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Cartan Subalgebras of a Semi-restricted Lie Algebra

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ABSTRACT. In this paper we show that if a semi-restricted Lie algebra L has an one dimensional toral Cartan subalgebra, then L is simple and $L \cong \mathrm{sl}(2)$ or $W(1:\underline{1})$. And we study that if L is semi-simple but not simple and H is 2-dimensional, then H is a torus.

1. Preliminaries.

Let F be a field of prime characteristic. If a Lie algebra L has a map $x \to x^p$ of L to L satisfying

1) $(\alpha x)^p = \alpha^p x^p$ for any $\alpha \in F, x \in L$
2) $(\mathrm{ad}\ x)^p = \mathrm{ad}\ (x^p)$
3) $(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x,y)$ where $is_i(x,y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}\ (\lambda x + y)^{p-1})x$,

then we say that L is *restricted* and the map is called *p-mapping.*

If K is any subalgebra of L, we define $\overline{K} = \langle x^{p^n} | x \in K, n \in \mathbb{N}\rangle$. Clearly $\overline{K}$ is contained in any restricted subalgebra of L which contains K. On the other hand, if $x, y \in K$ then for $m, n \in \mathbb{N}$,

$$[x^{p^n}, y^{p^m}] = (\mathrm{ad}\ x)^{p^n} y^{p^m} = -(\mathrm{ad}\ x)^{p^n-1}(\mathrm{ad}\ y)^{p^m} x \in [K,K]$$

Thus we have;

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LEMMA 1.1. *$\overline{K}$ is the smallest restricted subalgebra of L containing K. Furthermore $[\overline{K}, \overline{K}] \subseteq [K, K]$.*

Following [3, p 119] we define $x \in L$ to be nilpotent if $x^{p^n} = 0$ for some $n \in \mathbb{N}$, and define $x \in L$ to be semisimple if x satisfies some separable p-polynomial

$$x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_1 x^p + a_0 x, n \in \mathbb{N}, a_{n-1}, \ldots, a_0 \in F, a_0 \neq 0.$$

If $x^p = x$ (and hence is semisimple) we say x is toral. We say a subset K of L is *nil* if every $x \in K$ is nilpotent. A subalgebra T of L is called a *torus* (or a toral subalgebra) if every $x \in T$ is semisimple. A torus is necessarily abelian. By [3] (Theorem 5.13) a finite dimensional torus has a basis consisting of toral elements.

THEOREM 1.2 [4]. *Suppose that L is a restricted Lie algegra and $x \in L$. Then there exist unique elements s, z in L such that;*

(1) *s, z are p-polynomials in x.*
(2) *s is semisimple and z is nilpotent.*

Moreover s is a linear combination of p-polynomials $y_1, \ldots, y_n$ in x with $y_i^p = y_i$ for each i.

For any Lie algebra L, the derivation algebra Der$L$ is restricted where D^p is the p-th power of $D \in$ Der$L$. If L is semi-simple, we can identify L with ad L in the restricted algebra Der$L$. We will say H a Cartan subalgebra if H is nilpotent and self-normalized.

THEOREM 1.3 (Kaplansky, [2]). *Let L be a finite dimensional simple Lie algebra over F containing a one dimensional Cartan subalgebra H. Suppose that the additive subgroup of H^* generated by the roots has rank one (this always holds if L is restricted). Then $L \cong sl(2)$ or $W(1 : \underline{1})$.*

A Lie algebra L is called rank one provided that L has a one dimensional Cartan aubalgebra. In 1984, G.M. Benkart and J.M. Osborn proved a generalization of Kaplansky's Theorem;

THEOREM 1.4 [1]. *Let L be a finite-dimensional simple Lie albegra of rank one. Then L is sl(2) or an Albert-Zassenhaus algebra.*

Let $H \subseteq L$ be a Cartan subalgebra and $\overline{H}$ be the restricted subalgebra of Der$L$ generated by H. Let T denote the (unique) maximal torus of $\overline{H}$. We call $\dim T$ the *toral* rank of L with respect to H.

