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Results Concerning the Nilpotency of the Separating Ideal of a Banach Algebra

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ABSTRACT. The main goal of the present paper is to investigate conditions for the separating ideal of a Banach algebra to be nilpotent.

1. Introduction.

Johnson and sinclair proved that every derivation on a semisimple Banach algebra is continuous. But the answers to the following questions seem to be open.

(1) Is the separating space of a derivation on a Banach algebra nilpotent?

(2) Are derivations continuous on a semiprime Banach algebra?

(3) Are derivations continuous on a prime Banach algebra?

It is straightforward to notice that the above questions are equivalent (see Proposition 2.3). Throught this paper we suppose that A is a complex Banach algebra. R and L will denote, respectively, the Jacobson and prime radicals of A. L is also equal to the intersection of all prime ideals of A. If A is commutative, then L consists of all nilpotent elements of A. A is said to be semiprime if it has no nonzero nil ideals. Recall that A is said to be a prime algebra if $\{0\}$ is a prime ideal. For any derivation D on a Banach algebra A, let $S(D) = \{x \in A : \text{there is } x_n \to 0 \text{ with } Dx_n \to x\}$ be the separating

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space of D, and by the closed graph theorem it follows that D is continuous on A if and only if $S(D) = \{0\}$[2]. An ideal I of A is said to be nil if each element of I is nilpotent. If I is an ideal of A, let I^n denote the ideal of A that is the linear span of n-fold products of elements of I. An ideal of A that is said to be nilpotent if $I^n = \{0\}$ for some positive integer n. It is known that every closed nil ideal is nilpotent[5].

2. Preliminaries.

PROPOSITION 2.1 ([3]). *Let D be a derivation on a ring Q.*

Then D fixes each minimal prime ideal P of Q such that Q/P is torsion-free.

PROPOSITION 2.2 ([1]). *Let D be a derivation on a Banach algebra A with the radical R. If $R \cap S(D)$ is nilpotent, then $S(D)$ is nilpotent.*

PROPOSITION 2.3 ([3]). *The following statements are equivalent:*

(1) *The separating space of a derivation on a Banach algebra is always nilpotent.*
(2) *Derivations are continuous on semiprime Banach algebras.*
(3) *Derivations are continuous on prime Banach algebras.*

PROOF. The only thing we have to show is that (3) implies (1). Let A be a Banach algebra and $D : A \to A$ a derivation with non-nilpotent separating space $S(D)$. Then there is a minimal prime ideal $P \subset A$ that does not contain $S(D)$ and it is closed. Since A/P is a prime Banach algebra and $D(P) \subset P$ by Proposition 2.1, we can define a continuous derivation $\overline{D} : A/P \to A/P$ by $\overline{D}(x + P) = D(x) + P$ $(x \in A)$. Therefore we have $S(D) \subset P$. This is a contradiction.

PROPOSITION 2.4 (MITTAG-LEFFLER THEOREM[6]). *Let A be a Banach algebra satisfying $\overline{A^2} = A$. Then $\bigcap_{n\geq 1} A^n$ is dense in A.*

3. Main Results

The following stability Lemma [7] for the separating space is crucial tool for the automatic continuity theory.

Let X and Y be a Banach spaces and let $\{T_n\}$ and $\{R_n\}$ be sequences of continuous linear operators on X and Y, respectively. If $\theta$is a linear operator from X to Y such that $\theta T_n - R_n\theta$ is continuous for all n, then there is an integer N such that for all $n \geq N$ $(R_1...R_N S(\theta))^- = (R_1...R_n S(\theta))^-$.

DEFINITION 3.1 ([1]). A closed ideal J of a Banach algebra A is a separating ideal if for every sequence $\{a_n\}$in A,there is a positive integer N such that for all $n \geq N$,

$$(Ja_n...a_1)^- = (Ja_N...a_1)^-.$$

By Stability Lemma we see that every derivation on a Banach algebra has a separating space which is a separating ideal [7].

THEOREM 3.2. *Let A be a Banach algebra with a separating ideal J and I a closed ideal of A. Then $I \cap J$ is nilpotent if and only if $\bigcap_{n\geq 1}(I^n \cap J)$ is a nilpotent ideal.*

PROOF. One implication is obvious. Suppose that $\bigcap_{n\geq 1}(I^n \cap J)$ is nilpotent. Let $x \in I \cap J$. Then there is a positive integer N such that $\overline{Jx^n} = \overline{Jx^N}$ for all$n \geq N$. By the Mittag-Leffler theorem[6] $\bigcap_{n\geq 1} Jx^n$ is dense in $\overline{Jx^N}$. Since $Jx^n \subset I^n$ and $Jx^n \subset J$ for each $n \geq 1$, we see that $\bigcap_{n\geq 1} Jx^n \subset \bigcap_{n\geq 1}(I^n \cap J)$. By the nilpotency of $\bigcap_{n\geq 1}(I^n \cap J)$ we have

$$\overline{(Jx^N)}^m = \overline{(\bigcap_{n\geq 1} Jx^n)}^m \subset \overline{(\bigcap_{n\geq 1}(I^n \cap J))}^m \subset \overline{(\bigcap_{n\geq 1}(I^n \cap J))^m} = \{0\}$$

for some positive integer m. So we have $x^{(N+1)m} = 0$. Hence x is nilpotent. Thus $I \cap J$ is nilpotent.

