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Continuity of Derivations on Banach Algebras

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ABSTRACT. In this paper, we show that the module derivation D is continuous on the Banach algebra and the Silov algebra, and also that the derivation restricted by separating space and the radical on the semiprime Banach algebra is continuous.

Let $\mathfrak{A}$ be a Banach algebra. A linear map $D : \mathfrak{A} \to \mathfrak{A}$ is a *derivation* if $D(xy) = xD(y) + D(x)y, \ x, y \in \mathfrak{A}$.

Let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module, then the derivation $D : \mathfrak{A} \to \mathfrak{M}$ is called the module derivation.

If D is any linear operator from $\mathfrak{A}$ to the $\mathfrak{A}$-module $\mathfrak{M}$, then

$$\mathfrak{S}(D) = \{m \in \mathfrak{M} : \text{ there exist } \{x_n\} \subset \mathfrak{A}, x_n \to 0, \text{and } Dx_n \to m\},$$

is called the *separating space* for the operator D, measures the discontinuity of D.

By closed graph theorem $\mathfrak{S}(D) = 0$ if and only if D is continuous. When the operator D is undetstood, we abbreviate $\mathfrak{S}(D)$ by $\mathfrak{S}$.

The separating space $\mathfrak{S}$ is a submodule of $\mathfrak{M}$, and

$$I = I(D) = \{a \in \mathfrak{A} : a\mathfrak{S}(D) = 0\}$$

is an ideal of $\mathfrak{A}$ called the *continuity ideal* for the operator D. By ideal we will always mean two-sided ideal.

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If $E \subseteq \mathfrak{A}$, we call

$$E^{\perp} = \{m \in \mathfrak{M} |\ xm = mx = 0, \text{ for all } x \in E\},$$

that is called the *annihilator* of E in $\mathfrak{M}$.

For algebras without unit the simplest module condition is that $\mathfrak{A}^{\perp} = (0)$, i.e., $xm = mx = 0$ for all $x \in \mathfrak{A}$ implies $m = 0$. In this case, we called that $\mathfrak{M}$ is a *nondegenerate*. A closed subspace E of a Banach space X is *of finite codimension* (or cofinite) if $\dim(X/E) < \infty$, then in such condition if any operator is continuous on X/E, that is continuous.

Let $\text{rad}(\mathfrak{A})$ be the radical of the Banach algebra $\mathfrak{A}$, denoted by R. $\mathfrak{A}$ is *semisimple* if $\text{rad}(\mathfrak{A}) = \{0\}$. By [4], Let $\mathfrak{B}$ be a bi-ideal of $\mathfrak{A}$, then $\text{rad}(\mathfrak{B}) = \mathfrak{B} \cap \text{rad}(\mathfrak{A})$ and $\mathfrak{A}/\text{rad}(\mathfrak{A})$ is semisimple.

In the following theorem, let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module and also D be a module derivation from $\mathfrak{A}$ into $\mathfrak{M}$ with separating space $\mathfrak{S}(D)$ and continuity ideal I.

THEOREM 1. *If the continuity ideal I of a derivation $D : \mathfrak{A} \to \mathfrak{M}$ is of finite codimension in $\mathfrak{A}$ and $\mathfrak{M}$ is a nondegenerated I-module then the derivation D is continuous.*

PROOF. We restrict D to the continuity ideal I. Show that the restricted derivation $D|_I : \mathfrak{A} \to \mathfrak{M}$ is continuous. If $m \in \mathfrak{S}(D)$, then $xm = mx = 0$ for each $x \in I$ and hence $m = 0$. Thus $\mathfrak{S} = 0$ and consequently $D|_I$ is continuous. Therefore D is continuous on $\mathfrak{A}$ because I is of finite codimension.

Next we specialize to a certain commutative algebra. Let $\mathfrak{A}$ be a commutative semisimple Banach algebra considered as an algebra of functions on its structure space $\Phi_{\mathfrak{A}}$.

We call $\mathfrak{A}$ a *Silov algebra* if for each pair F_1, F_2 of disjoint closed sets in $\Phi_{\mathfrak{A}}$ with F_1 compact, there exists $x \in \mathfrak{A}$ with $x(F_1) = 1$, and $x(F_2) = 0$.

Let $\mathfrak{A}$ be a Silov algebra and $\phi \in \Phi_{\mathfrak{A}}$. The *primary ideal* $J(\phi)$ consists of those functions in $\mathfrak{A}$ each of which vanishes in some neighborhood of ϕ. i.e., $J(\phi) = \{x \in \mathfrak{A} | \ x(V_\phi = 0, V_\phi$ is a neighborhood of $\phi \in \Phi_{\mathfrak{A}}\}$.