THEOREM 1.5 [2]. *L is a simple Lie algebra over F and has toral rank one if and only if*

$$L \cong sl(2), \quad W(1:\underline{\mathbf{n}}), \quad \text{or} \quad H(2:\underline{\mathbf{n}}:\Phi)^{(2)}.$$

2. The semi-restricted Lie algebra.

As a generalization of restrictedness, Schue[4] introduced the notion of a semi-restricted Lie algebra.

DEFINITION 2.1. Suppose that L is a Lie algebra, H a Cartan subalgebra of L, and $L = H + \sum L_\alpha$ is the Cartan decomposition of L.

L is *semi-restricted* with respect to H if ;

(a) H is restricted

(b) $(\text{ad } h)^p x = [h^p, x]$ for $h \in H, x \in L$

(c) for each α with $x \in L_\alpha$, the semi-simple component of $\text{ad } x$ is equal to $\text{ad } s$ for some semi-simple element $s \in H$.

For a semi-simple Lie algebra L, embedding L into Der$L$ via adjoint mapping, we take a modification of the definition (c); for each root α with $x \in L_\alpha$, the semi-simple component of x is equal to s for some semi-simple element $s \in H$. Furthermore the decomposition $x = s + n$ is unique.

PROPOSITION 2.2. *If L is restricted and H is a Cartan subalgebra, then L is semi-restricted with respect to H.*

PROOF. H is restricted since $\overline{H}$ is a nilpotent subalgebra containing H. Let S be the subspace of semi-simple elements of H. Suppose α is a non-zero root and $x \in L_\alpha$. Then $[s, x^p] = 0$ for $s \in S$. Thus if $x = t + w$ is the decomposition of x with t semi-simple, then t^p is the semi-simple component for x^p so that $[S, t^p] = 0$ and hence $[S, t] = 0$. Thus $t \in H$.

The converse of the proposition is not true. For example, $W(1 : \underline{2})$ is not restricted. But $H =< xD, x^{p+1}D, \dots, x^{p(p-1)+1}D >$ is a Cartan subalgebra and $W(1 : \underline{2})$ is semi-restricted with respect to H. Any subalgebra of L containing H is also semi-restricted with respect to H and K is an ideal of L with $K_0 = K \cap H$ restricted, then L/K is semi-restricted with respect to H/K_0.

DEFINITION 2.3. Suppose L is a Lie algebra and H is a Cartan subalgebra. The maximal number of linearly independent roots in the Cartan decomposition of L relative to H, will called the *type* of L with respect to H.

PPRPOSITION 2.4 ([4] p 29). *Suppose L is semi-restricted with respect to H and the center of L is nil. Let S be the subspace of semi-simple elements of H. If L is of type d with respect to H, then $d = dimS$.*

Using the proposition we restate the Kaplansky's Theorem as follow; If L is semi-restricted and type 1, then L is either sl(2) or $W(1 : \underline{1})$.

For a semi-restricted Lie algebra L, let $x = s + n$ ($x \in L_\alpha$ a root vector) be the decomposition of x. Then for sufficiently large integer

k we have

$$x^{p^k} = (s+z)^{p^k} = s^{p^k} + z^{p^k} = s^{p^k} \in H.$$

LEMMA 2.5. *Let L be as above. If $x \in L_\alpha$ then $\alpha(x^{p^k}) = 0$ for some x^{p^k} in H.*

PROOF. For $x \in L_\alpha, x^{p^k} \in H$ for sufficiently large integer k. Since $0 = [x^{p^k}, x] = \alpha(x^{p^k})x$, we have $\alpha(x^{p^k}) = 0$.

Let L be a semi-simple Lie algebra over F with a Cartan subalgebra H and semi-restricted with respect to H.

