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L^2-Multiple Stochastic Integral and Ito Formula

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1. Introduction.

Let $T \subset \mathbf{R}$ be an open set, $\mathcal{B}(T)$ be its Borel subsets and $\mathcal{B}_c(T)$ its relatively compact subsets. Let μ be a σ-finite measure on T such that $\mu(A) < \infty$ for any $A \in \mathcal{B}_c(T)$ and $Z = (Z_1, Z_2)$ be a $\mathbf{R}^n$-valued Gaussian noise, that is, a family of random vectors $Z(A) = (Z_1(A), Z_2(A)), A \in \mathcal{B}_c(T)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that, for $A_i \in \mathcal{B}_c(T), i = 1, \dots k, k \geq 1$, and $A_i \cap A_j = \emptyset, i \neq j$,

$$(i)\ \ Z(A_1), \dots, Z(A_k) \text{ are independent}$$
$$(ii)\ Z(A_1) + \cdots + Z(A_k) = Z(\cup_{i=1}^k A_i).$$

Let $R = (r_{ij}(t))$ be a 2×2 symmetric strictly positive definite matrix. We say that Gaussian random measure Z corresponds to μ and have covariance matrix R if

$$r_{ij}(t) = \frac{E[Z_i(dt)Z_j(dt)]}{\mu(dt)},\ i, j = 1, 2$$

Introduce the Hilbert space $L^2(T)$ of functions $f : T \to \mathbf{C}^2$ such that

$$\| f \| = [\int_T (R(t)f(t), f(t))\mu(dt)]^{1/2} < \infty.$$

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Since R is strictly positive definite, $L^2(T)$ is complete if we identify as usual functions which are equal a.e.-μ . It can be shown that n-tuple tensor product $(\otimes L^2(T))^n$ can be identified with the Hilbert space $L^2(T^n)$ consisting of all functions

$$f : T^n \to (\otimes \mathbf{C}^2)^n, \ f = (f_{j_1,\dots,j_n}), \ j_1,\dots,j_n = 1,2$$

with finite norm

(2)

$$\| f \|_n = \Big[\int_{T^n} \sum_{j^{(n)},j'^{(n)}=1,2} f_{j^{(n)}}(t)\overline{f_{j'^{(n)}}(t)} \prod_{k=1}^{n} r_{j_k j'_k}(t_k) d\mu^n(t)\Big]^{1/2} < \infty,$$

where $t = (t_1,\dots,t_n), j^{(n)} = j_1,\dots,j_n$ and $j'^{(n)} = j'_1,\dots,j'_n$. The summation in (2) is taken over all possible cases with $j_i = 1$ or 2 and $j'_k = 1$ or 2.

In the following contex the variable t will be used in both of single and multidimentional cases if no confusion aries. Symmetric tensor product $[\widehat{\otimes} L^2(T)]^n$ can be identified with the subspace $\widehat{L^2(T^n)} \subset L^2(T^n)$ consisting of symmetric functions which satisfies $f = \mathrm{sym} f$. Symmetric function is defined, for fixed $j^{(n)}$,

$$(\mathrm{sym} f)_{j^{(n)}}(t) = \frac{1}{n!}\sum_{\sigma} f_{j_{\sigma(1)}\cdots,j_{\sigma(n)}}(t_{\sigma(1)},\dots,t_{\sigma(n)})$$

and the sum is taken over all permutations σ on $\{1,\dots,n\}$.

2. Multiple Stochastic Integral.

Referring to [8], we state the following theorem.

THEOREM 2.1. *For any* $n \geq 1$ *and any* $f = (f_{j^{(n)}})_{j^{(n)}=1,2} \in L^2(T^n)$ *there exists a random variable* $I^n(f)$ *called multiple stochastic integral (m.s.i.) of* f *with respect to* Z*, denoted also by*

$$I^n(f) = \int_{T^n} \sum_{j^{(n)}=1,2} f_{j^{(n)}}(t) Z_{j_1}(dt_j) \cdots Z_{j_n}(dt_j),$$

with the following properties :

(i) $I^n(f) = I^n(symf) \in L^2(\Omega)$

(ii) $E[I^n(f)] = 0$

(iii) $E[I^n(f)\overline{I^k(g)}] = \delta_{nk}\frac{1}{n!}\langle symf, g\rangle$ *for any* $k \geq 1$ *and* $g \in L^2(T^k)$*, where* δ_{nk} *is the Kronecker delta.*

PROOF. Let $\Delta(N) = \{\Delta_i : i = 1, \ldots, N\}$ be a partition of T. $f \in L^2(T^n)$ is said to be adapted to $\Delta(N)$ if

(i) f is constant on subsets $\Delta \subset T^n$of the form

$$\Delta = \Delta_{i_1} \times \cdots \times \Delta_{i_n}, \Delta_{i_\lambda} \in \Delta(N), \lambda = 1, \ldots, n.$$

(ii) f vanishes on hyperdiagonals.

