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Projectivity for 3-Dimensional Compact Nonsingular Toric Varieties

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ABSTRACT. There are some sufficient or necessary conditions about projectivity for toric varieties. We consider one of them and state some conditions about projectivity for a 3-dimensional compact nonsingular case which is obtained from a projective one by nonsingular equivariant blow-down.

1. Introduction.

We define a toric variety to be an irreducible normal scheme X over $\mathbf{C}$ which is separated and locally of finite type over $\mathbf{C}$, contains an algebraic torus T_N as an open set and is endowed with an algebraic action of T_N extending the group multiplication of T_N.

A toric variety can also be described in terms of a certain collection, which is called a fan, of cones. From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators.

There are some sufficient or necessary conditions about projectivity for toric varieties, which are stated in terms of cones and the relations of their generators.

Kleinschmidt and Sturmfels [5] proved that every r-dimensional compact toric variety with Picard number ≤ 3 must be projective, while Ewald [1] has constructed an r-dimensional nonsingular, non-projective toric variety with Picard number$= 4$. They used Gale

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diagram (cf. [6]). This concept is very useful in dealing with small Picard numbers. Shephard [12] also used it to give a combinatorial criterion for a toric variety to be projective. We have generalized this criterion to general cone decompositions (cf. Oda and Park [9], [10]).

In this paper, we state some conditions about projectivity for a 3-dimensional compact nonsingular case which is obtained from a projective one by nonsingular equivariant blow-down.

Now we introduce some basic definitions which are used throughout this paper. Let N be a free $\mathbf{Z}$-module of rank r over the ring $\mathbf{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ its dual $\mathbf{Z}$-module. We denote the scalar extensions of N and M to the field $\mathbf{R}$ of real numbers by $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, respectively.

A subset σ of $N_{\mathbf{R}}$ is called a *rational convex polyhedral cone* (or a *cone*, for short), if there exist a finite number of elements $n_1, n_2, ..., n_s$ in N such that

$$\begin{aligned}\sigma &= \mathbf{R}_{\geq 0} n_1 + \mathbf{R}_{\geq 0} n_2 + ... + \mathbf{R}_{\geq 0} n_s \\ &:= \{a_1 n_1 + ... + a_s n_s \mid a_i \in \mathbf{R}, a_i \geq 0 \text{ for all } i\},\end{aligned}$$

where we denote by $\mathbf{R}_{\geq 0}$ the set of nonnegative real numbers. σ is said to be *strongly convex* if it contains no nontrivial subspace of $\mathbf{R}$, that is, $\sigma \cap (-\sigma) = \{0\}$.

A finite collection Δ of strongly convex cones in $N_{\mathbf{R}}$ is called a *fan* if it satisfies the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The *support* of a fan Δ is defined to be $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$.

If a fan Δ is given, then there exists a toric variety $X := T_N\mathrm{emb}(\Delta)$ determined by Δ over the field $\mathbf{C}$ of complex numbers. For the precise definition of toric varieties, see [3], [7] and [8].

2. Definitions and some results.

Let (N, Δ) and (N', Δ') be two fans for $N \cong \mathbf{Z}^r$ and $N' \cong \mathbf{Z}^{r'}$. A *map of fans* $\phi : (N', \Delta') \to (N, \Delta)$ is a $\mathbf{Z}$-linear homomorphism $\phi : N' \to N$ whose scalar extension $\phi : N'_{\mathbf{R}} \to N_{\mathbf{R}}$ satisfies the following property: For each $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ such that $\phi(\sigma') \subset \sigma$.

An *equivariant* morphism from $T_N \subset X$ to another toric variety $T_{N'} \subset X'$ is a morphism $f : X \to X'$ which induces an algebraic group homomorphism $f : T_N \to T_{N'}$ and which satisfies the compatibility with respect to the action: $f(t \cdot x) = f(t) \cdot f(x)$ for any $t \in T_N$ and $x \in X$.

If $\phi : (N', \Delta') \to (N, \Delta)$ is a map of fans. Then ϕ gives rise to an equivariant holomorphic map $\phi_v : T_{N'}\text{emb}(\Delta') \to T_N\text{emb}(\Delta)$ between toric varieties.

Important algebro-geometry properties of toric varieties can be interpreted in terms of the corresponding fans. For example,

(1) The toric variety is nonsingular if and only if the fan Δ is *nonsingular*, that is, every cone $\sigma \in \Delta$ is nonsingular. We say that a cone σ is *nonsingular* if there exist a $\mathbf{Z}$-basis $\{n_1, ..., n_r\}$ of N and $s \leq r$ such that

$$\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_s.$$

(2) The toric variety is compact if and only if the corresponding fan is finite and complete, where a fan Δ is said to be *complete* if $|\Delta| = N_{\mathbf{R}}$.

