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亂流剪斷 흐름에서의 定常 水平 線汚染源의 擴散

Diffusion of a Steady Horizontal Line Source in a Turbulent Shear Flow

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Abstract

Diffusion of a steady horizontal line source in a turbulent shear flow is simulated by numerically solving a steady two-dimensional advective diffusion equation. The computational result is compared with the analytic solution for uniform velocity and diffusivity distributions over the depth. The analytic solution for constant velocity and diffusivity overestimates the degree of vertical mixing. The normalized equation indicates that friction factor is the only physical parameter that governs the vertical diffusion process. Sensitivities of the diffusion process to the friction factor and initial source position are analyzed. The rate of vertical mixing varies approximately as the square root of the friction factor. The optimal source position, which gives the most rapid mixing, lies above the mid-depth and moves toward the water surface as the friction factor increases.

요 지

정상 2차원 移送擴散 방정식의 수치해석에 의하여 亂流剪斷 흐름에서의 定常 水平 線汚染源의 擴散을 모의하였다. 유속과 亂流擴散係數의 수심에 따른 변화가 없을 경우에 대한 해석해와 비교한 결과, 일정 유속 및 亂流擴散係數를 가정할 경우 수심방향 擴散을 과대평가하는 것으로 나타났다. 무차원화된 지배방정식에 따르면 수심방향 擴散을 지배하는 물리적 변수는 摩擦係數뿐이다. 摩擦係數에 대한 확산과정의 敏感度 分析으로부터 수심방향 擴散速度는 대략 摩擦係數의 제곱근에 비례함을 알 수 있었다. 汚染源의 초기 방류위치에 따른 敏感度 分析 결과, 가장 신속한 수심방향 擴散을 가져오는 최적의 방류위치는 수심의 중간점보다 위쪽이며, 摩擦係數가 커질수록 그 위치가 수면쪽에 가까와지는 것으로 나타났다.

1. INTRODUCTION

Vertical mixing is the initial stage of mixing in rivers. So, the rapid vertical mixing contributes to the whole mixing process by shortening the time before the succeeding stages, i.e., lateral diffusion (in case of point source) and longitudinal dispersion. In addition, vertical mixing itself is important for the atmospheric diffusion process, where the boundary layer thickness is of the order of hundred meters.

For a laterally uniform steady flow and a unifo-

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rmly distributed continuous line source, the vertical diffusion problem can be described by the two-dimensional steady-state advective diffusion equation. Mathematical solutions for the problem with constant velocity and diffusivity have long been available.[1] Yeh and Tsai[2] obtained an analytic solution for a power-law velocity and power-law diffusivity, which is unrealistic since the measured velocity distribution is approximately logarithmic, and this implies a parabolic diffusivity distribution. McNulty and Wood[3] used Aris' method of moments for the case of logarithmic velocity and parabolic diffusivity distributions but the comparison they made with the solution for constant velocity and diffusivity is incorrect. Nokes et al.[4] solved the problem analytically by reducing it to an eigenvalue problem following the approach of Smith[5], who dealt with the problem on where to put the discharge in meandering rivers. On the other hand, Coudert[6] solved the problem numerically by using a finite difference method but his stated initial condition is dimensionally incorrect.

In this paper, a two-dimensional steady-state advective diffusion equation is solved numerically to simulate the diffusion process for turbulent shear flow in a channel. The results are compared with those for the case of constant velocity and diffusivity. The numerical model is used to investigate the sensitivity of the vertical diffusion process to the friction factor. The effect of the source position on the downstream concentration distribution is also studied to find the best position for the most rapid vertical mixing.

2. MATHEMATICAL FORMULATION

2.1 Governing Equation and Initial and Boundary Conditions

The advective diffusion equation for a steady horizontal line source in a turbulent shear flow (Fig. 1) can be written as

$$u\frac{\partial c}{\partial x}-\frac{\partial}{\partial z}(\varepsilon\frac{\partial c}{\partial z})=0 \qquad (1)$$

where $c=c(x, z)$ = mass concentration; $u=u(z)$ = longitudinal flow velocity; $\varepsilon=\varepsilon(z)$ = turbulent