If f is adapted to $\Delta(N)$, then $f_{j(n)}^{\Delta_{i_1},\cdots,\Delta_{i_n}} = 0$ if $\Delta_{i_\lambda} = \Delta_{i_\gamma}$, for $\lambda \neq \gamma, \lambda, \gamma = 1, \ldots, n, j^{(n)} = 1, 2$, where $f_{j(n)}^{\Delta_{i_1},\cdots,\Delta_{i_n}}$ is the value $f_{j^{(n)}}$ takes on $\Delta_{i_1} \times \cdots \times \Delta_{i_n}$. We call f, which is adapted to some partition $\Delta(N)$ of T, simple and denote the set of all simple functions in $L^2(T^n)$ by $L^2_s(T^n)$. For simple function $f \in L^2(T^n)$, define

(3)
$$I^n(f) = \frac{1}{n!} \sum_{\Delta_{i_1},\cdots,\Delta_{i_n} \in \Delta(N)} \sum_{j^{(n)}=1,2} f_{j(n)}^{\Delta_{i_1},\cdots,\Delta_{i_n}} Z_{j_1}(\Delta_{i_1}) \cdots Z_{j_n}(\Delta_{i_n}).$$

The right hand side of (3) is well defined for N sufficiently large and does not depend on N. It is easy to show that (3) satisfies $(i) - (iii)$

above. For arbitrary $f \in L^2(T^n)$, set $I^n(f) = l.i.m.I^n(f_k)$, where $(f_k)_{k=1}^\infty$ is a sequence of simple functions convergent to f in $L^2(T^n)$. By (iii) such a limit exists and satisfies $(i) - (iii)$ as well.

For simplicity in manipulation of the indices appeared in subscripts of functions in $L^2(T^n)$, we use the following notations:

$$j_{\hat{k}} = j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_n,$$
$$t_{\hat{k}} = t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n,$$
$$j_{k=i} = j_1, \ldots, j_{k-1}, i, j_{k+1}, \ldots, j_n.$$

We introduce two functions generated from $f(t) = (f_{j^{(n)}}(t))_{j^{(n)}=1,2} \in L^2(T^n)$ and $g(t) = (g_1(t), g_2(t)) \in L^2(T)$. For each $j_{\hat{k}} = 1,2$ and $j_1, \ldots, j_{n+1} = 1,2$, define

$$(4)\quad (f \otimes g)_{j_1,\ldots,j_{n+1}}(t_1, \ldots, t_{n+1}) = f_{j_1,\ldots,j_n}(t_1, \ldots, t_n) g_{j_{n+1}}(t_{n+1}),$$

and

$$(5)\qquad (f \underset{(k)}{\times} g)_{j_{\hat{k}}}(t_{\hat{k}}) = \int_T \sum_{\alpha,\beta=1,2} f_{j_k=\alpha}(t)\overline{g_\beta(t_k)} r_{\alpha\beta}(t_k) d\mu(t_k).$$

LEMMA 2.1. *If $f \in L^2(T^n)$ and $g \in L^2(T)$, then*

(i) $f \otimes g \in L^2(T^{n+1})$

(ii) $f \underset{(k)}{\times} g \in L^2(T^{n-1})$, *for* $k = 1, \ldots, n$,

also their norms satisfy $\| f \otimes g \|_{n+1} = \| f \|_n \cdot \| g \|_1$ *and* $\| f \underset{(k)}{\times} g \|_{n-1} \leq \| f \|_n \cdot \| g \|_1$.

PROOF. It is enough to show the latter parts on the norm. (i) is trivial by definiton. If $f_p \in L^2(T^n), g_q \in L^2(T)$, then we may show the following inequality without any difficulty

$$(6)\qquad \| f_p \underset{(k)}{\times} g_q \|_{n-1} \leq \| f_p \|_n \cdot \| g_q \|_1, \text{ for all } p, q.$$

Let $f \in L^2(T^n), g \in L^2(T)$, then there are sequences $\{f_p\}_p \in L^2_s(T^n)$ and $\{g_p\}_p \in L^2_s(T)$ such that $f_p \to f$ in $L^2(T^n)$ and $g_p \to g$ in $L^2(T)$ as $p \to \infty$. We can show that, for each q, $\{f_p \underset{(k)}{\times} g_q\}_p$ is a Cauchy sequence in $L^2(T^{n-1})$. Therefore it has a limit, $\lim_{p\to\infty} f_p \underset{(k)}{\times} g_q$ in $L^2(T^{n-1})$. But it is also a Cauchy sequence, with respect to q, in $L^2(T^{n-1})$. This implies that $f \underset{(k)}{\times} g_q \to f \underset{(k)}{\times} g$ as $q \to \infty$.

Since $\| f_p \underset{(k)}{\times} g_q \|_{n-1} \to \| f \underset{(k)}{\times} g \|_{n-1}$, $\| f_p \|_n \to \| f \|_n$ and $\| g_q \|_1 \to \| g \|_1$, we have,by (6), $\| f \underset{(k)}{\times} g \|_{n-1} \leq \| f \|_n \cdot \| g \|_1$.

LEMMA 2.2. *If* $f \in L^2(T^n), g \in L^2(T)$, *then*

$$nI^n(f)I^1(g) = n(n+1)I^{n+1}(f \otimes g) + \sum_{k=1}^{n} I^{n-1}(f \underset{(k)}{\times} g).$$

PROOF. We consider first the case when f and g are simple functions. Without loss of generality we may assume that f and g are adapted to the same partition $\Delta(N)$.