(3) The equivariant morphism $\phi_v : X' \to X$ corresponding to map of fans $\phi : (N', \Delta') \to (N, \Delta)$ is proper and birational if and only if $\phi : N' \to N$ is isomorphism and Δ' is a locally finite subdivision of Δ.

If X is an r-dimensional compact nonsingular toric variety, then we have the following exact sequences:

$$0 \longleftarrow N \longleftarrow (T_N\mathrm{Div}(X))^* \longleftarrow A^{r-1}(\Delta) \longleftarrow 0$$

$$0 \longrightarrow M \longrightarrow T_N\mathrm{Div}(X) \longrightarrow A^1(\Delta) \longrightarrow 0,$$

where $(T_N\mathrm{Div}(X))^*$ denotes the dual of the group of T_N-invariant divisors $T_N\mathrm{Div}(X)$ and $A^i(\Delta)$ denotes the homogeneous part of degree $i, 1 \leq i \leq r$, of the Chow ring $A(\Delta)$.

In this case, from [8] we have

$$\begin{aligned} T_N\mathrm{Div}(X) &= \bigoplus_{\rho\in\Delta(1)} \mathbf{Z}V(\rho) \\ A^1(\Delta) &= \sum_{\rho\in\Delta(1)} \mathbf{Z}v(\rho) \\ A^{r-1}(\Delta) &= \sum_{\tau\in\Delta(r-1)} \mathbf{Z}v(\tau), \end{aligned}$$

where $V(\rho)$ (resp. $V(\tau)$) is the closures of the codimension-one (resp. -two) T_N-orbit orb(ρ) (resp. orb(τ)) corresponding to each cone $\rho \in \Delta(1)$ (resp. $\tau \in \Delta(r-1)$), and $v(\rho)$ (resp. $v(\tau)$) is the linearly equivalent class of $V(\rho)$ (resp. $V(\tau)$).

PROPOSITION 2.1. *Let X be an r-dimensional nonsingular toric variety. Then X is non-projective if and only if there exist $c(\tau) \in \mathbf{Z}_{\geq 0}, \tau \in \Delta(r-1)$, not all zero, such that*

$$\sum_{\tau\in\Delta(r-1)} c(\tau)v(\tau) = 0.$$

PROOF. Reid [11] showed that for an r-dimensional nonsingular toric projective variety, the Mori cone $NE(X)$ is a convex polyhedral cone, and is in fact strongly convex, i.e., $NE(X) \cap (-NE(X)) = \{0\}$ holds and

$$NE(X) = \sum_{\tau \in \Delta(r-1)} \mathbf{R}_{\geq 0} v(\tau).$$

If we denote by $PA(X)$ the dual cone of $NE(X)$, then Kleiman [4] showed that X is a projective variety if and only if the interior $\mathrm{int}(PA(X))$ of $PA(X)$ is nonempty. Since we have that (cf. [8, Appendix])

$$\mathrm{int}(PA(X)) \neq \emptyset \quad \text{if and only if} \quad NE(X) \cap (-NE(X)) = \{0\},$$

it implies that X is non-projective if and only if there exist $\tau_1, ..., \tau_s \in \Delta(r-1)$ and $c_1, ..., c_s \in \mathbf{Z}_{>0}$, such that

$$-\sum_{i=1}^{k} c_i v(\tau_i) = \sum_{j=k+1}^{s} c_j v(\tau_j) ,$$

hence we get the conclusion.

3. Projectivity for a 3-dimensional case.

In this section, we fix $N_{\mathbf{R}} \cong \mathbf{R}^3$. Let $X = T_N\mathrm{emb}(\Delta)$ be a 3-dimensional compact nonsingular toric variety. The corresponding fan is nonsingular and complete. If we intersect Δ with a sphere $S \subset N_{\mathbf{R}}$ centered at 0, we get a triangulation of S.

