JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 6, June 1993

A Maximal Element of Condensing Multimaps (*)

WON KYU KIM

ABSTRACT. In this note, we shall give a maximal element existence theorem for condensing multimaps in a locally convex Hausdorff topological vector space.

Shafer-Sonnenschein extended the Debreu theorem on the existence of equilibrium in a generalized N-person game or an abstract economy. In fact, they maintained the sprits of the pioneering works of Debreu, Arrow-Debreu and Mas-Colell. In 1976, Borglin-Keiding [1] first introduced the majorized concept of multimaps and recently many authors have proved a number of general results on the existence of equilibrium point of generalized games with infinite dimemsional commodity spaces and infinite number of agents. Furthermore, Kim [4], Ding-Kim-Tan [2], Tarafdar [8] and Mehta [6] have investigated the existence of equilibrium for non-compact generalized games in locally convex Hausdorff topological vector spaces. The main tool for proving the existence of equilibrium is the maximal element existence theorem, e.g. see [4, 8].

The purpose of this note is to prove a new existence theorem of maximal element for condensing multimap in a non-compact subset of a locally convex Hausdorff topological vector space.

(*) This work was partially supported by KOSEF in 1993
Received by the editors on May 24, 1993.
1980 *Mathematics subject classifications*: Primary 47H09; Secondary 47N10.

If A is a set, we shall denote by 2^A the family of all subsets of A. If A is a non-empty subset of a topological vector space E, we shall denote by $co\,A$, $cl\,A$, $\overline{co}\,A$ the convex hull of A, the closure of A in E, the closed convex hull of A in E, respectively. If $T : A \to 2^E$ is a multimap, then $co\,T : A \to 2^E$ is a multimap defined by $(co\,T)(x) = co\,T(x)$ for each $x \in A$.

Let X be a non-empty subset of a topological vector space. A multimap $\phi : X \to 2^X$ is said to be of *class* L [9] if (i) for each $x \in X$, $x \notin co\,\phi(x)$, (ii) for each $y \in X$, $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. Let $\phi : X \to 2^X$ be a given multimap and $x \in X$; then a multimap $\phi_x : X \to 2^X$ is said to be an *L-majorant of ϕ at x* [9] if ϕ_x is of class L and there exists an open neighborhood N_x of x in X such that for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$. The multimap ϕ is said to be *L-majorized* if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an L-majorant of ϕ at x.

We remark here that the notions of a multimap of class L and an L-majorized multimap generalize the notions of KF-multimap and KF-majorized multimap introduced in [1], respectively. And those notions have been further generalized in [2, 4].

In order to relax the compactness assumption, we shall need the following defintions in [3]. Let E be a locally convex Hausdorff topological vector space and $\mathcal{B}$ be a basis of convex open neighborhoods of 0. If $A \subset E$, we define $Q(A)$ to be the collection of all $U \in \mathcal{B}$ such that $A \subset K + U$ for some precompact subset K of E. Then the set $Q(A)$ is called a *measure of nonprecompactness* of A, i.e. the larger $Q(A)$ is, the more nearly is A precompact.

When $A, B \subset E$ are non-empty bounded but not precompact subsets of E, the following properties of the measure of nonprecompactness are very essential in our proofs (for the proofs, see [3]) :

(i) A is precompact if and only if $Q(A) = \mathcal{B}$.
(ii) $Q(cl\, A) = Q(A)$.
(iii) $Q(co\, A) = Q(A)$.
(iv) $Q(A \cup B) = Q(A) \cap Q(B)$.
(v) if $A \subset B$, then $Q(B) \subset Q(A)$.

DEFINITION. [3] Let X be a non-empty subset of a locally convex Hausdorff topological vector space E. A multimap $T : X \to 2^E$ is called *condensing* if for some choice of basis $\mathcal{B}$ of convex open neighborhoods of 0, we have $Q(A) \subsetneqq Q(T(A))$ for every bounded but not precompact subset A of X.

The sense of the above definition is that the condensing multimaps take bounded but not precompact sets to sets which are more nearly precompact. And it is not difficult to prove that if E is a Banach space and if $\mathcal{B}$ is the collection of spherical neighborhoods of 0, then this definition of condensing multimap can be reduced to Sadovskii's original definiton [7]. In this case, the condensing condition means that the size of the image $T(A)$ is smaller than that of A. It is evident that a multimap T is condensing when either T is a compact multimap or T takes bounded sets to precompact sets.

We first prove the following, which generalizes Lemma in [5] to a locally convex Hausdorff topological vector space.

LEMMA 1. *Let X be a non-empty closed bounded convex subset of a locally convex Hausdorff topological vector space E and $T : X \to 2^X$ be a condensing multimap. Then there exists a non-empty compact convex subset C_o of X such that $T(C_o) \subset C_o$.*

PROOF. Let $x_o \in X$ and consider the family $\mathcal{F}$ of all closed convex subset C of X such that $x_o \in C$ and $T(C) \subset C$. Then $\mathcal{F} \neq \emptyset$ and let $C_o := \cap_{C \in \mathcal{F}} C$. Then C_o is non-empty closed convex and