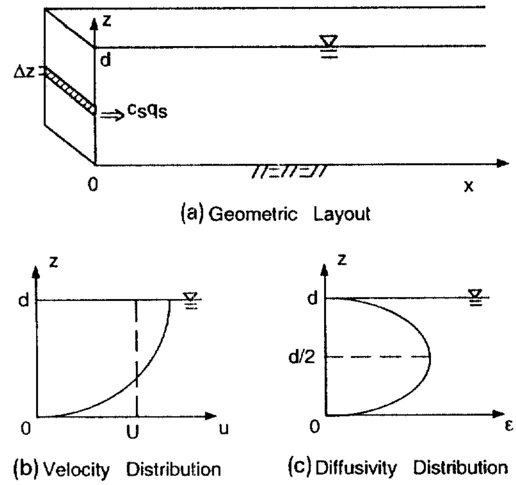

Fig. 1. Definition Sketch.

diffusion coefficient; z = vertical coordinate; and x = longitudinal coordinate. Diffusive transport in the x direction is not included in Eq. (1) because it is negligibly small compared with the advective transport.[7] Temporal mean flow velocity in the z direction is assumed to be zero. If a logarithmic velocity distribution is assumed except for the region very near the bottom, u can be written as

$$u(z)=U+\frac{u_*}{\kappa}[1+\ln(\frac{z}{d})] \quad \text{for } z_0\le z\le d \qquad (2)$$

where κ = von Karman constant; d = water depth; U = mean flow velocity; u_* = shear velocity($=\sqrt{\tau_0/\rho}$); τ_0 = bottom shear stress; and ρ = water density. For a hydraulically smooth channel, z_0 is the thickness of the laminar sublayer in which the velocity distribution is linear and given by

$$u(z)=\frac{z\,u_*^2}{\nu} \quad \text{for } 0\le z\le z_0 \qquad (3)$$

where ν is the kinematic viscosity of water. For a rough channel, z_0 can be taken as the distance to the position where the velocity given by Eq. (2) is zero, and the velocity is taken as zero for the region $0\le z\le z_0$.

The vertical turbulent diffusion coefficient can be obtained from the Reynolds analogy that the turbulent mass diffusivity is the same as the momentum diffusivity, which can be expressed as[7]:

$$\varepsilon(z)=\kappa u_* z(1-\frac{z}{d}) \quad (4)$$

In a laminar sublayer for a smooth channel, ε is replaced by the kinematic viscosity of water.

Boundary conditions at the bottom and on the water surface are given by no-flux conditions as

$$\frac{\partial c}{\partial z}(x,\ 0)=0 \quad (5)$$

$$\frac{\partial c}{\partial z}(x,\ d)=0 \quad (6)$$

The initial condition for the source introduced at $x=0$ through the area(Δz) between $z=d_s+\Delta z/2$ and $z=d_s-\Delta z/2$, can be expressed as

$$c(0,\ z)=c_0=\frac{c_s q_s}{u(d_s)\Delta z} \quad (d_s-\frac{\Delta z}{2}<z<d_s+\frac{\Delta z}{2}) \quad (7a)$$

$$=0 \quad (z=\text{elsewhere}) \quad (7b)$$

where $c_s q_s$ is the source strength in mass per unit volume per unit width.

2.2 Normalization of the Mathematical Model

To normalize the advective diffusion equation and the initial and boundary conditions, the following dimensionless variables are introduced:

$$z'=\frac{z}{d} \quad (8)$$

$$x'=\frac{x}{d} \quad (9)$$

$$u'=\frac{u}{u_*} \quad (10)$$

$$\varepsilon'=\frac{\varepsilon}{u_* d} \quad (11)$$

$$c'=\frac{c}{c_\infty} \quad (12)$$

where c_∞ is the equilibrium concentration, i.e., the concentration far downstream where the source mass is completely mixed over the depth:

$$c_\infty=\frac{c_s q_s}{Ud} \quad (13)$$

From Eqs. (2) and (4), the normalized equations for the velocity and diffusivity distributions are

$$u'=\sqrt{\frac{8}{f}}+\frac{1}{\kappa}(1+\ln\ z') \quad (14)$$

and

$$\varepsilon'=\kappa z'(1-z') \quad (15)$$

for $z_0'\leq z'\leq 1$. where the friction factor $f=8(u_*/U)^2$ and z_0' is z_0 normalized by the water depth. It can also be shown that for the laminar sublayer,

