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Fixed Point Theorems in Product Spaces

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ABSTRACT. Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that X is weakly compact, convex and has the fixed point property for a nonexpansive mapping, and Y has the fixed point property for a multivalued nonexpansive mapping. Then $(X \oplus Y)_p, 1 \leq p < \infty$ has the fixed point property for a multivalued nonexpansive mapping. Furthermore, if X has the generic fixed point property for a nonexpansive mapping, then $(X \oplus Y)_\infty$ has the fixed point property for a multivalued nonexpansive mapping.

1.Introduction.

Let E and F be Banach spaces. A mapping $T : E \to F$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$. A multivalued mapping $T : E \to F$ is nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, where H is the Hausdorff metric induced by the norm of F. We recall that a nonempty subset X of E is said to have the fixed point property for (multivalued) nonexpansive mappings if every (multivalued, resp.) nonexpansive mapping $T : X \to X$ has a fixed point, and the space E has the fixed point property if every weakly compact convex subset of X has the fixed point property.

Now suppose E and F are Banach spaces with $X \subset E$ and $Y \subset F$ and let $E \oplus F$ be the product spaces. For $1 \leq p < \infty$ and $(x, y) \in E \oplus F$, we set

$$\|(x, y)\|_p = (\|x\|_E^p + \|y\|_F^p)^{\frac{1}{p}}$$

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and

$$\|(x,y)\|_\infty = \max\{\|x\|_E, \|y\|_F\}.$$

In[1], W.A.Kirk and Y.Sternfeld showed that if X and Y are bounded, closed and convex subsets of a uniformly convex Banach space E and Y is separable, then the assumption that Y has the fixed point property for a nonexpansive mapping assures the same is true of $(X \oplus Y)_\infty$. In [2], if E and F be Banach spaces with $X \subset E$ and $Y \subset F$, and if both X and Y have the fixed point property for nonexpansive mappings, then $(X \oplus Y)_p$ has the fixed point property for nonexpansive mappings for $1 \leq p < \infty$. Also it was shown in [3] that if E has KK-norm, $\phi \neq X \subset E$ is weakly compact and convex, and if X and Y have the fixed point property for nonexpansive mappings, then $(X \oplus Y)_\infty$ has the fixed point property for nonexpansive mappings. Moreover, T.Kuczumow[4] improved [3] by assuming that X has the generic fixed point property.

In this paper we prove fixed point theorems for multivalued nonexpansive mappings in $(X \oplus Y)_p, 1 \leq p \leq \infty$.

2.Results.

Now we state our first theorem.

THEOREM 1. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that X is weakly compact, convex and has the fixed point property for nonexpansive mappings, and Y has the fixed point property for multivalued nonexpansive mappings. If $T_1 : (X \oplus Y)_p \to X$ is a nonexpansive mapping and $T_2 : (X \oplus Y)_p \to Y$ is a multivalued nonexpansive mapping with closed values, then a mapping $T = (T_1, T_2) : (X \oplus Y)_p \to (X \oplus Y)_p$ which is defined by $(T_1, T_2)(x, y) = (T_1(x, y), T_2(x, y))$ has a fixed point for $1 \leq p < \infty$.*

PROOF. For each fixed $y \in Y$, we define $T_y : X \to X$ by $T_y(x) =$

$T_1(x,y), x \in X$. Then T_y is nonexpansive and has a fixed point which we denote by y(1).

