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Some Remarks on Faithful Multiplication Modules

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ABSTRACT. Let R be a commutative ring with identity and let M be a nonzero multiplication R-module. In this note we prove that M is finitely generated if M is a faithful multiplication R-module.

1. Introduction.

In this note all rings are commutative rings with identity and all modules are unital. Let R be a ring and M a nonzero R-module. The annihilator of M is denoted $Ann(M)$. For any submodule N of M the annihilator of the factor module M/N will be denoted by $(N : M)$ so that $(N : M) = \{r \in R : rM \subseteq N\}$. A module M is called faithful if $Ann(M) = 0$. Following [1], A module M is called a multiplication module if for any submodule N of M, there exists an ideal I of R such that $N = IM$. It is well-known that M is a multiplication module if and only if $N = (N : M)M$ for evey submodule N of M. A proper submodule N of a module M over a ring R is said to be prime if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in (N : M)$. Following [7], the radical of N, denoted by $radN$, is defined to be the intersection of all prime submodules of M cotaining N. If I is an ideal of ring R, then the radical of I considered as a submodule of R-module R is denoted by $\sqrt{I}$ and consists of all elements r of R such that $r^n \in I$ for some positive integer n. $RadM$ is defined to be the intersection of all the maximal submodules of M. $J(R)$ is defined to be the intersection of all maximal ideals of R.

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A prime submodule P of a module M is called a minimal prime of A if $A \subseteq P$ and if there is no prime submodule Q of M such that $A \subset Q \subset P$. Following [4, corollay 2] or [3, corollary], it was proved that if R is an integral domain and M is a faithful multiplication R-module, then M is finitely generated. We shall show that if M is a faithful multiplication R-module, then M is finitely generated.

2. Faithful mulitplication modules.

PROPOSITION 2.1. *Let R be an integral domain. If M is a faithful multiplication R-module, then every non-zero submodule of M is faithful.*

PROOF. Suppose N is nonzero submodule of M. Then $N = IM$ for some ideal of R. Suppose $rN = 0$ for $r \in R$. Then $rN = rIM = 0$. Since M is a faithful module and R is an integral domain, $rI = 0$ and $r = 0$. Hence N is faithful.

LEMMA 2.2. *Let R be a commutative ring with identity, M a multiplication R-module with annihilator J and A and B ideals of R. Then $AM \subseteq BM$ if and only if $A \subseteq B + J$ or $M = ((B+J) : A)M$.*

PROOF. See [8, theorem 9].

THEOREM 2.3. *Let M be a multiplication module. If N is a prime submodule of M, then there exists a unique prime ideal P of R containing $Ann(M)$ such that $N = PM$.*

PROOF. Since M is a multiplication module and N is a prime submodule of $M, N = (N : M)M = PM$ for some prime ideal P of R with $Ann(M) \subsetneq P$. We show that $(N : M) = (PM : M) = P$ for the uniqueness. Clearly $P \subseteq (PM : M)$. If $r \in (PM : M)$, then $(r)M \subseteq PM$. By lemma 2.2 $(r) \subseteq P$ or $(P : (r))M = M$. Suppose $(r) \nsubseteq P$, then $(P : (r))M = M$. Clearly $P \subsetneq (P : (r))$. If

$a \in (P : (r))$, then $a(r) \subseteq P$ and so $ar \in P$. Since $r \notin P$ and P is prime ideal of R ,$a \in P$. Thus $(P : (r)) \subseteq P$. Hence $(P : (r)) = P$ and $M = (P : (r))M = PM$. It contradicts to $PM \neq M$. Thus $r \in P$ and $(PM : M) = P$.

COROLLARY 2.4. *Let M be a faithful multiplication module and let P be a prime ideal of R. Then PM is prime submodule if and only if $(PM : M) = P$.*

PROOF. Since M is faithful, $Ann(M) \subseteq P$. If PM is a prime submodule of M, then $PM = (PM : M)M$ and $(PM : M)$ is a prime ideal of R. By theorem 2.3, $P = (PM : M)$. Conversely, if $(PM : M) = P$, then $PM \neq M$. By [2, corollary 2.11] it holds.