$$u'=\frac{u_* d}{\nu}(1-z') \quad (16)$$

and

$$\varepsilon'=\frac{\nu}{u_* d} \quad (17)$$

for $0\leq z'\leq z_0'$. The normalized advective diffusion equation and the initial, and boundary conditions in terms of dimensionless variables defined above are

$$u'=\frac{\partial c'}{\partial x'}-\frac{\partial}{\partial z'}(\varepsilon'\frac{\partial c'}{\partial z'})=0 \quad (18)$$

$$\frac{\partial c'}{\partial z'}(x',\ 0)=0 \quad (19)$$

$$\frac{\partial c'}{\partial z'}(x',\ 1)=0 \quad (20)$$

$$c'(0,\ z)=c_0'(d_s'-\frac{\Delta z'}{2}<z'<d_s'+\frac{\Delta z'}{2}) \quad (21a)$$

$$=0 \quad (z'=\text{elsewhere}) \quad (21b)$$

where

$$c_0'=\frac{Ud}{u(d_s)\Delta z} \quad (22)$$

and d_s' and $\Delta z'$ are, respectively, d_s and Δz normalized by the water depth.

It is seen from the normalized governing equation that the friction factor is the only physical parameter on which the vertical diffusion process in a turbulent shear flow depends. This means that for a fixed friction factor, variation of the mean flow velocity does not affect the vertical concentration distribution at any position along the flow direction, as explained by the following reasons. If the mean flow velocity is doubled for a constant friction factor, the shear velocity is doubled. The turbulent velocity components in shear flows are proportional to the shear velocity as is known from a number of previous experimental works including those done by Laufer.[8,9] Thus, the vertical turbulent velocity components of eddies which cause the turbulent diffusion are also doubled. In this way, the longitudinal advection and vertical diffusion processes are scaled at the same rate; consequently, vertical concentration distribution at a given longitudinal position remains the same.

2.3 Description of the Numerical Method

A Crank-Nicholson scheme, which is unconditionally stable, was used to solve the turbulent diffusion equation with previously described initial and boundary conditions. Writing Eq. (18) in a finite difference form based on the Crank-Nicholson scheme, one obtains:

$$Fc_{j-1}^{i+1}+Gc_j^{i+1}+Hc_{j+1}^{i+1}=R^i \quad j=1,\ \cdots,\ J-1 \tag{23}$$

where

$$F=-\lambda\varepsilon_{j-0.5} \tag{24}$$

$$G=u_j+\lambda(\varepsilon_{j+0.5}+\varepsilon_{j-0.5}) \tag{25}$$

$$H=-\lambda\varepsilon_{j+0.5} \tag{26}$$

$$R^i=-Fc_{j-1}^i+(2u_j-G)c_j^i-Hc_{j+1}^i \tag{27}$$

$$J=\frac{1}{\Delta z'} \tag{28}$$

$$\lambda=0.5\frac{\Delta x'}{(\Delta z')^2} \tag{29}$$

where $\Delta x'$ = dimensionless increment in x-direction; $\Delta z'$ = dimensionless increment in z-direction; i = index for x-direction; j = index for z-direction. As a matter of notation, all the variables in the difference equations are dimensionless ones. The initial and boundary conditions in the finite difference form are

$$c_0^i=c_1^i \tag{30}$$

$$c_{J-1}^i=c_J^i \tag{31}$$

$$c_j^0=c_0 \quad (j=\frac{d_s}{\Delta z}) \tag{32a}$$

$$=0 \quad (j=\text{elsewhere}) \tag{32b}$$

Eqs. (23), (30) and (31) constitute a system of J+1 linear equations with three unknowns at each i. The coefficient matrix associated with this linear system has a tridiagonal structure, and is solved by the Thomas algorithm[10] which requires the least amount of calculations. The following values were used throughout the computation: $\Delta z'=0.005$, $\Delta x'=0.01$.

