2 \ \mathrm{Var}(\hat{x}_o) + (\partial X_T/\partial\alpha)^2 \ \mathrm{Var}(\hat{\alpha}) + (\partial X_T/\partial\beta)^2 \ \mathrm{Var}(\hat{\beta}) + 2(\partial X_T/\partial x_o)(\partial X_T/\partial\alpha)\mathrm{Cov}(\hat{x}_o, \hat{\alpha}) + 2(\partial X_T/\partial\alpha)(\partial X_T/\partial\beta)\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) + 2(\partial X_T/\partial x_o)(\partial X_T/\partial\beta)\mathrm{Cov}(\hat{x}_o, \hat{\beta}) \tag{46}$$

The derivatives of X_T with respect to the parameters x_o, α and β are

$$\left[\frac{\partial X_T}{\partial x_o}\right] = 1 \tag{47a}$$

$$\left[\frac{\partial X_T}{\partial\alpha}\right] = [-\log(1/T)]^{1/\beta} \tag{47b}$$

$$\left[\frac{\partial X_T}{\partial\beta}\right] = -\frac{\alpha}{\beta^2}\log[-\log(1/T)][-\log(1/T)]^{1/\beta} \tag{47c}$$

and the variance and covariance terms are obtained from Eq. (22) as

$$\mathrm{Var}(\hat{x}_o) = \alpha^2 b/N(\beta-1)^2 D \tag{48a}$$

$$\mathrm{Var}(\hat{\alpha}) = \alpha^2 a/N\beta^2 D \tag{48b}$$

$$\mathrm{Var}(\hat{\beta}) = \beta^2 c/ND \tag{48c}$$

$$\mathrm{Cov}(\hat{x}_o, \hat{\alpha}) = -\alpha^2 h/N\beta(\beta-1)D \tag{48d}$$

$$\mathrm{Cov}(\hat{x}_o, \hat{\beta}) = \alpha g/ND \tag{48e}$$

$$\mathrm{Cov}(\hat{x}_o, \hat{\beta}) = \alpha\beta f/N(\beta-1)D \tag{48f}$$

Thus, substituting Eqs. (47) and (48) into (46) yields

$$S_T^2 = \frac{\alpha^2}{ND}\left\{\frac{b}{(\beta-1)^2} - \frac{2B^{1/\beta}}{\beta(\beta-1)}[h + f\log(B)] + \frac{B^{1/\beta}}{\beta^2}[a - 2g\log(B) + (\log(B))^2]\right\} \tag{49}$$

4.3 Standard Error by Probability Weighted Moments

The asymptotic distribution of the sample PWMs, $\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$ can be written as[20,21,11]

$$\begin{bmatrix}\hat{A}_0\\ \hat{A}_1\\ \hat{A}_2\end{bmatrix} \sim \mathrm{TVN}\left(\begin{matrix}A_0\\ A_1\\ A_2\end{matrix} ; \begin{bmatrix}A_{00}/N & A_{01}/N & A_{02}/N\\ A_{01}/N & A_{11}/N & A_{12}/N\\ A_{02}/N & A_{12}/N & A_{22}/N\end{bmatrix}\right) \tag{50}$$

where $\sim$ reads "is asymptotically distributed as" and TVN is an abbreviation for trivariate normal distribution and A_{ij} are given by Heo et al.[11]

$$A_{00}=\alpha^2[\Gamma(1-2/\beta)-\Gamma^2(1+1/\beta)] \quad (51a)$$

$$A_{01}=(\alpha^2/2)[2^{-2/\beta}\Gamma(1+2/\beta) + (1-2^{1-1/\beta})\Gamma^2(1+1/\beta)] \quad (51b)$$

$$A_{02}=(\alpha^2/2)[\{3^{-2/\beta}-2^{-2/\beta}H(1/2)\}\Gamma(1+2/\beta) -2(3^{-1/\beta}-2^{-1/\beta})\Gamma^2(1+1/\beta)] \quad (51c)$$

$$A_{11}=\alpha^2 2^{-2/\beta}[H(1/2)\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta)] \quad (51d)$$

$$A_{12}=(\alpha^2/2)[3^{-2/\beta}H(1/3)\Gamma(1+2/\beta) -(2\cdot 6^{-1/\beta}-2^{-1/\beta})\Gamma^2(1+1/\beta)] \quad (51e)$$

$$A_{22}=\alpha^2 3^{-2/\beta}[H(2/3)\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta)] \quad (51f)$$

where H(z) is a hypergeometric function.