COROLLARY 2.5. *Let M be a multiplication module and let P be a maximal ideal of R. Then PM is a maximal submodule of M if and only if $(PM : M) = P$.*

PROOF. Since $PM \neq M$, $Ann(M) \subseteq P$. It is obvious by theorem 2.3 and [2, theorem 2.5]. Let R be an integral domain and let M be a faithful multiplication module. Then M is finitely generated by [4, corollary 2] or [3, 2 corollary]. We shall show that if M is a faithful multiplication module, then M is finitely generated.

THEOREM 2.6. *Let M be a faithful multiplication module, then M is finitely generated.*

PROOF. Suppose M is not finitely generated. By [2, theorem 3.1] there exists some maximal ideal P of R such that $M = PM$. Since M is a multiplication module, there exists a maximal submodule Q of M such that $Q = qM \neq M$ with a maximal ideal q of R by [2, theorem 2.5]. Since a maximal submodule is a prime submodule and $Q = qM = qPM = PqM$. $q = Pq$ and $q = Pq \subseteq P$ by theorem 2.3. Since P and q are maximal ideals of $R, P = q$. Therefore $M = PM = qM = Q$. It contradicts to $M \neq Q$. Hence M is finitely generated.

COROLLARY 2.7. *Let M be a faithful multiplication module. Then $M \neq AM$ for any proper ideal A of R.*

PROOF. See the proof of the theorem 2.6.

THEOREM 2.8. *M is a faithful multiplication R-module if and only if for each submodule N of M, there exists a unique ideal I of R such that $N = IM$.*

PROOF. Suppose M is a faithful multiplication module. Then M is finitely generated by theorem 2.6. Since M is a multiplication module, for each submodule N of M, there exists an ideal I of R such that $N = IM$. It is sufficient to prove that $I = (N : M)$ for the uniqueness. Clearly $I \subseteq (N : M)$. If $r \in (N : M)$, then $rM \subseteq N = IM$. By lemma 2.2 $(r) \subseteq I$ or $(I : r)M = M$. Suppose $(r) \nsubseteq I$. Then $(I : r)M = M$. By corollary 2.7, $I : (r) = R$. Hence $(r) \subseteq I$, a contradiction. Therefore $(r) \subseteq I$ and $I = (N : M)$. Conversely suppose the condition holds. Then M is a multiplication module. Suppose $rM = 0$. Then $(r)M = 0$. By uniqueness $(r) = 0$. Thus $r = 0$ and M is faithful.

COROLLARY 2.9. *If M is faithful multiplication R-module. Then $RadM = J(R)M$.*

COROLLARY 2.10. *Let M be a faithful multiplication R-module. and let A be an ideal of R and N a submodule of M. Then*

(1) *N is a multiplication R-module if and only if $(K : N)(N : M) = (K : M)$ for each submodule K of N.*
(2) *$I = (IM : M)$ for each ideal I of R.*

(3) *N is finitely generated if and only if $(N : M)$ is finitely generated.*

(4) *N is faithful if and only if $(N : M)$ has zero annihilator.*

3. Radicals of submodules in modules.

THEOREM 3.1. *Let M be a multiplication module and let N and L be submodules of M. Then $radN + radL = M$ if and only if $N + L = M$*

PROOF. Clearly, if $N + L = M$, then $radN + radL = M$. Suppose $N + L \neq M$. There is a maximal submodule P of M containing $N + L$ by [2,theorem 2.5].

Therefore $radN \subseteq P$, and $radL \subseteq P$. It is that $radN + radL \subseteq P \neq M$.