3. SIMULATION RESULTS AND SENSITIVITY ANALYSIS

3.1 Comparison with the Analytic Solution for Constant Velocity and Diffusivity

The analytic solution to Eq. (1) with the mean velocity(U) and the depth-averaged turbulent diffusivity(ε_m) in place of u(z) and ε(z), respectively, and with the initial and boundary conditions, Eq (5), (6) and (7), can be written as[7]

$$c=\frac{c_s q_s/U}{\sqrt{4\pi\varepsilon_m x/U}}\sum_{n=-\infty}^{\infty}[\exp(-\frac{(z+2nd+d_s)^2}{4\varepsilon_m x/U})+\exp(-\frac{(z+2nd-d_s)^2}{4\varepsilon_m x/U})] \tag{33}$$

where

$$\varepsilon_m=\frac{u_* d}{6}. \tag{34}$$

In Eq. (33) it usually suffices to use only a few terms, for instance, n=0, ±1, and ±2.[7] In this study, n=0, ±1, ···, ±20 was used. It can also be normalized in terms of previously defined dimensionless variables and ε_m' as

$$c'=\frac{1}{\sqrt{4\pi\varepsilon_m' x'}}\sum_{n=-\infty}^{\infty}[\exp(-\frac{(z'+2n+d_s')^2}{4\varepsilon_m' x'})+\exp(-\frac{(z'+2n-d_s')^2}{4\varepsilon_m' x})] \tag{35}$$

where ε_m' is ε_m normalized by Ud, which can be expressed as

$$\varepsilon_m'=\frac{1}{6}\sqrt{\frac{f}{8}} \tag{36}$$

The numerical model was tested by comparing the result computed for constant velocity(U) and diffusivity(ε_m) with the analytic solution. The comparison showed a good agreement between the numerical and the analytic solutions. Fig. 2 illustrates the comparison of the concentration profiles at $x'=5$, where the source position is the mid-depth and $f=0.02$ is used. The normalized concentration computed by the numerical model is 1.005 at far downstream positions where the concentrations at all depths are the same to the fourth digit. This means that the computation by the numerical model has an error of 0.5% in terms of the concentration of complete mixing.

Fig. 3 and Fig. 4 shows comparisons between the numerical model for the case of logarithmic velocity and parabolic diffusivity and the analytic solution for the case of constant velocity and diffusivity. Illustrated in Fig. 3 are concentration distributions at two downstream locations for the source released at $d_s'=0.01$ and the friction factor $f=0.02$. The source position was taken as $d_s'=0.01$ not to place it within the laminar sublayer or the roughness height. Fig. 4 represents the normalized crossing distance (X_c') and mixing distance(X_m') for friction factors ranging from 0.01 to 0.1. The crossing distance is defined as the longitudinal distance required for the source mass to spread across the whole depth and the mixing distance is the distance for the mass to be completely mixed. These definitions are essentially the same as those defined by Holley et al.[11] for the case of transverse mixing. X_c' and X_m' in Fig. 4 are taken as the distances where the concentration at the water surface first becomes larger than 2% and 98%, respectively, of the concentration at the bottom.

It is seen that the analytic results for constant velocity and diffusivity overestimate the rate of vertical mixing. The mixing distance from the analytic solution is about one half as short as that from the numerical model. This contradicts the conclusion of McNulty and Wood[3] that the pollutant reaches the complete mixing earlier in case of logarithmic velocity profile than it does under the assumption of constant velocity. But they used the wrong value, u_*d, for the average diffusivity instead of the correct value, $u_*d/6$. Also observed

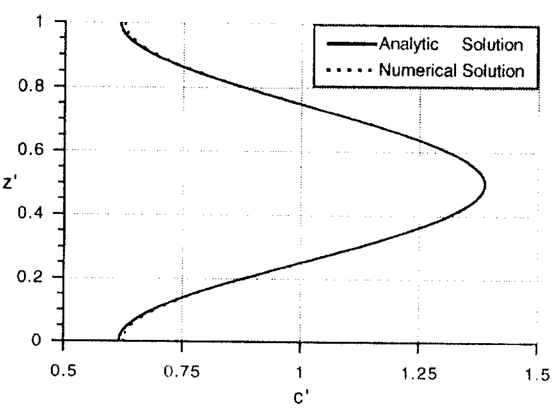

Fig. 2. Concentration Distribution for Constant Velocity and Diffusivity($x'=5$).