Now define $\overline{T_2} : Y \to Y$ by $\overline{T_2}(y) = T_2(y(1), y)$. Then for $y, z \in Y$,

$$\begin{aligned} H(\overline{T_2}(y), \overline{T_2}(z))^p &= H(T_2(y(1),y), T_2(z(1),z))^p \\ &= H((T_1,T_2)(y(1),y),(T_1,T_2)(z(1),z))^p \\ &\qquad - \|T_1(y(1),y) - T_1(z(1),z)\|^p \\ &\le \|y(1)-z(1)\|^p + \|y-z\|^p - \|y(1)-z(1)\|^p \\ &= \|y-z\|^p \end{aligned}$$

Therefore, $\overline{T_2}$ is a multivalued nonexpansive mapping, and hence $\overline{T_2}$ has a fixed point $y_0 \in Y$, that is $y_0 \in \overline{T_2}(y_0) = T_2(y_0(1), y_0)$. It follows that $(y_0(1), y_0)$ is a fixed point of T, since $(y_0(1), y_0) \in (T_1(y_0(1), y_0), T_2(y_0(1), y_0)) = (T_1, T_2)(y_0(1), y_0) = T(y_0(1), y_0)$.

A nonempty, convex and weakly compact subset X of a Banach space E has the generic fixed point property for nonexpansive mappings if for every nonexpansive mapping $T : X \to X$ and every nonempty convex closed subset $X_0 \subset X$ with $TX_0 \subset X_0$, we have $X_0 \cap \mathrm{Fix}(T) \neq \phi$, where $\mathrm{Fix}(T) = \{x \in X; Tx = x\}$.

The following lemma is needed to prove our second result.

LEMMA (T.Kuczumow [4]). *Let E be a Banach space and let Y be a metric space. Suppose that $X \subset E$ is weakly compact, convex and has the generic fixed point property for nonexpansive mappings. If $F : (X \oplus Y)_\infty \to X$ is a nonexpansive mapping, then there exists a nonexpansive mapping $r : (X \oplus Y)_\infty \to X$ such that $F(r(x,u),u) = r(x,u)$ for $(x,u) \in (X \oplus Y)_\infty$ and r(x,u)=x when F(x,u)=x.*

THEOREM 2. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that X is weakly compact and convex, and has* *the generic fixed point property for nonexpansive mappings and Y has the fixed point property for multivalued nonexpansive mappings. If $T_1(X \oplus Y)_\infty \to X$ is a nonexpansive mapping and $T_2 : (X \oplus Y)_\infty \to Y$ is a multivalued nonexpansive mapping, then a mapping $T = (T_1, T_2) : (X \oplus Y)_\infty \to (X \oplus Y)_\infty$ which is defined by $(T_1, T_2)(x, y) = (T_1(x, y), T_2(x, y))$ has a fixed point.*

PROOF. Let $T = (T_1, T_2) : (X \oplus Y)_\infty \to (X \oplus Y)_\infty$ be a mapping such that $T_1(X \oplus Y)_\infty \to X$ is nonexpansive. Then by the Lemma, we can obtain a nonexpansive mapping $r : (X \oplus Y)_\infty \to X$ such that $T_1(r(x, y)) = r(x, y)$ for $(x, y) \in (X \oplus Y)_\infty$ and $r(x, y) = x$ when $T_1(x, y) = x$. Now for a fixed $x_0 \in X$, define $\overline{T_2} : Y \to Y$ by $\overline{T_2}(y) = T_2(r(x_0, y), y), y \in Y$. Then for $y, z \in Y$,

$$\begin{aligned} H(\overline{T_2}(y), \overline{T_2}(z)) &= H(T_2(r(x_0, y), y), T_2(r(x_0, z), z) \\ &\le \max\{\|r(x_0, y) - r(x_0, z)\|, \|y - z\| \\ &\le \|y - z\|. \end{aligned}$$

Therefore $\overline{T_2}$ is a multivalued nonexpansive mapping, and hence T_2 has a fixed point $y_0 \in Y$, that is $y_0 \in \overline{T_2}(y_0) = T_2(r(x_0, y_0), y_0)$. This implies that

$$(r(x_0, y_0), y_0) \in (T_1(r(x_0, y_0), y_0), T_2(r(x_0, y_0), y_0)) = T(r(x_0, y_0), y_0).$$

This completes the proof.

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