Suppose they have the following form, for fixed $j^{(n)}$ and j,

$$f_{j^{(n)}}(t^{(n)}) = \begin{cases} f_{j^{(n)}}^{\Delta_{\mathfrak{i}^{(n)}}}, & \text{if } t^{(n)} \in d(\Delta_{\mathfrak{i}^{(n)}}) \\ 0, & \text{othrewise,} \end{cases}$$

$$g_j(t) = \begin{cases} g_j^{\Delta_i}, & \text{if } t \in \Delta_i \\ 0, & \text{otherwise,} \end{cases}$$

where $d(\Delta_{\mathfrak{i}^{(n)}}) = \Delta_{i_1} \times \cdots \times \Delta_{i_n}$ and $\Delta_{i_1}, \ldots, \Delta_{i_n} \in \Delta(N)$ are disjoint. Denote $D^n = D(\Delta(N))$ be the set of all possible sets of type $d(\Delta_{\mathfrak{i}^{(n)}})$.

Let

$$A = \max_{j^{(n)}\mathfrak{i}^{(n)}} |f_{j^{(n)}}^{\Delta_{\mathfrak{i}^{(n)}}}| \quad \text{and} \quad B = \max_{j,i} |g_j^{\Delta_i}|.$$

Then $A < \infty$ and $B < \infty$. We may assume that, for $\alpha, \beta = 1, 2$,

$$|r_{\alpha\beta}(\Delta_i)\mu(\Delta_i)| < \varepsilon, \quad i = 1, \ldots N, \tag{7}$$

for an arbitray small $\varepsilon > 0$, by subdividing each Δ_i, if necessary. In (7), $r_{\alpha\beta}(\Delta_i)$ denotes $r_{\alpha\beta}(t_i) - r_{\alpha\beta}(t_{i-1})$ or $r_{\alpha\beta}(t_i - t_{i-1})$, if $\Delta_i = (t_{i-1}, t_k)$. A and B remain invariant by this subdivision.

Now we define a simple function $\mathcal{X}_\varepsilon$ by

$$(\mathcal{X}_\varepsilon)_{\substack{j^{(n+1)} \\ j_{n+1}=j}}(t^{(n)}, t) = \begin{cases} f_{j^{(n)}}^{\Delta_{i^{(n)}}} \cdot g_j^{\Delta_i}, & \text{if } (t^{(n)}, t) \in d(\Delta_{i^{(n)}} \times \Delta_i) \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned}
&I^n(f)I^1(g) \\
&= [\frac{1}{n!}\int_{T^n} \sum_{j^{(n)}=1,2} f_{j^{(n)}}(t^{(n)}) dZ_{j_1}(t_1) \cdots dZ_{j_n}(t_n)] [\sum_{j=1,2} \int_T g_j(t) dZ_j(t)] \\
&= \frac{1}{n!}[\sum_{j^{(n)}=1,2} \sum_{d(\Delta_{i^{(n)}}) \in D^n} f_{j^{(n)}}^{\Delta_{i^{(n)}}} \prod_{\lambda=1}^{n} Z_{j_\lambda}(\Delta_{i_\lambda})] [\sum_{j=1,2} \int_T g_j(t) dZ_j(t)] \\
&= \frac{1}{n!}[\sum_{j^{(n)}, j=1,2} \sum_{d(\Delta_{i^{(n)}} \times \Delta_i) \in D^{n+1}} f_{j^{(n)}}^{\Delta_{i^{(n)}}} g_j^{\Delta_i} \prod_{\lambda=1}^{n} Z_{j_\lambda}(\Delta_{i_\lambda}) Z_j(\Delta_i)] \\
&+ \frac{1}{n!}[\sum_{j^{(n)}, j=1,2} \sum_{k=1}^{n} \sum_{d(\Delta_{i^{(n)}}) \in D^n} f_{j^{(n)}}^{\Delta_{i^{(n)}}} g_j^{\Delta_{i_k}} \prod_{\lambda=1}^{n} Z_{j_\lambda}(\Delta_{i_\lambda}) Z_j(\Delta_{i_k})].
\end{aligned}$$

Let

$$\xi_1 = \sum_{j^{(n)}, j=1,2} \sum_{d(\Delta_{i^{(n)}} \times \Delta_i) \in D^{n+1}} f_{j^{(n)}}^{\Delta_{i^{(n)}}} g_j^{\Delta_i} \prod_{\lambda=1}^{n} Z_{j_\lambda}(\Delta_{i_\lambda}) Z_j(\Delta_i)$$

$$\xi_2 = \sum_{j^{(n)}, j=1,2} \sum_{k=1}^{n} \sum_{d(\Delta_{i^{(n)}}) \in D^n} f_{j^{(n)}}^{\Delta_{i^{(n)}}} g_j^{\Delta_{i_k}} Z_{j_1}(\Delta_{i_1}) \cdots Z_{j_n}(\Delta_{i_n}) Z_j(\Delta_{i_k}).$$

By definition of $\mathcal{X}_\varepsilon$, $\xi_1 = (n+1)! \cdot I^{n+1}(\mathcal{X}_\varepsilon)$. We will show that $\xi_2 = \sum_{k=1}^n (n-1)! \cdot I^{n-1}(f \underset{(k)}{\times} g) + \sum_{k=1}^n R_k$, where

$$R_k = \sum_{\substack{j^{(n)},\\ j=1,2}} \sum_{d(\Delta_{\mathbf{i}(n)}) \in D^n} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_j^{\Delta_{i_k}} Z_{j_1}(\Delta_{i_1}) \cdots Z_{j_{k-1}}(\Delta_{i_{k-1}}) Z_{j_{k+1}}(\Delta_{i_{k+1}}) \cdots Z_{j_n}(\Delta_{i_n}) Z_j(\Delta_{i_k}) \cdot [Z_{j_k}(\Delta_{i_k}) Z_j(\Delta_{i_k}) - r_{j_k j}(\Delta_{i_k}) \mu(\Delta_{i_k})]$$