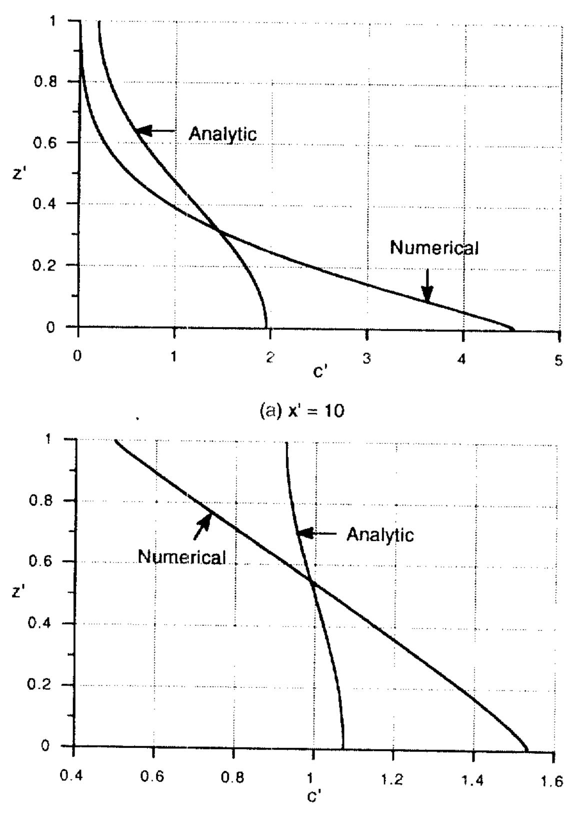

Fig. 3. Concentration Distribution at Downstream Locations.

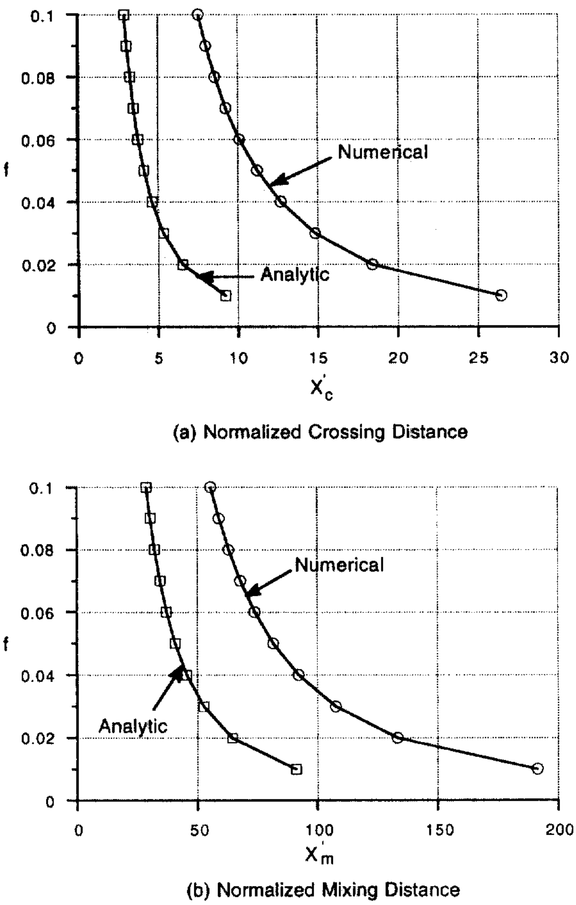

Fig. 4. Crossing and Mixing Distances.

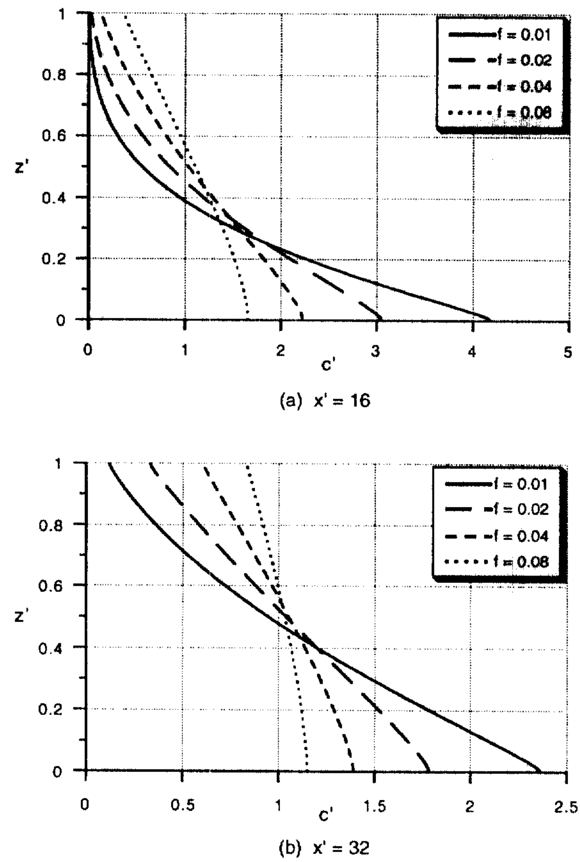

Fig. 5. Concentration Distributions for Various Friction Factors.

in Fig. 4 is that the crossing and mixing distances become shorter for larger friction factor. This means that mixing occurs more rapidly for a larger friction factor.