COROLLARY 3.2. *Let M be a multiplication module and let N and L be submodules of M. Then*

(1) *$N + L = M$ if and only if $radN + L = M$.*

(2) *$radN = M$ if and only if $N = M$.*

PROPOSITION 3.3. *Let M be a multiplication R-module and let I be an ideal of R. If $I \subseteq J(R)$, then IM is small in M.*

PROOF. Suppose $IM + N = M$. If $N \neq M$, then there exists a maximal submodule P of M containing N and $P = (P : M)M \neq M$ where $(P : M)$ is a maximal ideal of R, by [2,theorem 2.5]. Since $I \subseteq J(R)$, $IM + N \subseteq P \neq M$.

PROPOSITION 3.4. *Let M be a multiplication module and let N be a submodule of M. P is a minimal prime submodule of N if and only if there exisits a minimal prime ideal I of $(N : M)$ such that $P = IM \neq M$.*

PROOF. Suppose P is a minimal prime submodule of N. Then $(N : M)M \subseteq (P : M)M$ and $(N : M) \subseteq (P : M)$ with a prime ideal $(P : M)$ of R. We show that $(P : M)$ is a minimal prime ideal of $(N : M)$. If $(N : M) \subseteq I \subseteq (P : M)$ with a prime ideal I, then $N \subseteq IM \subseteq P$. Since P is a minimal prime of N, $IM = P$ or $N = IM$. By theorem 2.3, $I = (P : M)$ or $(N : M) = I$, Therefore $(P : M)$ is a minimal prime ideal of $(N : M)$.

Conversely, if I is minimal prime ideal of $(N : M)$ such that $IM \neq M$, then $N \subseteq IM$. Let $IM = P$. Then P is a prime submodule of M containing N. If Q is a prime submodule of M containing N such that $N \subseteq Q \subseteq P$, then $(N : M) \subseteq (Q : M) \subseteq (P : M)$. Since $(P : M) = (IM : M) = I$ by theorem 2.3, $(P : M)$ is a minimal prime ideal of $(N : M)$ Therefore $(Q : M) = (N : M)$ or $(Q : M) = (P : M)$. Hence $Q = N$ or $Q = P$. Thus P is a minimal prime submodule of N.

COROLLARY 3.5. *Let A be an ideal of R containing $Ann(M)$ and M a multiplication module.*

Then P is a minimal prime ideal of A if and only if PM is a minimal prime submodule of AM.

PROOF. By corollary 2.7, $M \neq PM$. By proposition 3.4, PM is a minimal prime submodule of AM. Conversely, by proposition 3.4, there is a minimal prime ideal Q of $(AM : M)$ such that $PM = QM \neq M$. By theorem 2.3 and proposition 3.4, $P = Q$.

COROLLARY 3.6. *Let M be a multiplication module. Then rad $AM = \sqrt{A}M$ for every ideal A containing $Ann(M)$.*

PROOF. By corollary 3.5 and [2, corollary 1.7], it is obvious.

COROLLARY 3.7. *Let M be a multiplication module and let N be a submodule of M. Then $radN = \sqrt{(N : M)}M$.*

References

1. A. Barnard, *Multiplication Modules,*, J. Algebra **71** (1981), 174-178.
2. Z.A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. in Algebra **16** (1988), 755-779.
3. V.Erdogdu, *Multiplication modules which are distributive,* J. Pure and Applied Algebra **54** (1988), 209-213.
4. E.S. Kim and C.W. Choi, *On multiplication modules,* Kyungpook Math.J. **32(1)** (1992), 97-102.
5. G.M. Low and P.F. Smith, *Multiplication modules and ideals*, Comm. in Algebra **18(12)** (1990), 4353-4375.
6. C.P.Lu, *M-radicals of submodules in modules*, Math. Japonica **34(2)** (1989), 211-219.
7. R.L. McCasland and M.E.Moore, *On radicals of submodules of finitely generated modules*, Canad. Math. Bull. **29** (1986), 37-39.
8. P.F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50** (1988), 223-235.

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