Therefore it follows that from the argument above, by substituting j by j_{n+1},

$$I^n(f) I^1(g) = \frac{1}{n!} \; [\, (n+1)! \;\; I^{n+1}(\mathcal{X}_\varepsilon) + \xi_2 \,].$$

We compute ξ_2:

$$\begin{aligned}
\xi_2 &= \sum_{\substack{j^{(n)},j\\ =1,2}} \sum_{k=1}^n \sum_{d(\Delta_{\mathbf{i}(n)}) \in D^n} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_j^{\Delta_{i_k}} \prod_{\lambda=1}^n Z_{j_\lambda}(\Delta_{i_\lambda}) Z_j(\Delta_{i_k}) \\
&= \sum_{\substack{j^{(n)},j\\ =1,2}} \sum_{k=1}^n \Bigg\{ \sum_{d(\Delta_{\mathbf{i}(n)}) \in D^n} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_j^{\Delta_{i_k}} \prod_{\substack{\lambda=1,\\ \lambda \neq k}}^n Z_{j_\lambda}(\Delta_{i_\lambda}) r_{j_k j}(\Delta_{i_k}) \mu(\Delta_{i_k}) \\
&\quad + \sum_{d(\Delta_{\mathbf{i}(n)}) \in D^n} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_j^{\Delta_{i_k}} \cdot \\
&\quad \prod_{\substack{\lambda=1,\\ \lambda \neq k}}^n Z_{j_\lambda}(\Delta_{i_\lambda}) [Z_{j_k}(\Delta_{i_k}) Z_j(\Delta_{i_k}) - r_{j_k j}(\Delta_{i_k}) \mu(\Delta_{i_k})] \Bigg\} \\
&= \sum_{\substack{j^{(n)},j\\ =1,2}} \sum_{k=1}^n \Bigg\{ \sum_{d(\Delta_{\mathbf{i}(n)}) \in D^n} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_j^{\Delta_{i_k}} \prod_{\substack{\lambda=1\\ \lambda \neq k}}^n Z_{j_\lambda}(\Delta_{i_\lambda}) r_{j_k j}(\mu)(\Delta_{i_k}) \Bigg\} \\
&\quad + \sum_{k=1}^n R_k
\end{aligned}$$

$$= \sum_{k=1}^{n} (n-1)! \; I^{n-1}(f \underset{(k)}{\times} g) \; + \; \sum_{k=1}^{n} R_k.$$

The last equality holds by the definition (5). Therefore we have

$$n\, I^n(f)\, I^1(g) = \; n(n+1)\, I^{n+1}(\mathcal{X}_\varepsilon) \\ + \; [\sum_{k=1}^{n} \; I^{n-1}(f \underset{(k)}{\times} g) \; + \; \frac{1}{n!} R_k\,].$$

We can show the following :

$$\text{(8)} \qquad \| \, I^{n+1}(\mathcal{X}_\varepsilon) - I^{n+1}(f \otimes g) \, \|^2_{L^2} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

and

$$\text{(9)} \qquad \| \, \frac{1}{n!} R_k \, \|^2_{L^2} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

The norm in (8) and (9) is the one on the space $L^2(\Omega)$. By (8) and (9) we have

$$\text{(10)} \quad n\, I^n(f)\, I^1(g) = \; n(n+1)\, I^{n+1}(f \otimes g) \; + \; \sum_{k=1}^{n} \; I^{n-1}(f \underset{(k)}{\times} g).$$

Proof of (8). Let $\Phi_\varepsilon = \mathcal{X}_\varepsilon - f \otimes g$, that is, for fixed $j^{(n)}, j$,

$$(\Phi_\varepsilon)_{j^{(n)}j}(t^{(n+1)}) = (\mathcal{X}_\varepsilon)_{j^{(n)}j}(t^{(n+1)}) - (f \otimes g)_{j^{(n)}j}(t^{(n+1)}) \\ = \begin{cases} f_{j^{(n)}}^{\Delta_{i^{(n)}}} g_j^{\Delta_{i_\ell}}, & (t^{(n)}, t_{n+1}) \in d(\Delta_{i^{(n)}}) \times \Delta_{i_\ell} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
&I^{n+1}(\Phi_\varepsilon) \\
&= \frac{1}{(n+1)!}\int_{T^{n+1}} \sum_{j^{(n+1)}=1,2} (\Phi_\varepsilon)_{j^{(n)}j}(t^{(n+1)}) \prod_{k=1}^{n+1} Z_{j_k}(dt_k) \\
&= \frac{1}{(n+1)!} \sum_{d(\Delta_{\mathbf{i}(n)})\in D^n} \sum_{\ell=1}^{n} \sum_{j^{(n+1)}=1,2} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_{j_{n+1}}^{\Delta_{i_\ell}} \cdot \\
&\qquad Z_{j_1}(\Delta_{i_1})\cdots Z_{j_\ell}(\Delta_{i_\ell})\cdots Z_{j_n}(\Delta_{i_n})Z_{j_{n+1}}(\Delta_{i_\ell}).
\end{aligned}$$