3.2 Sensitivity to Friction Factor

A series of numerical simulations was carried out to see the effect of friction factor on the vertical diffusion in turbulent shear flows. Friction factors of 0.01, 0.02, 0.04, and 0.08 were tried. The source position was taken as $d_s'=0.01$. Fig. 5 shows concentration distributions at downstream locations, $x'=16$ and $x'=32$ for various friction factors. The initial concentration is higher for larger friction factor because given the same mean velocity, the velocity near the bottom is lower for a larger friction factor consequently giving the higher initial concentration as it is given by Eq. (7a). However, as the source mass travels downstream, it is mixed more rapidly in case of larger friction factor. One can observe that the concentration distribution at $x'=16$ for $f=0.08$ is very close to that at $x'=32$ for $f=0.02$, which means that the degree of vertical mixing for $f=0.08$ is twice as rapid as it is for $f=0.02$. Similar relationship is observed between $f=0.04$ and $f=0.01$. The vertical turbulent velocity component has the same order of magnitude as the shear velocity, which varies as the square root of friction factor for a given mean flow velocity. Thus, increasing a friction factor by 4 times would give twice as large a turbulent velocity, resulting in approximately twice as rapid a diffusion process as observed in the above comparison. By comparing concentration distributions for various friction factors at two different downstream locations, it is concluded that the rate of vertical diffusion varies approximately as the square root of the friction factor.

3.3 Sensitivity to Source Position

The effect of source position on the vertical diffusion was investigated by computing concentration distributions at downstream locations for various source positions. Fig. 6 shows the simulation results for the following three different initial source positions: (1) $d_s'=0.01$ (near the bottom); (2) $d_s'=0.99$ (near the water surface); (3) $d_s'=0.50$ (at the mid-depth). For all cases $f=0.04$ was used. The high initial concentration for $d_s'=0.01$ in Fig. 6-(a) is due to the low flow velocity near the bottom. As shown in Fig. 6-(b) and Fig. 6-(c), $d_s'=0.50$ is the best among the three for the rapid mixing , and $d_s'=0.99$ results in the slowest mixing. The vertical diffusivity, which has a parabolic distribution, is equally low both near the surface and near the bottom of the channel, but the longitudinal advection is higher at the surface due to higher flow velocity. Hence, to achieve the same degree of mixing, source mass introduced near the water surface needs more time than that introduced near the bottom. Moreover, the position of the maximum concentration for $d_s'=0.50$, due to the slow longitudinal advection as well as the low vertical diffusion near the bottom, moves toward the channel bottom. The fact that the concentration is higher in the region closer to the bottom implies that the best source position, which gives the most rapid complete mixing, exists somewhere between the mid-depth and the water surface.

3.4 Optimum Source Position

The best source position for rapid mixing was found for friction factors ranging from 0.01 to 0.09 . Various source positions were tried for each friction factor and the one which gives the shortest mixing distance was taken as the optimum source position. The result is summarized in Table 1 and the best source position for each friction factor is plotted in Fig. 7.