Each spherical edge is of the form $\tau \cap S$ for a 2-dimensional cone $\tau \in \Delta$. There exist exactly two 3-dimensional cones $\sigma, \sigma' \in \Delta$ satisfying $\sigma \cap \sigma' = \tau$. Let $\{n, n_1, n_2\}$ and $\{n', n_1, n_2\}$ be the primitive elements in N which generate σ and σ', respectively. Since σ and σ' are nonsingular, there exist $a_1, a_2 \in \mathbf{Z}$ such that

$$n + n' + a_1 n_1 + a_2 n_2 = 0.$$

We attach a pair (a_1, a_2) to the edge $\tau \cap S$ as a *double* $\mathbf{Z}$*-weight*, with a_1 on the side of $\mathbf{R}_{\geq 0}n_1 \cap S$ and a_2 on the side of $\mathbf{R}_{\geq 0}n_2 \cap S$. If we denote $\rho' = \mathbf{R}_{\geq 0}n', \rho'' = \mathbf{R}_{\geq 0}n''$ and $\rho_j = \mathbf{R}_{\geq 0}n_j$ for $j = 1, 2$, which all along to $\Delta(1)$, then we have

$$(V(\rho).V(\tau)) = \begin{cases} 1 & \text{if } \rho = \rho' \text{ or } \rho = \rho'' \\ a_j & \text{if } \rho = \rho_j \text{ with } j = 1, 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho \in \Delta(1)$ and $\tau = \rho_1 + \rho_2 \in \Delta(2)$.

Ewald [2] has shown that any 3-dimensional compact nonsingular toric variety can be turned into a 3-dimensional compact nonsingular *projective* toric variety by a finite sequence of T_N-invariant blowing-ups along T_N-stable nonsingular centers. Here we consider the last blowing-up above. We prove the projectivity for a 3-dimensional compact nonsingular toric variety obtained from a projective one by blow-down. We use the following proposition (cf. [8]):

PROPISITION 3.1. *Let N be a free* $\mathbf{Z}$*-module of rank r, and Δ a fan for N. For a nonsingular toric variety $X = T_N\mathrm{emb}(\Delta)$ and $\tau \in \Delta$, the* equivariant blowing-up *of X along the closure $V(\tau)$ of the orbit* orb(τ) *is the equivariant birational morphism $T_N\mathrm{emb}(\Delta^*(\tau)) \to X = T_N\mathrm{emb}(\Delta)$ corresponding to the following nonsingular* star subdivision *$\Delta^*(\tau)$ of Δ with respect to τ: Denote $\Delta(1) := \{\rho \in \Delta \mid \dim \rho = 1\}$ and let $\{\rho \in \Delta | \rho \prec \tau\} =: \{\rho_1, \rho_2, ..., \rho_k\}$. Each ρ_j contains a unique primitive element $n_j := n(\rho_j) \in N$. Let $n_0 := n_1 + \cdots + n_k$ and*

$$\tau_j := \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_{j-1} + \mathbf{R}_{\geq 0}n_0 + \mathbf{R}_{\geq 0}n_{j+1} + \cdots + \mathbf{R}_{\geq 0}n_k$$

for $1 \leq j \leq k$. Each $\sigma \in \Delta$ with $\tau \prec \sigma$ can be written as

$$\sigma = \tau + \sigma' \quad \text{with} \quad \sigma' \in \Delta \qquad \text{and} \quad \sigma' \cap \tau = \{0\}.$$

We then let

$$\sigma_j := \tau_j + \sigma' \quad \textit{for} \quad 1 \le j \le k$$

and

$$\Delta^*(\tau) := (\Delta \setminus \{\sigma \in \Delta | \tau \prec \sigma\})$$
$$\cup \{\textit{the faces of}\, \sigma_j | \sigma \in \Delta, \tau \prec \sigma, 1 \le j \le k\}.$$

THEOREM 3.2. *Let X and X' be 3-dimensional compact nonsingular toric varieties, and $f : X' \to X$ be an equivariant blowing-up with T_N-fixed point $V(\sigma)$ as a center, where $\sigma = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2 + \mathbf{R}_{\geq 0}n_3 \in \Delta(3)$. If X' is projective, then X is projective.*

PROOF. Let Δ and Δ' be the corresponding fans to X and X', respectively. By Toric Nakai criterion (cf. [8]), X' is projective if and only if there exist $D' := \sum_{\rho \in \Delta'(1)} c'(\rho)V(\rho)$ with

$$(D'.V(\tau)) > 0 \qquad \text{for any} \qquad \tau \in \Delta(2).$$

Let $n_0 := n_1 + n_2 + n_3$ and $\rho_0 := \mathbf{R}_{\geq 0}n_0$. Then by Proposition 3.1, $\rho_0 \in \Delta'(1) \setminus \Delta(1)$. Define $D := \sum_{\rho \in \Delta(1)} c'(\rho)V(\rho)$. For any $\tau \in \Delta(2)$ with $\tau + \rho_0 \notin \Delta'(3)$, we have

$$(D.V(\tau)) = (D'.V(\tau)) > 0.$$

So we may consider only the case when $\tau + \rho_0 \in \Delta'(3)$, that is,

$$\tau = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2, \text{ or } \mathbf{R}_{\geq 0}n_2 + \mathbf{R}_{\geq 0}n_3, \text{ or } \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_3$$