Note that

$$\begin{aligned}
&E\big[Z_{j_1}(\Delta_{i_1})\cdots Z_{j_\ell}(\Delta_{i_\ell})\cdots Z_{j_n}(\Delta_{i_n})Z_{j_{n+1}}(\Delta_{i_\ell}) \\
&\qquad \cdot \overline{Z_{j'_1}(\Delta_{i'_1})\cdots Z_{j'_{\ell'}}(\Delta_{i'_{\ell'}})\cdots Z_{j'_n}(\Delta_{i'_n})Z_{j'_{n+1}}(\Delta_{i'_{\ell'}})}\big] = \\
&\begin{cases}
[\prod_{\alpha\neq\ell} r_{j_\alpha j'_{m(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})] \\
\qquad \cdot \big[r_{j_\ell j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell}) r_{j_{n+1}j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell}) \\
\qquad + r_{j_\ell j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell}) r_{j_{n+1}j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})\big], \\
\qquad\qquad \text{if } \{i_1,\dots,i_{n+1}\} = \{i'_1,\dots,i'_{n+1}\} \\
0, \qquad\qquad \text{otherwise},
\end{cases}
\end{aligned} \tag{11}$$

where $i_\alpha = i'_{m(\alpha)}$ and hence $\Delta_{i_\alpha} = \Delta_{i'_{m(\alpha)}}$. In other words m is a permutation on $\{1,2,\dots,n+1\}$ such that $\ell' = m(\ell)$. By the relation (11) above

$$\begin{aligned}
&\| I^{n+1}(\Phi_\varepsilon) \|_{L^2}^2 = E|I^{n+1}(\Phi_\varepsilon)|^2 = \frac{1}{((n+1)!)^2} \cdot \\
&E\Big[\Big\{ \sum_{d(\Delta_{\mathbf{i}(n)})\in D^n} \sum_{\ell=1}^{n} \sum_{j^{(n+1)}=1,2} f_{j^{(n)}}^{\Delta_{\mathbf{i}(n)}} g_{j_{n+1}}^{\Delta_{i_\ell}} \Big(\prod_{k=1}^{n} Z_{j_k}(\Delta_{i_k})\Big) Z_{j_{n+1}}(\Delta_{i_\ell}) \Big\}
\end{aligned}$$

$$\cdot\Big\{\sum_{i'(n)}\sum_{\ell=1'}^{n}\sum_{j'(n+1)=1,2}\overline{f_{j'(n)}^{\Delta_{i'(n)}}g_{j'_{n+1}}^{\Delta_{i'_{\ell'}}}}$$

$$\overline{Z_{j'_1}(\Delta_{i'_1})\cdots Z_{j'_{\ell'}}(\Delta_{i'_{\ell'}})\cdots Z_{j'_n}(\Delta_{i'_n})Z_{j'_{n+1}}(\Delta_{i'_{\ell'}})}\Big\}\Big]$$

$$= \frac{1}{((n+1)!)^2}\sum_{\substack{\{i_1,\cdots,i_n\}\\=\{i'_1,\cdots,i'_n\}}}\sum_{\ell,\ell'=1}^{n}\sum_{\substack{j^{(n+1)},\\ j'^{(n+1)}=1,2}} f_{j(n)}^{\Delta_{i(n)}}g_{j_{n+1}}^{\Delta_{i_\ell}}\overline{f_{j'(n)}^{\Delta_{i'(n)}}g_{j'_{n+1}}^{\Delta_{i'_{\ell'}}}}$$

$$\cdot\Big[\prod_{\alpha\neq\ell}r_{j_\alpha j'_{m(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})\Big]\cdot\Big[r_{j_\ell j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})r_{j_{n+1}j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})$$

$$+ r_{j_\ell j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})r_{j_{n+1}j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})\Big].$$

Therefore

$$E|I^{n+1}(\Phi_\varepsilon)|^2 \leq \frac{1}{[(n+1)!]^2}\cdot$$

$$\sum_{\substack{\{i_1,\cdots,i_n\}\\=\{i'_1,\cdots,i'_n\}}}\sum_{\ell,\ell'=1}^{n}\sum_{\substack{j^{(n+1)},\\ j'^{(n+1)}=1,2}}\Big[\max_{j(n)}|f_{j^{(n)}}|^2\Big]\Big[\max_{j_{n+1}}|g_{j_{n+1}}|^2\Big]$$

$$\cdot\Big[\prod_{\alpha\neq\ell}|r_{j_\alpha j'_{k(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})|\Big]\cdot\Big[|r_{j_\ell j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})||r_{j_{n+1}j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})|$$

$$+ |r_{j_\ell j'_{n+1}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})||r_{j_{n+1}j'_{\ell'}}(\Delta_{i_\ell})\mu(\Delta_{i_\ell})|\Big]$$

$$\leq n!\cdot\frac{1}{[(n+1)!]^2}\cdot n^2\cdot 2^nA^2B^2\cdot\varepsilon(\varepsilon^2+\varepsilon^2)$$

$$\leq M\varepsilon A^2B^2,$$

because $n!\cdot n^2\cdot 2^n/((n+1)!)^2 \leq 1$ for all n. M is a finite positive number. Since the last term converges to 0 as ε approaches 0, it completes the proof of (8). Now we present the proof of (9) :