Since the asymptotic variance of the PWM estimator of quantile, $\hat{X}_T$ can not be found directly, the following transformations are used

$$\begin{matrix} \hat{A}_0 & & \hat{A}_0 & & \hat{A}_0 & & \hat{x}_o & & \\ \hat{A}_1 & \rightarrow & \hat{A}_1 & \rightarrow & \hat{A}_1 & \rightarrow & \hat{\alpha} & \rightarrow & \hat{X}_T \\ \hat{A}_2 & & \hat{A}_2 & & \hat{A}_2 & & \hat{\beta} & & \\ & & R & & \hat{\beta} & & & & \end{matrix}$$

where $R=(3\hat{A}_2-\hat{A}_0)/(2\hat{A}_1-\hat{A}_0)$ in the first transformation and $\hat{\beta}$ is given implicitly by $(1-3^{-2/\hat{\beta}})/(1-2^{-1/\hat{\beta}})=R$ in the second transformation. Finally, the asymptotic variance of $\hat{X}_T$ is given by

$$S_T^2=\frac{1}{N}\Big[\mathrm{Var}(\hat{x}_o)+B^{2/\beta}\mathrm{Var}(\hat{\alpha}) + \frac{\alpha^2}{\beta^4}(\log B)^2B^{2/\beta}\mathrm{Var}(\hat{\beta}) + 2B^{1/\beta}\mathrm{Cov}(\hat{x}_o,\ \hat{\alpha})-\frac{2\alpha}{\beta^2}\log B\ B^{1/\beta} [\mathrm{Cov}(\hat{x}_o,\ \hat{\beta})+B^{1/\beta}\mathrm{Cov}(\hat{\alpha},\ \hat{\beta})]\Big] \quad (52)$$

where

$$\mathrm{Var}(\hat{x}_o)=W_0^2A_{00}+2W_0W_1A_{01}+W_1^2A_{11}+2W_0W_\beta C_1 H+2W_1W_\beta C_2H+W_\beta^2CH^2 \quad (53a)$$

$$\mathrm{Var}(\hat{\alpha})= T_0^2A_{00}+2T_0T_1A_{01}+T_1^2A_{11}+2T_0T_\beta C_1H+ 2T_1T_\beta C_2H+T_\beta^2CH^2 \quad (53b)$$

$$\mathrm{Var}(\hat{\beta})=CH^2 \quad (53c)$$

$$\mathrm{Cov}(\hat{x}_o,\ \hat{\alpha})=W_0T_0A_{00}+W_0T_1A_{01}+W_1T_0A_{01}+W_1T_1 A_{11}+W_0T_\beta C_1H+W_1T_\beta C_2H+W_\beta T_0C_1 H+W_\beta T_1C_2H+W_\beta T_\beta CH^2 \quad (53d)$$

$$\mathrm{Cov}(\hat{x}_o,\ \hat{\beta})=W_0C_1H+W_1C_2H+W_\beta CH^2 \quad (53e)$$

$$\mathrm{Cov}(\hat{\alpha},\ \hat{\beta})=T_0C_1H+T_1C_2H+T_\beta CH^2 \quad (53f)$$

and

$$W_0=\partial x_o/\partial A_0=-2^{-1/\beta}//(1-2^{-1/\beta}) \quad (54a)$$

$$W_1=\partial x_o/\partial A_1=2/(1-2^{-1/\beta}) \quad (54b)$$

$$W_\beta=\partial x_o/\partial\beta=-\alpha\Gamma(1+1/\beta)\log(2)2^{-1/\beta}/[\beta^2(1-2^{-1/\beta})] \quad (54c)$$

$$T_0=\partial\alpha/\partial A_0=1/[(1-2^{-1/\beta})\Gamma(1+1/\beta)] \quad (54d)$$

$$T_1=\partial\alpha/\partial A_1=-2/[(1-2^{-1/\beta})\Gamma(1+1/\beta)] \quad (54e)$$

$$T_\beta=\partial\alpha/\partial\beta=[(1-2^{-1/\beta})\psi(1+1/\beta)-\log(2)2^{-1/\beta}]/ [\beta^2(1-2^{-1/\beta})^2\Gamma(1+1/\beta)] \quad (54f)$$