$x_o \in C_o$. We know that $T(C_o) \subset C_o$. In fact, for any $x \in C_o$, $x \in C$ for every $C \in \mathcal{F}$, and hence $T(x) \subset C$ for all $C \in \mathcal{F}$. Therefore $T(x) \subset \bigcap_{C \in \mathcal{F}} C = C_o$. Suppose that C_o is not precompact; then $Q(C_o) \subsetneqq \mathcal{B}$. Since T is condensing, we have $Q(T(C_o)) \supsetneqq Q(C_o)$. Let $C_1 = \overline{co}(\{x_o\} \cup T(C_o))$; then $C_1 \subset C_o$ and hence $T(C_1) \subset T(C_o) \subset C_1$. Therefore we have $C_1 = C_o$. However,

$$\begin{aligned} Q(C_1) &= Q(\overline{co}(\{x_o\} \cup T(C_o))) \\ &= Q(\{x_o\} \cup T(C_o)) \\ &= Q(\{x_o\}) \cap Q(T(C_o)) \\ &= Q(T(C_o)) \qquad (\text{since } Q(\{x_o\}) = \mathcal{B}) \\ &\supsetneqq Q(C_o), \end{aligned}$$

which is a contradiction. Therefore C_o is precompact. Since C_o is closed, C_o is the desired compact convex set. This completes the proof.

As a consequence of Lemma 1, we can obtain a fixed point theorem for a condensing upper semicontinuous multimap as in [3].

The following is a special case of Lemma 1 in [2] :

LEMMA 2. *Let X be a non-empty convex subset of a topological vector space and $P : X \to 2^X$ be L-majorized. If every open subset of X containing the set $\{x \in X : P(x) \neq \emptyset\}$ is paracompact, then there exists a multimap $S : X \to 2^X$ of class L such that $P(x) \subset S(x)$ for all $x \in X$.*

We are ready to prove the following existence theorem of maximal elements for condensing multimaps.

THEOREM. *Let X be a non-empty closed bounded convex subset of a locally convex Hausdorff topological vector space E and $T : X \to 2^X$*

be a condensing multimap. If T is L-majorized, then there exists a maximal element, i.e. there exists $\hat{x} \in X$ such that $T(\hat{x}) = \emptyset$.

PROOF. Suppose the contrary, i.e. $T(x) \neq \emptyset$ for each $x \in X$. By Lemma 1, there exists a non-empty compact convex subset K of X such that $T(K) \subset K$. Since the set $\{x \in K : T(x) \neq \emptyset\} = K$ is compact, by Lemma 2, we can find a multimap $S : K \to 2^K$ satisfying the following

(i) $T(x) \subset S(x)$ for each $x \in K$,

(ii) for each $x \in K$, $x \notin co\, S(x)$,

(iii) for each $y \in K$, $S^{-1}(y)$ is open in K.

By Proposition 5.1 in [9], for each $y \in K$, $(co\, S)^{-1}(y)$ is also open in K. By applying the Fan-Browder fixed point theorem [6, p.7] to $co\, S$, there exists a point $\tilde{x} \in K$ such that $\tilde{x} \in co\, S(\tilde{x})$, which contradicts the condition (ii). Therefore there exists a maximal element $\hat{x} \in X$ such that $T(\hat{x}) = \emptyset$. This completes the proof.

REMARKS. (i) Theorem generalizes Theorem in [5] and Theorem 7. 15 in [6] to locally convex Hausdorff topological vector spaces.

(ii) If X is compact, then $T : X \to 2^X$ is automatically a condensing multimap. And in this case, Theorem is a generalization of Corollary 1 in [1].

Maximal element existence theorems are essential tools in proving the existence of equilibrium in generalized games, e.g. see [1, 2, 4, 8, 9]. As an application of Theorem, we can obtain an existence theorem of equilibrium for a non-compact 1-person game with L-majorized preference multimap and condensing constraint multimap in a locally convex Hausdorff topological vector space, e.g. see [10].

REFERENCES

1. A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilib-*

rium : A note on the 'new' existence theorem, J. Math. Econom. **3** (1976), 313–316.

2. X. P. Ding, W. K. Kim and K.-K.Tan, *Equilibria of non-compact generalized games with L^*-majorized preferences*, J. Math. Anal. Appl. **164** (1992), 508–517.
3. C. J. Himmelberg, J. R. Porter and F. S. Van Vleck, *Fixed point theorems for condensing multifunctions*, Proc. Amer. Math. Soc. **23** (1969), 635–641.
4. W. K. Kim, *Existence of maximal element and equilibrium for non-paracompact N-person game*, Proc. Amer. Math. Soc. **116** (1992), 797–807.
5. G. Mehta, *Maximal elements of condensing preference maps*, Appl. Math. Lett. **3** (1990), 69–71.
6. G. Mehta, *Topological methods in equilibrium analysis*, Ph. D. thesis, Dept. of Math., Univ. of Queensland, 1987.
7. B. N. Sadovskii, *A fixed point principle*, Functional Anal. Appl. **1** (1967), 151–153.
8. E. Tarafdar, *Equilibrium in abstract economies with a non-compact infinite demensional strategy space, an infinite number of agents and without ordered preferences*, J. Math. Econom. **33** (1990), 203–206.
9. N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), 233–245.
10. W. K. Kim, *Maximal elements and equilibria of condensing multimpas*, Numer. Funct. Anal. Optimiz. (1993) (to appear).

Department of Mathematics Education
Chungbuk National University
Cheongju 360–763, Korea