Proof of (9).

$$\|\frac{1}{n!}R_k\|^2 = \frac{1}{(n!)^2}E|R_k|^2$$

$$= E\Big[\sum_{j^{(n+1)}=1,2} \sum_{d(\Delta_{i(n)})\in D^n} f_{j(n)}^{\Delta_{i(n)}} g_{j_{n+1}}^{\Delta_{i_k}} \prod_{m\neq k}^{n} Z_{j_m}(\Delta_{i_m})$$

$$\cdot \big\{ Z_{j_k}(\Delta_{i_k}) Z_{j_{n+1}}(\Delta_{i_k}) - r_{j_k j_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k})\big\}\Big]$$

$$\cdot \Big[\sum_{j'^{(n+1)}=1,2} \sum_{d(\Delta_{i(n)})\in D^n} \overline{f_{j'(n)}^{\Delta_{i'(n)}} g_{j'_{n+1}}^{\Delta_{i'_k}}} \prod_{m\neq k}^{n} Z_{j'_m}(\Delta_{i'_k})$$

$$\cdot \overline{\big\{ Z_{j'_k}(\Delta_{i'_k}) Z_{j'_{n+1}}(\Delta_{i'_k}) - r_{j'_k j'_{n+1}}(\Delta_{i'_k})\mu(\Delta_{i'_k})\big\}}\Big]$$

$$= \frac{1}{(n!)^2} \sum_{\substack{\{i_1,\cdots,i_n\} \\ =\{i'_1,\cdots,i'_n\}}} \sum_{\substack{j^{(n+1)}, \\ j'^{(n+1)}=1,2}} f_{j(n)}^{\Delta_{i(n)}} g_{j_{n+1}}^{\Delta_{i_k}} \overline{f_{j'(n)}^{\Delta_{i'(n)}} g_{j'_{n+1}}^{\Delta_{i'_k}}}$$

$$\cdot \Big[\prod_{\alpha\neq \ell} r_{j_\alpha j'_{m(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})\Big] \cdot \big[r_{j_k j'_k}(\Delta_{i_k})\mu(\Delta_{i_k}) r_{j_{n+1} j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k})$$

$$+ r_{j_k j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k}) r_{j_{n+1} j'_k}(\Delta_{i_k})\mu(\Delta_{i_k})\big]$$

$$+ \frac{1}{(n!)^2} \sum_{\substack{\{i_1,\cdots,i_n\} \\ =\{i'_1,\cdots,i'_n\}}} \sum_{\substack{j^{(n+1)} \\ j'^{(n+1)}=1,2}} f_{j(n)}^{\Delta_{i(n)}} g_{j_{n+1}}^{\Delta_{i_k}} \overline{f_{j'(n)}^{\Delta_{i'(n)}} g_{j'_{n+1}}^{\Delta_{i'_k}}}$$

$$\cdot r_{j_k j_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k}) \overline{r_{j'_k j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k})} \cdot \Big[\prod_{\alpha\neq \ell} r_{j_\alpha j'_{m(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})\Big]$$

$$= \frac{1}{(n!)^2} \sum_{\substack{\{i_1,\cdots,i_n\} \\ =\{i'_1,\cdots,i'_n\}}} \sum_{\substack{j^{(n+1)}, \\ j'^{(n+1)}=1,2}} f_{j(n)}^{\Delta_{i(n)}} g_{j_{n+1}}^{\Delta_{i_k}} \overline{f_{j'(n)}^{\Delta_{i'(n)}} g_{j'_{n+1}}^{\Delta_{i'_k}}}$$

$$\cdot \Big[\prod_{\alpha\neq \ell} r_{j_\alpha j'_{m(\alpha)}}(\Delta_{i_\alpha})\mu(\Delta_{i_\alpha})\Big] \cdot \big[r_{j_k j'_k}(\Delta_{i_k})\mu(\Delta_{i_k}) r_{j_{n+1} j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k})$$

$$+ r_{j_k j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k}) r_{j_{n+1} j'_k}(\Delta_{i_k})\mu(\Delta_{i_k})$$

$$+ r_{j_k j_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k}) \overline{r_{j'_k j'_{n+1}}(\Delta_{i_k})\mu(\Delta_{i_k})}\big].$$

Note that we have used the following property in the argument above

$$E[Z_{j_k}(\Delta_{i_k}) Z_{j'_k}(\Delta_{i_k})] = 0.$$

Therefore we have

$$\begin{aligned}\left\| \frac{1}{n!} R_k \right\|^2 &\leq \frac{1}{(n!)^2} \cdot n! \cdot 2^n \cdot A^2 \cdot B^2 \cdot \varepsilon(3\varepsilon^2) \\ &< M\varepsilon \cdot A^2 \cdot B^2.\end{aligned}$$

Since the last term converges to 0 as ε approaches, 0, it completes the proof of Lemma 2.2 for simple function case. Now assume f is an arbitrary function in $L^2(T^n)$ and g is an arbitrary function in $L^2(T)$. Then we can show that Lemma 2.2 holds true by the following arguments. Let $f \in L^2(T^n)$. Then, by Theorem 1.1,

$$\begin{aligned}\| I^n(f) \|^2 &= E|I^n(f)|^2 \\ &= \frac{1}{n!} \langle symf, f \rangle \\ &\leq \frac{1}{n!} |\langle symf, f \rangle| \\ &\leq \frac{1}{n!} \| f \|_n^2 .\end{aligned}$$