LEMMA 2.6. *Let I be an H-invariant ideal of L. If $\overline{I} \cap H = (0)$, then I is nil.*

PROOF. Since I is H-invariant, we have $I = \sum_{\alpha \in H*} I_\alpha$, where $I_\alpha = I \cap L_\alpha$. If $x \in L_\alpha$ then $x^{p^k} \in \overline{I} \cap H = (0)$ for some k, i.e., x is nilpotent. Then by Engel-Jacobson Theorem, I is nil.

THEOREM 2.7. *If H is one-dimensional toral, then L is simple and $L \cong sl(2)$ or $W(1:\underline{1})$.*

PROOF. Let I be a nonzero ideal of L. I is not nil since L is semi-simple. Lemma 2.6 shows that $(0) \neq \overline{I} \cap H \subseteq H$. As $\dim H = 1, \overline{I} \cap H = H$. So $H \subseteq \overline{I}$, but then $L_\alpha = [H, L_\alpha] \subseteq [\overline{I}, L_\alpha] \subseteq I$. Since H is one dimensional toral, $L = H + \sum L_{i\alpha}$ for some nonzero root α. If $I = \sum_{j=1}^{p-1} L_{i\alpha}$ then for $x \in L_{i\alpha}, i\alpha(x^{p^k}) = 0$ for sufficiently large integer k. So for j in $\mathbb{Z}_p^*, j\alpha(x^{p^k}) = 0$. Hence x is nilpotent and by Engel-Jacobson Theorem, I is nil. This contradict to the semi-simplicity of L. Hence $I = L$, so L is simple. The Kaplansky's Theorem proves the remaining conclusion.

Now we indroduce a property of nonsolvable semi-restricted Lie algebra. This has been proved by Schue [4].

LEMMA 2.8. *Suppose L is semirestricted, H a Cartan subalgebra, and $L = \sum L_\alpha$ is the Cartan decomposition with $H = L_0$. Let*

$$W = \sum [L_\alpha, L_{-\alpha}] + \sum L_\alpha, \alpha \neq 0$$

If L is not solvable and all root vectors are nilpotent, then $\beta(\sum [L_\alpha, L_{-\alpha}]) \neq 0$ for some root $\beta \neq 0$.

THEOREM 2.9. *Let L be a non-simple semi-simple Lie algebra with a two-dimensional Cartan subalgebra H and L is semi-restricted with respect to H. Then H is a torus.*

PROOF. . Assume H is not a torus. Since L is semi-restricted, H must have a maximal torus of dimension 1. Then $L = H + \sum L_{i\alpha}$ where α is a non-zero root of L. Let $x \in L_{i\alpha}, i \in \mathbb{Z}_p^*$, then $[x^{p^k}, x] = 0$ implies $\alpha(x^{p^k}) = 0$ for some k. So x is nilpotent.

Let K be an ideal of L, then we have $K = K_0 + \sum K_{i\alpha}$ with $K_{i\alpha} = L_{i\alpha} \cap J$.

If K_0 is nil, by the Engel-Jacobson Theorem, K is nilpotent. This contradict to the semisimplicity of L. Thus K_0 is not nil and so $K_{i\alpha} = L_{i\alpha}$ since K is an ideal of L.

Now let I be the ideal of L, generated by $\sum L_{i\alpha}$. Then $I = I_0 + \sum L_{i\alpha}$. (and I_0 is not nil). Since any ideal of I is also an ideal of L, I is simple and we have $I \neq L$.

From the non-solvability of L, Lemma 2.8 shows $I_0 = \sum_{i \in z_p^*} [L_{i\alpha}, L_{-i\alpha}] \neq (0)$. Hence the dimension of the Cartan subalgebra I_0 of I is one. Thus the Kaplansky theorem shows that $I \cong \mathrm{sl}(2)$ or $W(1 : \underline{1})$. But the derivations of I are all inner. This contradict to $I \neq L \subseteq \mathrm{Der}L$.

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