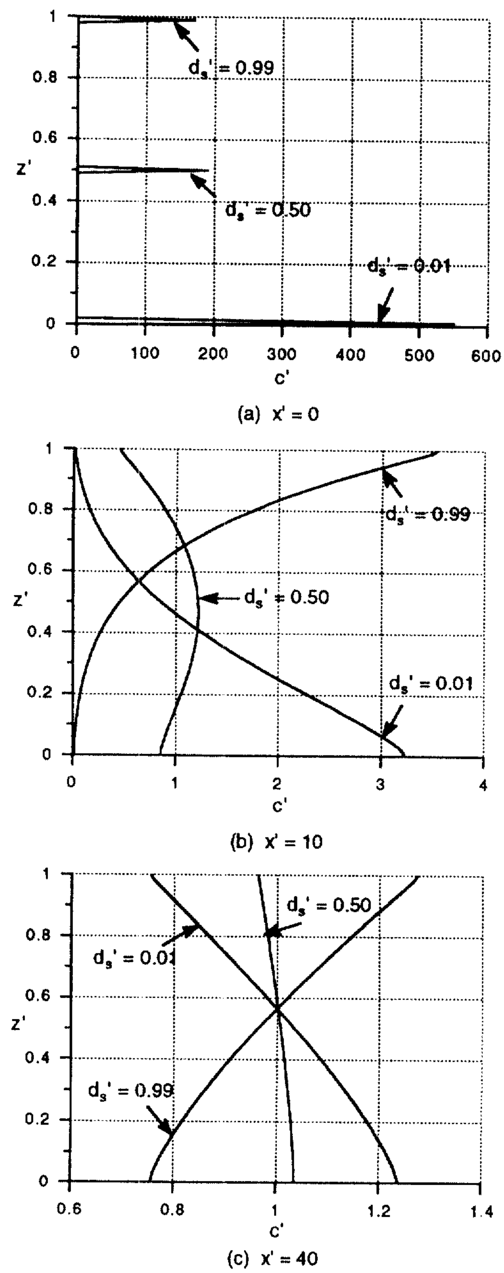

Fig. 6. Concentration Distributions for Various Source Positions.

One can see that the optimum source position is located slightly above the mid-depth as expected. Furthermore, the best source position moves toward the water surface as the friction factor increases since the velocity profile becomes steeper

Table 1. Mixing Distances for Various Friction Factors and Source Positions

d_s' \ f	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.535	60.3								
0.540	59.2	44.1	45.0	45.4	44.1	42.5	40.9	39.3	37.9
0.545	63.1	41.3	39.2	41.8	41.7	40.7	39.5	38.2	36.9
0.550	68.9	42.1	35.7	37.2	38.8	38.6	37.9	36.9	35.8
0.555	78.9	54.2	33.4	32.6	35.2	36.1	36.0	35.4	34.6
0.560	87.7	50.2	33.9	30.0	30.6	33.1	33.8	33.7	33.2
0.565	94.7	57.7	36.4	28.2	27.6	29.2	31.1	31.6	31.6
0.570	100.5	63.4	40.6	29.6	25.7	25.7	27.7	29.2	29.2
0.575	105.5	68.7	46.7	32.1	25.4	23.7	24.1	26.0	27.3
0.580	109.8	68,0	46.7	32.1	25.4	23.7	24.1	26.0	27.3
0.585	113.7	71.8	51.4	36.9	27.3	22.6	22.1	22.5	24.3
0.590	117.1	75.2	55.1	41.7	30.4	24.1	20.7	20.6	21.1
0.595	120.3	78.1	58.2	45.4	35.3	26.5	21.9	19.3	19.2
0.600	123.1	80.7	60.9	48.5	39.0	30.9	23.9	20.3	18.0
0.605									20.9

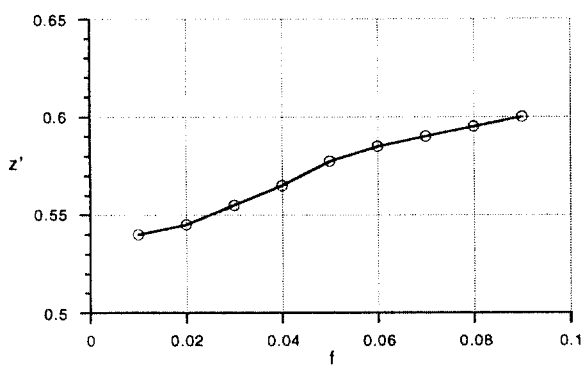

Fig. 7. Optimum Source Position for Various Friction Factors.

for a larger friction factor, which means a larger difference between the longitudinal advection near the surface and that near the bottom.

4. CONCLUSIONS

As is seen from a normalization of the advective diffusion equation, the vertical diffusion process depends solely on the friction factor. The analytic solution for constant velocity and diffusivity overestimates the degree of vertical mixing. The rate of vertical mixing varies approximately as the square root of the friction factor, which is reasonable considering that the turbulent velocity component varies with the square root of friction factor. The best source position, which gives the most rapid mixing moves toward the water surface as the friction factor increases since the velocity profile becomes steeper for a larger friction factor.

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