Suppose that $\tau = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2 \in \Delta(2)$ (the remaining cases are the same). By Proposition 3.1, the double $\mathbf{Z}$-weights of

$$\tau_1 = \mathbf{R}_{\geq 0}n_0 + \mathbf{R}_{\geq 0}n_1 \quad \text{and} \quad \tau_2 = \mathbf{R}_{\geq 0}n_0 + \mathbf{R}_{\geq 0}n_2$$

are both $-1, 1$, and the link of ρ_0 has weights $1, 1, 1$. Since Δ is complete, there exists exactly one $\rho_4 := \mathbf{R}_{\geq 0}n_4 \in \Delta(1)$ such that $\tau + \rho_4 \in \Delta(3)$ Let the double $\mathbf{Z}$-weight of τ in Δ is a, b, then the double $\mathbf{Z}$-weight of τ in Δ' becomes $a-1, b-1$, that is,

$$n_2 + n_4 + an_1 + bn_3 = 0$$
$$n_0 + n_4 + (a-1)n_1 + (b-1)n_3 = 0.$$

Hence we have

$$\begin{aligned}
(D.V(\tau)) &= c'(\rho_2) + c'(\rho_4) + ac'(\rho_1) + bc'(\rho_3) \\
&= [\ c'(\rho_4) + c'(\rho_0) + (a-1)c'(\rho_1) + (b-1)c'(\rho_2)\ ] \\
&\quad\ + [\ c'(\rho_2) + c'(\rho_1) + c'(\rho_3) - c'(\rho_0)\ ] \\
&= (D'.V(\tau)) + (D'.V(\rho_0 + \rho_1)) > 0,
\end{aligned}$$

where $\rho_i := \mathbf{R}_{\geq 0}n_i, 1 \leq i \leq 4$.

THEOREM 3.3. *Let X and X' be 3-dimensional compact nonsingular toric varieties, and $f : X' \to X$ be an equivariant blowing-up along T_N-stable one-dimensional closed subvariety $V(\tau_0)$, where $\tau_0 = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2 \in \Delta(2)$. If X' is projective and X is non-projective, then the double $\mathbf{Z}$-weight of τ_0 is both negative.*

PROOF. Let Δ and Δ' be the corresponding fans to X and X', respectively, and the double $\mathbf{Z}$-weight of τ_0 be a, b. Suppose on the contrary that $a \geq 0$ (if $b \geq 0$, then it is similar). Since X' is projective, by Toric Nakai criterion (cf. [8]) there exist $D' := \sum_{\rho\in\Delta'(1)} c'(\rho)V(\rho)$ such that $(D'.V(\tau)) > 0$ for any $\tau \in \Delta(2)$.

Define

$$D := \sum_{\rho\in\Delta(1)} c'(\rho)V(\rho)$$

$$n_0 := n_1 + n_2$$

$$\rho_i := \mathbf{R}_{\geq 0} n_i \qquad 0 \leq i \leq 4$$

For any $\tau \in \Delta(2)$, there are three cases :

(1) If $\tau + \rho_0 \notin \Delta'(3)$ and $\tau \neq \tau_0$, then we have

$$(D.V(\tau)) = (D'.V(\tau)) > 0.$$

(2) If $\tau + \rho_0 \in \Delta'(3)$ and $\tau \neq \tau_0$, then by proposition 3.1, $\tau = \mathbf{R}_{\geq 0}n_i + \mathbf{R}_{\geq 0}n_j$ for any two i, j among $\{1, 2, 3, 4\}$. Suppose that $\tau = \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_4$ and c, d be the double **Z**-weight of τ with c (resp. d) on the side of ρ_1 (resp. ρ_4). Then the double **Z**-weight of τ in Δ' becomes $c - 1, d$ in this order. Since the double **Z**-weight of $\rho_0 + \rho_4$ is $-1, 0$, we have

$$(D.V(\tau)) = (D'.V(\tau)) + (D'.V(\rho_0 + \rho_4)) > 0.$$

(3) Now we consider the case $\tau = \tau_0$. Since the double **Z**-weight of $\rho_0 + \rho_4 \in \Delta'(2)$ is $-1, 0$ and that of $\rho_0 + \rho_2 \in \Delta'(2)$ is $a, b - a$ in this order, we have

$$(D.V(\tau_0)) = (D'.V(\rho_0 + \rho_2)) + a \cdot (D'.V(\rho_0 + \rho_4)) > 0,$$

because $a \geq 0$. Hence X is projective, a contradiction.

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