and

$$C=[A_{00}(3^{-1/\beta}-2^{-1/\beta})-2A_{01}(3^{-1/\beta}-1) +3A_{02}(2^{-1/\beta}-1)]/M \quad (54g)$$

$$C_1=[A_{01}(3^{-1/\beta}-2^{-1/\beta})-2A_{11}(3^{-1/\beta}-1) +3A_{12}(2^{-1/\beta}-1)]/M \quad (54h)$$

$$C_2=[A_{02}(3^{-1/\beta}-2^{-1/\beta})-2A_{12}(3^{-1/\beta}-1) +3A_{22}(2^{-1/\beta}-1)]/M \quad (54i)$$

$$C_3=[C_1(3^{-1/\beta}-2^{-1/\beta})-2C_2(3^{-1/\beta}-1) +3C_3(2^{-1/\beta}-1)]/M \quad (54j)$$

$$M=\alpha\Gamma(1+1/\beta)(1-2^{-1/\beta})^2 \quad (54k)$$

$$H=\frac{\beta^2(1-2^{1/\beta})^2}{\log(2)\,2^{-1/\beta}(1-3^{-1/\beta})-\log(3)\,3^{-1/\beta}(1-2^{-1/\beta})} \quad (54l)$$

5. Data Application

The annual flood data of Rock River at Afton, Wisconsin (1930-1983) is used to compute the parameter estimates, quantiles and confidence limits for the Weibull distribution. The sample mean, standard deviation and skewness coefficient of the annual flood data are 178.99, 73.44 and 0.3839, respectively. The moment estimates can be obtained from Eqs. (13), (15) and (16). The maximum likelihood estimates are evaluated from two numerical procedures: (1) Newton-Raphson method and (2) Jenkinson procedure. Note that the Jenki-

Table 1. Parameter Estimates of the Weibull Distribution

Method		Parameter		
		x_0	α	β
	MOM	8.72255	191.9884	2.44457
ML	Newton-Raphson	23.48409	175.6373	2.24087
	Jenkinson	23.48550	175.6357	2.24085
	PWM	28.99386	169.3316	2.08271

Table 2. The Quantiles and 95% Confidence Limits for the Method of Moments

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2.	.50000	152.2363	173.9800	195.7236
5.	.80000	214.5857	241.9716	269.3575
10.	.90000	245.3957	278.7747	312.1538
20.	.95000	268.3896	309.4684	350.5472
50.	.98000	291.4799	344.1593	396.8387
100.	.99000	305.3629	367.3067	429.2505
500.	.99800	330.2424	414.0814	497.9204

nson procedure is valid only for the flood data whose shape parameter $\beta>2$. Usually, the Newton-Raphson method converges faster than the Jenkinson approach[22]. For example, the Newton-Raphson method converged at 5th iteration, but the Jenkinson procedure converged at 59th iteration for this annual flood data. Therefore, the Newton-Raphson method is recommended unless there is a convergence problem. The probability weighted moments estimates can be found from Eqs. (30) through (32). These parameter estimates are given in Table 1. For given these parameter estimates, the quantiles and 95% confidence limits, corresponding to return periods T=2, 5, 10, 20, 50, 100 and 500 of the Weibull distribution are given in Tables 2, 3 and 4, respectively. As shown in Tables 2, 3, and 4 the confidence limits of the ML show the narrower bands than the MOM and PWM for all return periods.

Table 3. The Quantiles and 95% Confidence Limits for the Method of Maximum Likelihood (Newton-Raphson)

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2.	.50000	158.3950	172.6207	186.8464
5.	.80000	223.1970	240.6773	258.1576
10.	.90000	256.7949	278.3181	299.8413
20.	.95000	283.9023	310.0725	336.2428
50.	.98000	313.6935	346.3179	378.9423
100.	.99000	333.1546	370.6953	408.2359
500.	.99800	371.6443	420.3851	469.1259

Table 4. The Quantiles and 95% Confidence Limits for the Method of Probability Weighted Moments

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2.	.50000	149.2500	171.0014	192.7528
5.	.80000	215.5745	241.7937	268.0130
10.	.90000	249.1854	281.7220	314.2586
20.	.95000	274.8350	315.7600	356.6850
50.	.98000	301.3204	354.9625	408.6046
100.	.99000	317.6934	381.5200	445.3465
500.	.99800	348.0656	436.0844	524.1031