COROLLARY 3.3. *Let A be a semiprime Banach algebra with the radical R and D a derivation on A. Then D is continuous if and only if* $\bigcap_{n\geq 1}(R^n \cap S(D)) = \{0\}$.

PROOF. One way implication is clear. If $\bigcap_{n\geq 1}(R^n \cap S(D)) = \{0\}$, then, by Theorem 3.2, $R \cap S(D)$ is nilpotent and so $S(D)$ is nilpotent by Proposition 2.2. Since A is semiprime, $S(D) = \{0\}$ and D is continuous.

THEOREM 3.4. *Let D be a derivation on a Banach algebra A with a finite dimensional radical R. Then $S(D)$ is nilpotent.*

PROOF. Let $y \in S(D) \cap R$. Then there is a sequence $\{x_n\}$ in A with $x_n \to 0$ such that $D(x_n) \to y$. Then $x_n y \to 0$ in R and $D(x_n y) = D(x_n)y + x_n D(y)$. Thus we have $D(x_n y) \to y^2$, and since D is continuous on R, we get $y^2 = 0$. Hence $S(D) \cap R$ is nilpotent and so $S(D)$ is nilpotent by Proposition 2.2.

COROLLARY 3.5. *Derivations on a semiprime Banach algebra with a finite dimensional radical are continuous.*

THEOREM 3.6. *Let D be a derivation on a Banach algebra A in which every closed prime ideal has a finite codimension. Then $S(D)$ is nilpotent.*

PROOF. We suppose that $S(D)$ is non-nilpotent. Then there is a minimal prime ideal P such that P is closed and $S(D) \not\subset P$. Note that $D(P) \subset P$ by Proposition 2.1. Thus we can define a derivation $\overline{D} : A/P \to A/P$ by $\overline{D}(x+P) = D(x)+P$ $(x \in A)$. By the assumption $\dim(A/P) < \infty$. So $\overline{D}$ is continuous. Hence we have $S(D) \subset P$. This is a contradiction.

COROLLARY 3.7. *Derivations on a semiprime Banach algebra in which every closed prime ideal has a finite codimension are continuous.*

THEOREM 3.8. *Let A be a commutative Banach algebra and J a separating ideal of A. Then J is nilpotent if and only if $\bigcap_{n\geq 1} J^n$ is a nil ideal.*

PROOF. Let L be the prime radical of A. One implication is obvious, so suppose that $\bigcap_{n\geq 1} J^n$ is a nil ideal and J is not nilpotent. By Cusack's theorem[1, Theorem 2.5] there are closed prime ideals $P_1, P_2, ..., P_k$ that do not contain J such that $J \cap L = J \cap P_1 \cap P_2 \cap \cdots \cap P_k$. Since each P_i is closed, $J \cap L$ is closed. Let x be an element of J that is not nilpotent. Since J is a separating ideal of A, there is a positive integer N such that for each $n \geq N$, $\overline{Jx^n} = \overline{Jx^N}$. By the Mittag-Leffler theorem $\bigcap_{n\geq 1} Jx^n$ is dense in $\overline{Jx^N}$. Since $J \cap L$ is closed,

$$\overline{Jx^N} = \overline{\bigcap_{n\geq 1} Jx^n} \subset \overline{\bigcap_{n\geq 1} J^n} \subset J \cap L.$$

Therefore‘ we see that $\overline{Jx^N} \subset P_i$ for $i = 1, 2, ..., k$. But each P_i is a prime ideal. Hence $x \in P_i$ for $i = 1, 2, ..., k$. So $x \in J \cap L$. This implies that x is nilpotent, which is a contradiction.

COROLLARY 3.9. *Let A be a semiprime commutative Banach algebra. Then D is continuous if and only if $\bigcap_{n\geq 1}(S(D))^n = \{0\}$.*

COROLLARY 3.10. *Let A be a commutative Banach algebra with the radical R and D a derivation on A. If $\bigcap_{n\geq 1} R^n$ is a nil ideal, then $S(D)$ is nilpotent.*

PROOF. Let $\bigcap_{n\geq 1} R^n$ be a nil ideal. Then $\bigcap_{n\geq 1} (S(D))^n$ is also a nil ideal because $S(D) \subset R$[4]. Hence S(D) is nilpotent by Theorem 3.8.

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