Let $\{f_p\}_{p \geq 1}$ and $\{g_p\}_{p \geq 1}$ be sequences in $L_s^2(T^n)$ and $L_s^2(T)$ such that $f_p \to f$ and $g_p \to g$, respectively. Since

$$n(n+1) I^{n+1}(f_p \otimes g_p) = n I^n(f_p) I^1(g_p) - \sum_{k=1}^{n} I^{n-1}(f_p \underset{(k)}{\times} g_p),$$

we can make use of the following properties to complete the proof of Lemma 2.2

$$\| f \otimes g \|_{n+1} = \| f \|_n \cdot \| g \|_1,$$

$$\| f \underset{(k)}{\times} g \|_{n-1} \leq \| f \|_n \cdot \| g \|_1,$$

$$\| I^n(f) \| \leq \frac{1}{n!} \| f \|_n . \tag{12}$$

By the relation (12), we obtain

$$
\begin{aligned}
&\| I^{n+1}(f_p \otimes g_p) - I^{n+1}(f \otimes g) \|_{L^1} \\
&\quad = \| I^{n+1}((f_p \otimes g_p - f \otimes g)) \|_{L^1} \\
&\quad \leq \| I^{n+1}((f_p \otimes g_p - f \otimes g)) \|_{L^2} \\
&\quad \leq \frac{1}{\sqrt{n!}} \| f_p \otimes g_p - f \otimes g \|_{n+1} \\
&\quad \leq \frac{1}{\sqrt{n!}} \| f_p \otimes [g_p - g] + [f_p - f] \otimes g \|_{n+1} \\
&\quad \leq \frac{1}{\sqrt{n!}} \| f_p \|_n \| g_p - g \|_1 + \frac{1}{\sqrt{n!}} \| f_p - f \|_n \| g \|_1,
\end{aligned}
$$

$$
\begin{aligned}
&\| I^n(f_p)I^1(g_p) - I^n(f)I^1(g) \|_{L^1} \\
&\quad = \| I^n(f_p)I^1(g_p - g) + I^n(f_p - f)I^1(g) \|_{L^1} \\
&\quad \leq \| I^n(f_p)I^1(g_p - g) \|_{L^1} + \| I^n(f_p - f)I^1(g) \|_{L^1} \\
&\quad \leq \| I^n(f_p) \|_{L^2} \| I^1(g_p - g) \|_{L^2} + \| I^n(f_p - f) \|_{L^2} \| I^1(g) \|_{L^2} \\
&\quad \leq \frac{1}{\sqrt{n!}} \| f_p \|_n \| g_p - g \|_1 + \frac{1}{\sqrt{n!}} \| f_p - f \|_n \| g \|_1
\end{aligned}
$$

and

$$
\begin{aligned}
&\| I^{n-1}(f_p \underset{(k)}{\times} g_p) - I^{n-1}(f \underset{(k)}{\times} g) \|_{L^1} \\
&\quad \leq \| I^{n-1}(f_p \underset{(k)}{\times} g_p - f \underset{(k)}{\times} g) \|_{L^2} \\
&\quad \leq \frac{1}{\sqrt{(n-1)!}} \| f_p \underset{(k)}{\times} g_p - f \underset{(k)}{\times} g \|_{n-1} \\
&\quad = \frac{1}{\sqrt{(n-1)!}} \| (f_p - f) \underset{(k)}{\times} g_p + f \underset{(k)}{\times} (g_p - g) \|_{n-1} \\
&\quad \leq \frac{1}{\sqrt{(n-1)!}} \| (f_p - f) \underset{(k)}{\times} g_p \|_{n-1} + \frac{1}{\sqrt{(n-1)!}} \| f \underset{(k)}{\times} (g_p - g) \|_{n-1} \\
&\quad \leq \frac{1}{\sqrt{(n-1)!}} \| f_p - f \|_n \| g_p \|_1 + \frac{1}{\sqrt{(n-1)!}} \| f \|_n \| g_p - g \|_1 .
\end{aligned}
$$

By letting p tend to ∞ we can complete the proof of Lemma 1.2.

3. Ito's Formula.

The following result is a well-known recursion formula for Hermite polynomials. We state it as a lemma.

LEMMA 3.1. *Let* $H_n(x) = (-1)^n exp(\frac{x^2}{2})\frac{d^n}{dx^n}[exp(-\frac{x^2}{2})]$ *be a Hermite polynomial of leading coefficitnt 1. Then*

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x),$$

for $n = 1, 2, \ldots$

Consider the following problem: For given $n_1, n_2, \ldots, n_m$ (positive integers), how can we express the product of Hermite polynomials of multiple stochastic integrals of orthogonal functions in $L^2(T)$?