DEFINITION. Let $\mathfrak{A}$ be a commutative semi-simple regular Banach algebra with unit and let ν be an arbitrary homomorphism of $\mathfrak{A}$ into a Banach algebra. A *singularity* set of ν is a finite set F of points of $\Phi_{\mathfrak{A}}$ which satisfies $||\nu(g)|| \leq M||g||||h||$ for all functions g and h in $\mathfrak{A}$ having carriers in $\Phi_{\mathfrak{A}} \sim F$ and such that $gh = g$.

The following two theorem proved in [3].

THEOREM 2. *Let $\mathfrak{A}$ be a Silov algebra and $\mathfrak{M}$ be an $\mathfrak{A}$-module satisfying* $\dim J(\phi)^{\perp} < \infty$ *for each $\phi \in \Phi_{\mathfrak{A}}$. If $D : \mathfrak{A} \to \mathfrak{M}$ is a discontinuous derivation, then $\mathfrak{S}$ is finite dimensional and I has finite codimension in $\mathfrak{A}$.*

THEOREM 3. *Let $\mathfrak{A}$ be a Silov algebra and $D : \mathfrak{A} \to \mathfrak{M}$ be a discontinuous derivation. Suppose* $\dim(\mathfrak{A}/I) < \infty$. *Then for each ϕ in the finite singularity set F for D there exists a nonzero vector m in $\mathfrak{S}$ which annihilators the maximal ideal $M(\phi)$ associated with the homomorphism ϕ.*

THEOREM 4. *Let $\mathfrak{A}$ be Silov algebra satisfying* $\dim J(\phi)^{\perp} < \infty$ *and $\mathfrak{M}$ be an $\mathfrak{A}$-module with $M(\phi)$-nondegenerate for each ϕ in the finite singularity set F for D. Then $D : \mathfrak{A} \to \mathfrak{M}$ is continuous.*

PROOF. Assume that D is discontinuous. By Theorem 2, then $\mathfrak{S}$ is finite dimensional and I has finite codimension. By Theorem 3, there exists a nonzero vector m in $\mathfrak{S}$ such that $m \cdot M(\phi) = 0$ for each $\phi \in F$. Since $\mathfrak{M}$ is a $M(\phi)$-nondegenerate, so $m = 0$. It is a contradiction.

The following lemma is due to Garimella [5].

LEMMA. *$\mathfrak{S}(D)$ is nilpotent if and only if $\mathfrak{S}(D) \cap R$ is nilpotent.*

THEOREM 6. *Let D be a derivation on a semiprime Banach algebra $\mathfrak{A}$ with radical R. If $D|_{\mathfrak{S}(D) \cap R}$ is continuous, then D is continuous.*

PROOF. Let $y \in \mathfrak{S}(D) \cap R$. Then there is a sequence x_n in $\mathfrak{A}$ with $x_n \to 0$ such that $Dx_n \to y$. Then $x_n y \to 0$ in $R \cap \mathfrak{S}(D)$ and $D(x_n y) = x_n(Dy) + (Dx_n)y$. Thus $D(x_n y) \to y^2$ and since $D|_{\mathfrak{S}(D) \cap R}$ is continuous, $y^2 = 0$. Therefore $\mathfrak{S}(D) \cap R$ is nilpotent and by Lemma 5, $\mathfrak{S}(D)$ is nilpotent. Since $\mathfrak{A}$ is semiprime, $\mathfrak{S}(D) = \{0\}$, thus D is continuous.

COROLLARY 7. *Let $\mathfrak{A}$ be a Banach algebra with the radical R which is an integral domain. If $D|_{\mathfrak{S}(D) \cap R}$ is continuous, then D is continuous.*

PROOF. Let $y \in \mathfrak{S}(D)$, then there is a sequence x_n in $\mathfrak{A}$ with $x_n \to 0$ such taht $Dx_n \to y$. For $0 \neq r \in R \cap \mathfrak{S}(D), rx_n \to 0$ in $R \cap \mathfrak{S}(D)$ and $D(rx_n) = rD(x_n) + (Dr)x_n$. Thus $D(rx_n) \to ry$ and so $ry = 0$. Since R is an integral domain, $y = 0$. Thus $\mathfrak{S}(D) = \{0\}$. Therefore D is continuous.

The following lemma is due to Garimella [5].

LEMMA 8. *If $\bigcap_{n \geq 1} R^n = \{0\}$, then $\mathfrak{S}(D)$ is nilpotent.*

THEOREM 9. *Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{B}$ be a semiprime Banach $\mathfrak{A}$-module with radical R such that $\bigcap_{n \geq 1} R^n = \{0\}$. Then every derivation from $\mathfrak{A}$ into $\mathfrak{B}$ is continuous.*

PROOF. It is immediate from Lemma 8.

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