6. Conclusions

The parameter estimation techniques for the two and three parameter Weibull distribution are applied based on the methods of moments (MOM), maximum likelihood (ML) and probability weighted moments (PWM). The results of the parameter estimations are as follows: The iterative procedures are always required for each estimation method to estimate the parameters for the three parameter Weibull distribution. The method of probability weighted moments is the simplest estimation technique especially for the two parameter Weibull model because no iterative procedure is necessary. Therefore, the PWM estimates can be used as initial values of the iterative procedure for the MOM and ML estimates. For the method of maximum likelihood, the Newton-Raphson method converges faster than the Jenkinson approach.

As a major contribution of this study, the asymptotic variances of the MOM, ML and PWM quantile estimators are derived as function of sample size, return period and the parameters to obtain the confidence limits of these quantiles for the three parameter Weibull distribution. These formulae do not have simple and nice forms but can be evaluated numerically. As shown in data applications, the ML method shows the tight confidence limits.

The derived asymptotic variances of the MOM, ML and PWM quantile estimators are based on the three parameter Weibull distribution. For a future study, the derivation of the asymptotic variances for the two parameter Weibull model, which are easily obtainable, can be useful in flood frequency analysis because the parameter estimation based on the three parameter model sometimes cannot be applicable depending on the data.

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Appendix A Second Partial Derivatives of the Log-Likelihood Function of the Weibull Distribution.

$$-\partial^2 LL/\partial x_o^2 = [(\beta-1)/\alpha^2][\Sigma y_i^{-2} + \beta\Sigma y_i^{\beta-2}] \quad \text{(A1)}$$

$$-\partial^2 LL/\partial\alpha\partial x_o = (\beta/\alpha)^2 \Sigma y_i^{\beta-1} \quad \text{(A2)}$$

$$-\partial^2 LL/\partial x_o\partial\beta = \Sigma(x_i - x_o)^{-1} - (\beta/\alpha)\Sigma y_i^{\beta-1}\log(y_i) - (1/\alpha)\Sigma y_i^{\beta-1} \quad \text{(A3)}$$

$$-\partial^2 LL/\partial\alpha^2 = -(\beta/\alpha^2)[N-(\beta+1)\Sigma y_i^{\beta}] \quad \text{(A4)}$$

$$-\partial^2 LL/\partial\alpha\partial\beta = (1/\alpha)[N - \Sigma y_i^{\beta} - \beta\Sigma y_i^{\beta}\log(y_i)] \quad \text{(A5)}$$

$$-\partial^2 LL/\partial\beta^2 = N/\beta^2 + \Sigma y_i^{\beta}[\log(y_i)]^2 \quad \text{(A6)}$$

where $y_i = (x_i - x_o)/\alpha$, Σ represents summation from 1 to N, and log represents the natural logarithm.

Appendix B Expected values of the second partial derivatives of the log-likelihood function of the Weibull distribution.

$$E\left[-\frac{\partial^2 LL}{\partial x_o^2}\right] = \frac{N(\beta-1)^2}{\alpha^2}\Gamma(1-2/\beta) \quad \text{(B1)}$$

$$E\left[-\frac{\partial^2 LL}{\partial \alpha^2}\right] = \frac{N\beta^2}{\alpha^2} \quad \text{(B2)}$$

$$E\left[-\frac{\partial^2 LL}{\partial \beta^2}\right] = \frac{N}{\beta^2}[1+\Gamma''(2)] \quad \text{(B3)}$$

$$E\left[-\frac{\partial^2 LL}{\partial x_o\partial\alpha}\right] = \frac{N\beta^2}{\alpha^2}\Gamma(2-1/\beta) \quad \text{(B4)}$$

$$E\left[-\frac{\partial^2 LL}{\partial x_o\partial\beta}\right] = -\frac{N}{\alpha}(1-1/\beta)\Gamma(1-1/\beta)[1+\psi(1-1/\beta)] \quad \text{(B5)}$$

$$E\left[-\frac{\partial^2 LL}{\partial\theta\partial\beta}\right] = -\frac{N}{\alpha}\Gamma'(2) \quad \text{(B6)}$$

where $\Gamma(w)$ and $\psi(w)$ are gamma and digamma function with argument w and Γ' (2) and Γ'' (2) are the first and second partial derivatives of a gamma function with argument 2, respectively.