DEFINITION 3.1. $f^{(1)} = (f_1^{(1)}, f_2^{(1)}), \cdots, f^{(m)} = (f_1^{(m)}, f_2^{(m)}) \in L^2(T)$ are said to be orthonormal if, for any $i \neq j$, $\langle f^{(i)}, f^{(j)} \rangle = 0$, that is,

$$\int_T \sum_{\alpha,\beta=1,2} f_\alpha^{(i)}(t)\overline{f_\beta^{(j)}(t)} r_{\alpha\beta}(\mu)(dt) = 0$$

and

$$\| f^{(i)} \|^2 = \langle f^{(i)}, f^{(i)} \rangle = 1.$$

THEOREM 3.1. *Suppose* $f^{(1)}, \cdots, f^{(m)} \in L^2(T)$ *are orthonormal in* $\in L^2(T)$. *Then*

$$\begin{aligned} & H_{n_1}\left[I^1(f^{(1)})\right] \cdots H_{n_m}\left[I^1(f^{(m)})\right] \\ (13) \quad & = (n_1 + \cdots + n_m)! \cdot I^{(n_1+\cdots+n_m)}\left[(\otimes f^{(1)})^{n_1} \otimes \cdots \otimes (\otimes f^{(m)})^{n_m}\right]. \end{aligned}$$

PROOF. Let $N = n_1 + n_2 + \cdots + n_m$. We prove (13) by induction on N. It is true for $N = 1$. Assume it is true for $N - 1$. Let

$$\begin{aligned} &g^{(i)} = f^{(s)} \quad for \quad n_1 + \cdots + n_{s-1} < i \leq n_1 + \cdots n_s, \\ &f(t_1, \ldots, t_{N-1}) = g^{(1)}(t_1) \otimes \cdots \otimes g^{(N-1)}(t_{N-1}), \\ &h(t) = g^{(N)}(t). \end{aligned}$$

Then

$$\mathcal{D} = (N-1)N \cdot I^N(f \otimes h) = (N-1) \cdot I^{N-1}(f) I^1(h) - \sum_{k=1}^{N-1} I^{N-2}(f \underset{(k)}{\times} h),$$

by Lemma 2.2. By induction we have

$$\begin{aligned} \mathcal{D} = \frac{N-1}{(N-1)!} \cdot H_{n_1}\left[I^1(f^{(1)})\right] \cdots H_{n_{m-1}}\left[I^1(f^{(m-1)})\right] \\ \cdot H_{n_m - 1}\left[I^1(f^{(m)})\right] \cdot \left[I^1(h)\right] \\ - (n_m - 1) \cdot \frac{1}{(N-2)!} H_{n_1}[I^1(f^{(1)})] \cdots H_{n_m - 2}[I^1(f^{(m)})], \end{aligned}$$

because

$$\begin{aligned} (f \underset{(k)}{\times} h)_{j_k^{(n)}}(t_k^{(n)}) &= \int_T \sum_{i,j=1,2} f_{j^{(n)}_{(k=i)}}(t^{(n)}) \overline{h_j(t_k)} r_{ij}(t_k) d\mu(t_k) \\ &= \begin{cases} 0, & \text{if } k \leq N - n_m \\ g_{j_1}^{(1)}(t_1) \cdots \widehat{g_{j_k}^{(k)}(t_k)} \cdots g_{j_{N-1}}^{(N-1)}(t_{N-1}), & N - n_m < k \leq N - 1 \end{cases} \end{aligned}$$

and

$$\sum_{k=1}^{N-1} I^{N-2}(f \underset{(k)}{\times} h) = \sum_{k=N-n_m+1}^{N-1} I^{N-2}(f \underset{(k)}{\times} h)$$

$$= \sum_{k=N-n_m+1}^{N-1} I^{N-2}(g^{(1)}(t_1) \otimes \cdots \otimes \widehat{g^{(k)}} \otimes \cdots \otimes g^{N-1}(t_{N-1}))$$

$$= \sum_{k=N-n_m+1}^{N-1} I^{N-2}((\otimes f^{(1)})^{n_1} \otimes \cdots \otimes (\otimes f^{(m-1)})^{n_{m-1}} \otimes (\otimes f^{(m)})^{n_m-2})$$

$$= (n_m - 1) \cdot I^{N-2}[(\otimes f^{(1)})^{n_1} \otimes \cdots \otimes (\otimes f^{(m-1)})^{n_{m-1}} \otimes (\otimes f^{(m)})^{n_m-2}]$$

$$= \frac{n_m - 1}{(N-2)!} H_{n_1}[I^1(f^{(1)})] \cdots H_{n_{m-1}}[I^1(f^{(m-1)})] H_{n_m-2}[I^1(f^{(m)})].$$

Therefore

$$\mathcal{D} = \frac{1}{(N-2)!} H_{n_1}[I^1(f^{(1)})] \cdots H_{n_{m-1}}[I^1(f^{(m-1)})]$$

$$\cdot \{H_{n_m-1}[I^1(f^{(m)})] \cdot I^1(f^{(m)}) - (n_m - 1) H_{n_m-2}[I^1(f^{(m)})]\}$$

$$= \frac{1}{(N-2)!} (\prod_{s=1}^{m-1} H_{n_s}[I^1(f^{(s)})]) \cdot \{H_{n_m}[I^1(f^{(m)})]\}$$

$$= \frac{1}{(N-2)!} \prod_{s=1}^{m} H_{n_s}[I^1(f^{(s)})].$$

The second equality holds by Lemma 3.1 and this implies that

$$\prod_{s=1}^{m} H_{n_s}[I^1(f^{(s)})] = (N-2)! \cdot \mathcal{D}$$

$$= (N-2)! \cdot (N-1) N \, I^N(f \otimes h)$$

$$= N! \, I^N \, ((\otimes f^{(1)})^{n_1} \otimes \cdots \otimes (\otimes f^{(m)})^{n_m}).$$

This completes the proof of Theorem 3.1.

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